8.286 Lecture 10
October 15, 2018

INTRODUCTION TO NON-EUCLIDEAN SPACES

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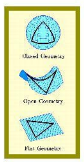
Ants on a Pringle

Mustafa Amin
18.10.2011



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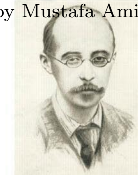
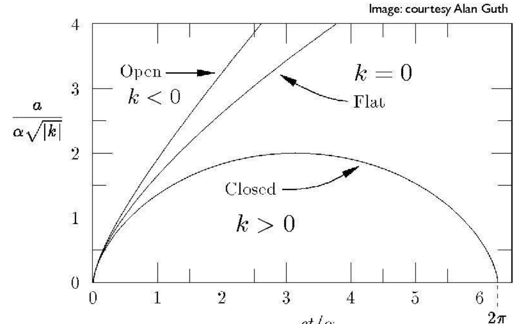
Recap



$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}$$

Image: courtesy Alan Guth



Note: matter dominated universes only!

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k ?

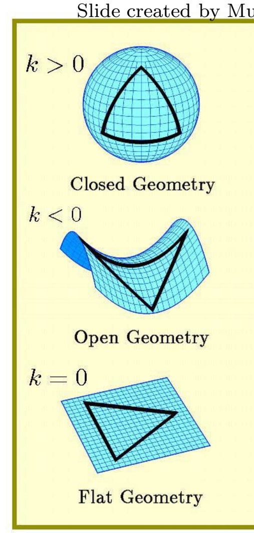

Image: courtesy Alan Guth

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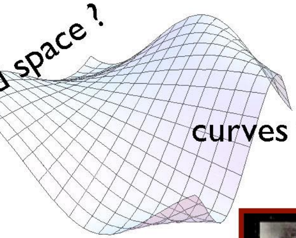
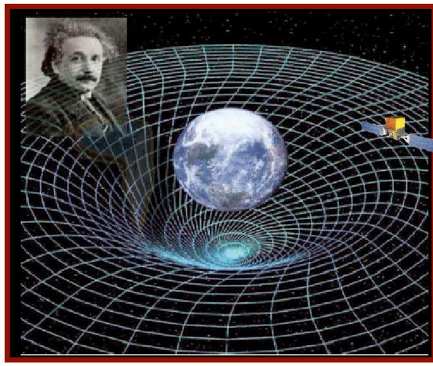


curved space ?

curves with respect to what ?

curved spacetime ?



non-Euclidean geometry

Note on image: I have added the "Dali" clock and Fermi satellite images to the original image created for the Gravity Probe B collaboration

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Euclid's *Postulates*

1. A straight line segment can be drawn joining any two points.
2. Any straight line segment can be extended indefinitely in a straight line.
3. Given a straight line segment , a circle can be drawn having the segment as radius and one endpoint as center.
4. All right angles are congruent.

5. If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines if produced indefinitely meet on that side on which the angles are less than two right angles

Corrected 10/10/13

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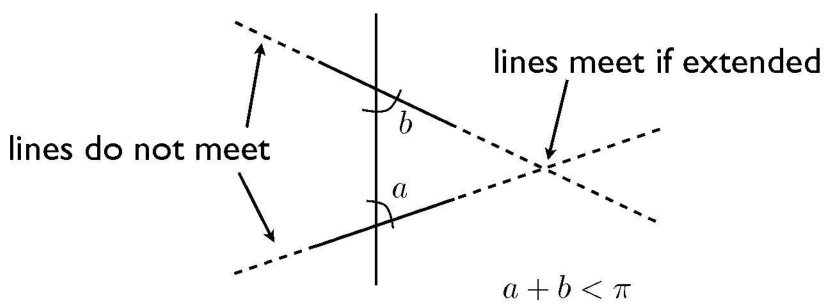
5th Postulate

