8.286 Lecture 9
October 10, 2018

DYNAMICS OF HOMOGENEOUS EXPANSION, PART IV

Summary of Lecture 8

Age of a Flat Matter-Dominated Universe:

$$a(t) \propto t^{2/3} \quad \Longrightarrow \quad t = \frac{2}{3} H^{-1}$$

For $H = 67.7 \pm 0.5$ km-s^{-1}-Mpc^{-1}, age $= 9.56 - 9.70$ billion years — but stars are older. Conclusion: our universe is nearly flat, but not matter-dominated.

The Big Bang Singularity: $a(0) = 0$, with infinite density, is a feature of our model, but not necessarily the real universe.

Horizon Distance: the present distance of the furthest particles from which light has had time to reach us.

$$\ell_{\rm phys,horizon}(t) = a(t) \int_0^t \frac{c}{a(t')} {\rm d}t' \ .$$

$$a(t) \propto t^{2/3} \quad \Longrightarrow \quad \ell_{\rm phys,horizon} = 3ct = 2cH^{-1} \ .$$

Equations for a Matter-Dominated Universe

("Matter-dominated" = dominated by nonrelativistic matter.)

Friedmann equations:

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{a^2} \ ,$$

$$\ddot a = -\frac{4\pi}{3} G\rho(t) a \ .$$

Matter conservation:

$$\rho(t) \propto \frac{1}{a^3(t)} \ , \text{ or } \rho(t) = \left[\frac{a(t_1)}{a(t)}\right]^3 \rho(t_1) \text{ for any } t_1 .$$

Any two of the above equations can allow us to find the third.

Evolution of a Closed Universe

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{a^2}\ , \qquad \rho(t)a^3(t) = \text{constant}\ ,\ k > 0\ .$$

Recall $[a(t)] = \text{meter/notch}$, $[k] = 1/\text{notch}^2$.

Define new variables:

$$\tilde{a}(t) \equiv \frac{a(t)}{\sqrt{k}}\ , \qquad \tilde{t} \equiv ct \qquad \text{(both with units of distance)}$$

Multiplying Friedmann eq by $a^2/(kc^2)$:

$$\frac{1}{kc^2}\left(\frac{da}{dt}\right)^2 = \frac{8\pi}{3}\frac{G\rho a^2}{kc^2} - 1\ .$$

$$\begin{aligned}\frac{1}{kc^2}\left(\frac{da}{dt}\right)^2 &= \frac{8\pi}{3}\frac{G\rho a^2}{kc^2} - 1 \\ &= \frac{8\pi}{3}\frac{G\rho a^3}{k^{3/2}c^2}\frac{\sqrt{k}}{a} - 1\ .\end{aligned} \qquad (4.15)$$

Rewrite as

$$\left(\frac{d\tilde{a}}{d\tilde{t}}\right)^2 = \frac{2\alpha}{\tilde{a}} - 1\ ,$$

where

$$\alpha \equiv \frac{4\pi}{3}\frac{G\rho\tilde{a}^3}{c^2}\ .$$

$[\alpha] = \text{meter}$. α is constant, since ρa^3 is constant.

$$\left(\frac{d\tilde{a}}{d\tilde{t}}\right)^2 = \frac{2\alpha}{\tilde{a}} - 1 \quad \Longrightarrow \quad d\tilde{t} = \frac{\tilde{a}\,d\tilde{a}}{\sqrt{2\alpha\tilde{a} - \tilde{a}^2}}\ .$$

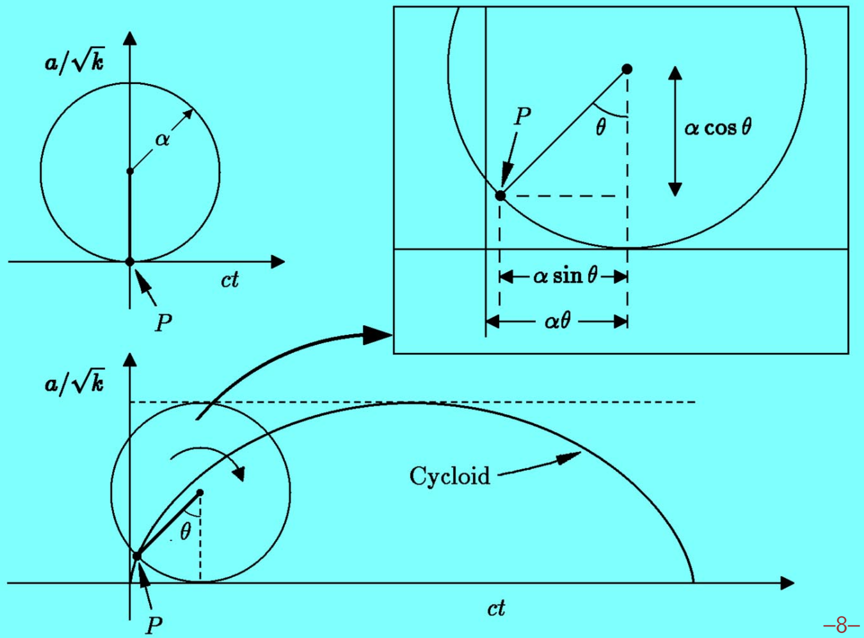
Then

$$\tilde{t}_f = \int_0^{\tilde{t}_f} d\tilde{t} = \int_0^{\tilde{a}_f} \frac{\tilde{a}\,d\tilde{a}}{\sqrt{2\alpha\tilde{a} - \tilde{a}^2}}\ ,$$

