8.286 Lecture 8
October 1, 2018

THE DYNAMICS OF NEWTONIAN COSMOLOGY, PART 3

Announcements

- Quiz 1 is Wed Oct 3, two days from now.
- Review Problems for Quiz 1, with solutions, are posted on the website. It also includes a formula sheet, which you will be given at the quiz. There are 24 problems, 8 of which are highlighted by stars. The 1up version is bookmarked.
- The coverage of the quiz is listed on the website and on the Review Problems: Lecture Notes 1-3, Problem Sets 1-3, Weinberg chapters 1-3, Ryden chapters 1, 2, and section 3.1.
- One problem on the quiz will be taken verbatim, or almost verbatim, from either the homework assignments or from the starred Review Problems. For the homework, extra credit problems are eligible to be the problem used on the quiz.

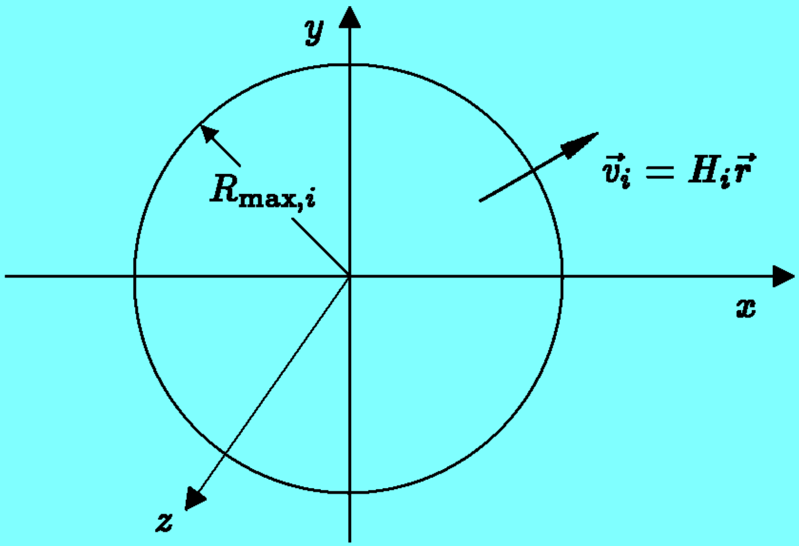
- Old exams are on the web, going back to 1994. Follow the link "8.286 Web Page Information from Previous Years".
- Office hours:

    Me: Today, Monday 7:30–8:30 pm, 6-322.

    Honggeun: Tomorrow, Tues at 5–6 pm, 8-308.

    No office hours Wed or Thurs.

Summary: Mathematical Model



$$t_i \equiv \text{time of initial picture}$$

$$R_{\max,i} \equiv \text{initial maximum radius}$$

$$\rho_i \equiv \text{initial mass density}$$

$$\vec{v}_i = H_i\vec{r} \ .$$

☆ Newtonian gravity of a shell:

Inside: $\vec{g} = 0$.

Outside: Same as point mass at center, with same M.

☆ Let $r(r_i, t) \equiv$ radius at t of shell initially at r_i.

☆ Let $M(r_i) \equiv$ mass inside r_i-shell $= \frac{4\pi}{3} r_i^3 \rho_i$ at all times.

☆ $$\vec{g} = -\frac{GM(r_i)}{r^2}\hat{r} \quad \Longrightarrow \quad \ddot{r} = -\frac{4\pi}{3}\frac{G r_i^3 \rho_i}{r^2}\ ,$$ where $r \equiv r(r_i, t)$.

☆ Initial conditions: $r(r_i, t_i) = r_i\ , \quad \dot{r}(r_i, t_i) = H_i r_i\ .$

☆ Rescaling: Let $u(r_i, t) \equiv \dfrac{r(r_i, t)}{r_i} \equiv a(t)$, where $r = a(t) r_i$ and

$$\boxed{\ddot{a} = -\frac{4\pi}{3}\frac{G\rho_i}{a^2}\ , \quad a(t_i) = 1\ , \quad \dot{a}(t_i) = H_i\ ,}$$ and

$$\boxed{\ddot{a} = -\frac{4\pi}{3} G\rho(t) a\ .}$$

Summary: A Conservation Law

$$\ddot{a} = -\frac{4\pi}{3}\frac{G\rho_i}{a^2} \quad \Longrightarrow \quad \dot{a}\left\{\ddot{a} + \frac{4\pi}{3}\frac{G\rho_i}{a^2}\right\} = 0 \quad \Longrightarrow \quad \frac{dE}{dt} = 0\ ,$$