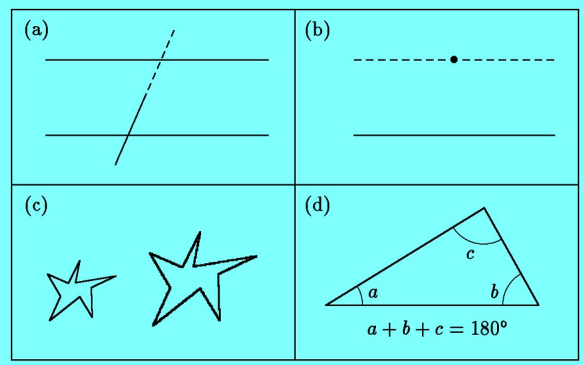


$$a + b < \pi$$

5. If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines if produced indefinitely meet on that side on which the angles are less than two right angles

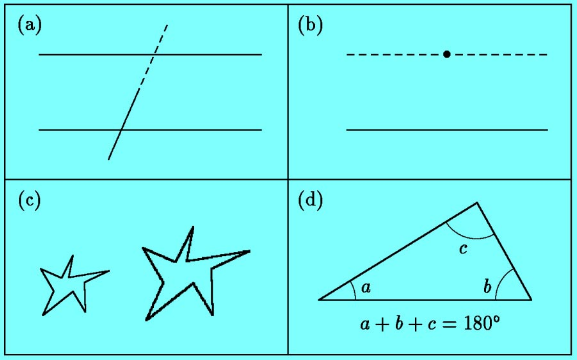
Corrected 10/10/13

Equivalent Statements of the 5th Postulate



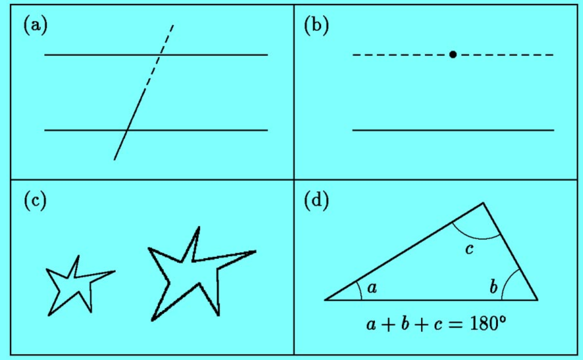
(a) "If a straight line intersects one of two parallels (i.e, lines which do not intersect however far they are extended), it will intersect the other also."

Equivalent Statements of the 5th Postulate



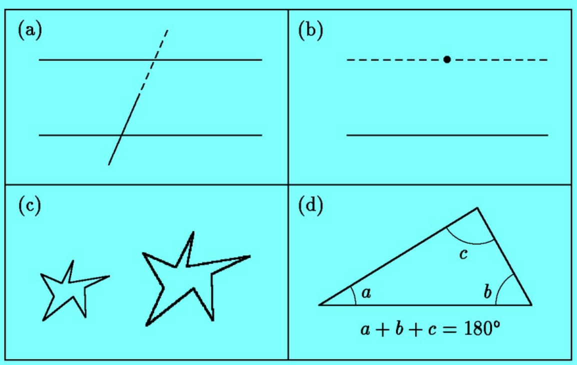
(b) "There is one and only one line that passes through any given point and is parallel to a given line."

Equivalent Statements of the 5th Postulate



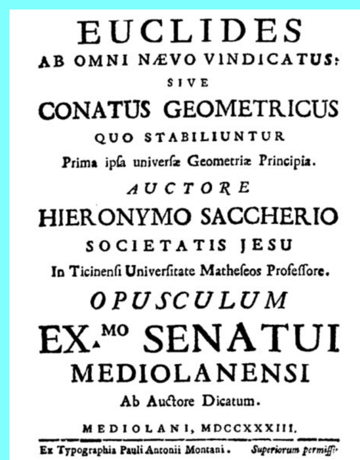
(c) "Given any figure there exists a figure, similar to it, of any size." (Two polygons are similar if their corresponding angles are equal, and their corresponding sides are proportional.)