where $\tilde{t}_f$ is an arbitrary choice for a "final time" for the calculation, and $\tilde{a}_f$ is the value of $\tilde{a}$ at time $\tilde{t}_f$.

Evolution of a Closed Universe

$$ct = \alpha(\theta - \sin\theta)\ ,$$
$$\frac{a}{\sqrt{k}} = \alpha(1 - \cos\theta)\ .$$



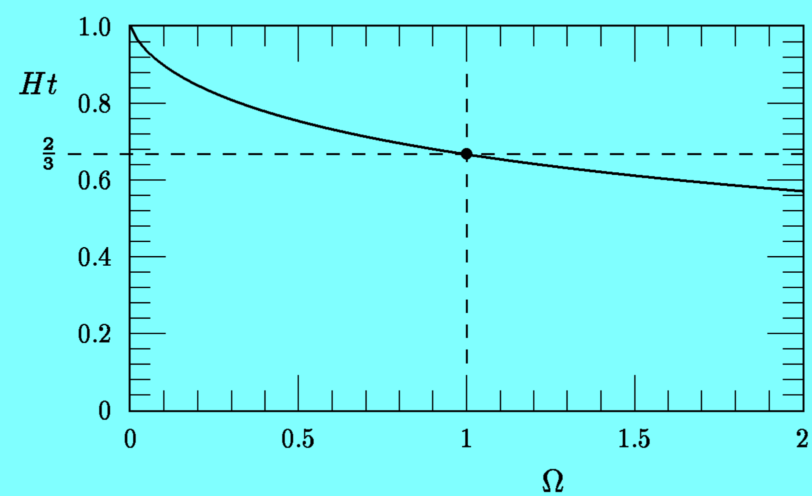
Evolution of a Closed Universe

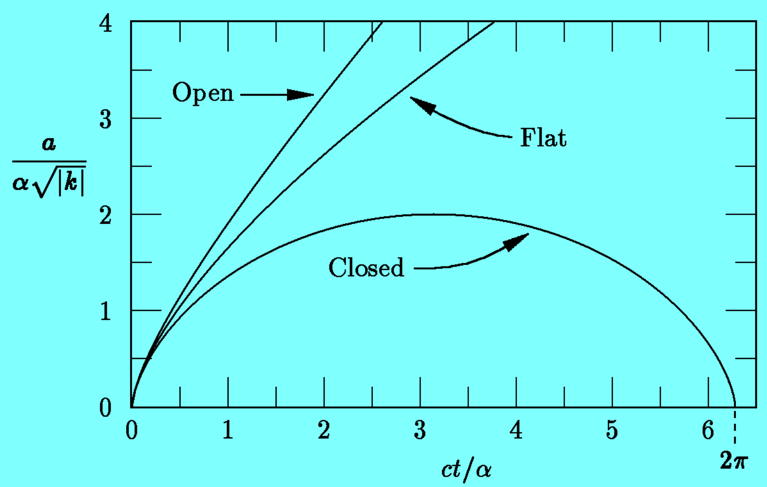
$$ct = \alpha(\theta - \sin\theta) \; ,$$
$$\frac{a}{\sqrt{k}} = \alpha(1 - \cos\theta) \; .$$

$$t = \frac{\Omega}{2|H|(\Omega-1)^{3/2}} \left\{ \arcsin\left(\pm \frac{2\sqrt{\Omega-1}}{\Omega} \right) \mp \frac{2\sqrt{\Omega-1}}{\Omega} \right\} \; .$$

$$t = \frac{\Omega}{2|H|(\Omega-1)^{3/2}} \left\{ \arcsin\left(\pm \frac{2\sqrt{\Omega-1}}{\Omega} \right) \mp \frac{2\sqrt{\Omega-1}}{\Omega} \right\} \; .$$

Quadrant	Phase	Ω	Sign Choice	$\sin^{-1}()$
1	Expanding	1 to 2	Upper	0 to $\frac{\pi}{2}$
2	Expanding	2 to ∞	Upper	$\frac{\pi}{2}$ to π
3	Contracting	∞ to 2	Lower	π to $\frac{3\pi}{2}$
4	Contracting	2 to 1	Lower	$\frac{3\pi}{2}$ to 2π




4
3
2
1
0
$\frac{a}{\alpha\sqrt{|k|}}$
Open
Flat
Closed
0
1
2
3
4
5
6
2π
ct/α