where

$$E = \frac{1}{2}\dot{a}^2 - \frac{4\pi}{3}\frac{G\rho_i}{a}\ .$$

Summary: Equations

Want: $r(r_i, t) \equiv$ radius at t of shell initially at r_i

Find: $r(r_i, t) = a(t) r_i$, where

Friedmann Equations
$$\begin{cases} \ddot{a} = -\dfrac{4\pi}{3} G\rho(t) a \\ H^2 = \left(\dfrac{\dot{a}}{a}\right)^2 = \dfrac{8\pi}{3} G\rho - \dfrac{kc^2}{a^2} \quad \text{(Friedmann Eq.)} \end{cases}$$

and

$$\rho(t) \propto \frac{1}{a^3(t)} \;, \quad \text{or} \quad \rho(t) = \left[\frac{a(t_1)}{a(t)}\right]^3 \rho(t_1) \text{ for any } t_1 .$$

Conventions

Us: Notch is arbitrary (free to be redefined each time we use it).

[Our construction used $a(t_i) = 1$ m/notch, but we can interpret this equation as the definition of t_i. But we can forget the definition of t_i if we don't intend to use it.]

Ryden: $a(t_0) = 1$ (where $t_0 =$ now). (For us, Ryden's convention is $a(t_0) = 1$ m/notch.)

Many Other Books: if $k \neq 0$, then $k = \pm 1$.

(For us, this means $k = \pm 1/\text{notch}^2$.)

Types of Solutions

$$\dot{a}^2 = \frac{8\pi G}{3} \frac{\rho(t_1) a^3(t_1)}{a(t)} - kc^2 \quad \text{(for any } t_1) \; .$$

For intuition, remember that $k \propto -E$, where E is a measure of the energy of the system.

Types of Solutions:

1) $k < 0$ $(E > 0)$: unbound system. $\dot{a}^2 > (-kc^2) > 0$, so the universe expands forever. **Open Universe.**

2) $k > 0$ $(E < 0)$: bound system. $\dot{a}^2 \geq 0 \quad \Longrightarrow$

$$a_{\max} = \frac{8\pi G}{3} \frac{\rho(t_1) a^3(t_1)}{kc^2} \; .$$

Universe reaches maximum size and then contracts to a Big Crunch. **Closed Universe.**

3) $k = 0$ ($E = 0$): critical mass density.

$$H^2 = \frac{8\pi G}{3}\rho - \underbrace{\frac{kc^2}{a^2}}_{=0} \quad \Longrightarrow \quad \boxed{\rho \equiv \rho_c = \frac{3H^2}{8\pi G}\ .}$$

Flat Universe.

Summary: $\rho > \rho_c \Longleftrightarrow$ closed, $\rho < \rho_c \Longleftrightarrow$ open, $\rho = \rho_c \Longleftrightarrow$ flat.

Numerical value: For $H = 68$ km-s^{-1}-Mpc^{-1} (Planck 2015 plus other experiments),

$$\rho_c = 8.7 \times 10^{-27}\ \mathrm{kg/m^3} = 8.7 \times 10^{-30}\ \mathrm{g/cm^3}$$

$$\approx 5 \text{ proton masses per m}^3.$$

Definition: $\Omega \equiv \dfrac{\rho}{\rho_c}$.

Evolution of a Flat Universe

If $k = 0$, then

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho = \frac{\text{const}}{a^3} \implies \frac{da}{dt} = \frac{\text{const}}{a^{1/2}}$$

$$\implies a^{1/2} a\, da = \text{const}\, dt \implies \frac{2}{3} a^{3/2} = (\text{const})t + c' \ .$$

Choose the zero of time to make $c' = 0$, and then

$$a(t) \propto t^{2/3} \ .$$

Age of a Flat Matter-Dominated Universe:

$$a(t) \propto t^{2/3} \quad \Longrightarrow \quad t = \frac{2}{3} H^{-1}$$

For $H = 67.7 \pm 0.5$ km-s^{-1}-Mpc^{-1}, age $= 9.56$ – 9.70 billion years — but stars are older. Conclusion: our universe is nearly flat, but not matter-dominated.

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The Big Bang Singularity: $a(0) = 0$, with infinite density, is a feature of our model, but not necessarily the real universe.