Equivalent Statements of the 5th Postulate



(d) "There is a triangle in which the sum of the three angles is equal to two right angles (i.e., 180°)."

Giovanni Geralamo Saccheri (1667–1733)


EUCLIDES
AB OMNI NÆVO VINDICATUS:
SIVE
CONATUS GEOMETRICUS
QUO STABILIUNTUR
Prima ipsa universæ Geometriæ Principia.
AUCTORE
HIERONYMO SACCHERIO
SOCIETATIS JESU
In Ticinensi Universitate Matheseos Professore.
OPUSCULUM
EX.MO SENATUI
MEDIOLANENSI
Ab Auctore Dicatum.
MEDIOLANI, MDCCXXXIII.
Ex Typographia Pauli Antonii Montani. Superiorum permissu.

In 1733, Saccheri, a Jesuit priest, published *Euclides ab omni naevo vindicatus (Euclid Freed of Every Flaw).*

The book was a study of what geometry would be like if the 5th postulate were false.

He hoped to find an inconsistency, but failed.

Carl Friedrich Gauss (1777–1855)

German mathematician and physicist.

Born as the son of a poor working-class parents. His mother was illiterate and never even recorded the date of his birth.

His students included Richard Dedekind, Bernhard Riemann, Peter Gustav Lejeune Dirichlet, Gustav Kirchhoff, and August Ferinand Möbius.

János Bolyai (1802–1860)

Hungarian mathematician and army officer.

Son of Farkas Bolyai, a teacher of mathematics, physics, and chemistry at the Calvinist College in Marosvásárhely, Hungary.

Attended Marosvásárhely College and later studied military engineering at the Academy of Engineering at Vienna, because that is what his family could afford.

Served 11 years in the army engineering corps; during this time he developed his non-Euclidean geometry, which was published as an appendix to a book written by his father.

Retired from the army at age 31 due to poor health, and died in relative poverty at age 57, from pneumonia.

Nikolai Ivanovich Lobachevsky (1792–1856)

Russian mathematician and college teacher.

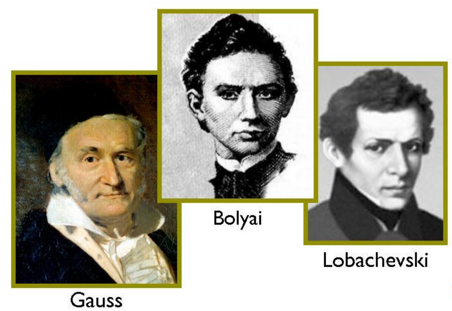
Born in Russia from Polish parents; father was a clerk in a land-surveying office, but died when Nikolai was only seven.

Moved to Kazan, attending Kazan Gymnasium and later was given a scholarship to Kazan University. He remained at Kazan University on the faculty.

Work on non-Euclidean geometry published in the *Kazan Messenger* in 1829, but was rejected for publication by the St. Petersburg Academy of Sciences.

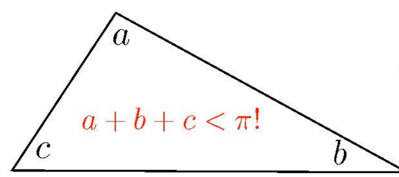
He was "asked to retire" at age 54, and died 10 years later in poor health and in poverty. His work was never appreciated during his lifetime.

~1750-1850



- infinite
- constant negative curvature

~~5th Postulate~~



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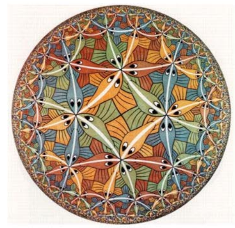
GBL geometry with Klein

1. constant negative curvature
2. infinite
3. ~~5th postulate~~



(x_1, y_1)

$x^2 + y^2 < 1$

$$d(1,2) = a\cosh^{-1}\left[\frac{1 - x_1x_2 - y_1y_2}{\sqrt{1 - x_1^2 - y_1^2}\sqrt{1 - x_2^2 - y_2^2}}\right]$$

(x_2, y_2)

Note: no global embedding in 3D Euclidean space possible

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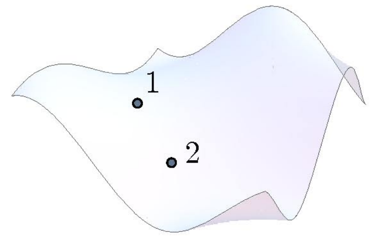
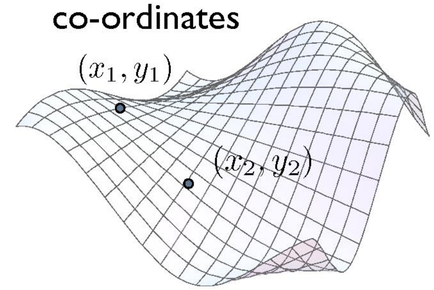
Geometry (after Klein)



co-ordinates



distance function

$d[(x_1, y_1), (x_2, y_2)]$

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Intrinsic Geometry

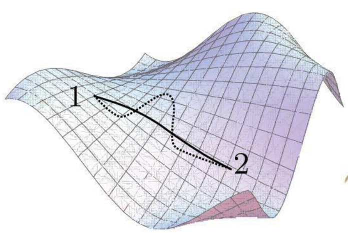

Image of ant from: http://www.termiteterry.com/

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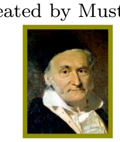
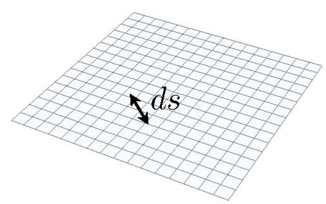
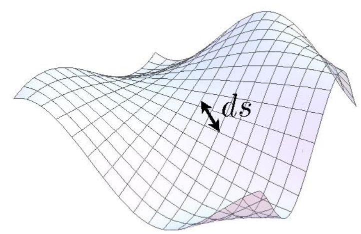
tiny distances







$ds^2 = dx^2 + dy^2$

$ds^2 = g_{xx}(x,y)dx^2 + 2g_{xy}(x,y)dxdy + g_{yy}(x,y)dy^2$

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quadratic form

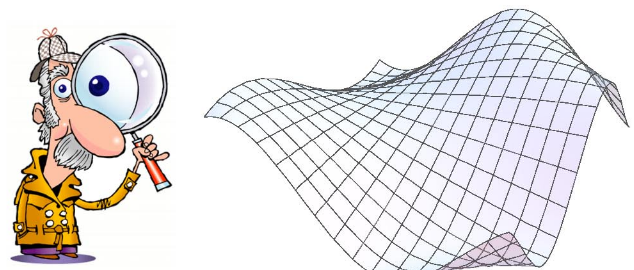

Image: www.easternct.edu/career/webresources.htm

$$ds^2 = g_{xx}(x,y)dx^2 + 2g_{xy}(x,y)dxdy + g_{yy}(x,y)dy^2$$

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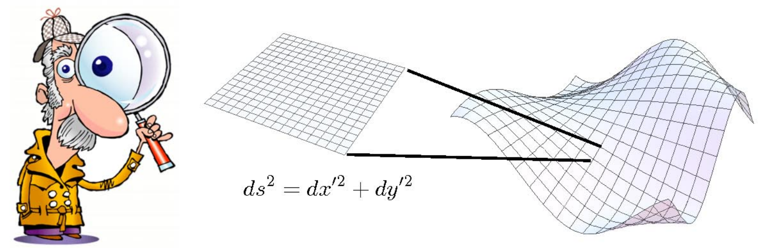
locally Euclidean



$$ds^2 = g_{xx}(x,y)dx^2 + 2g_{xy}(x,y)dxdy + g_{yy}(x,y)dy^2$$

$$g_{xx}g_{yy} - g_{xy}^2 > 0$$

–21–

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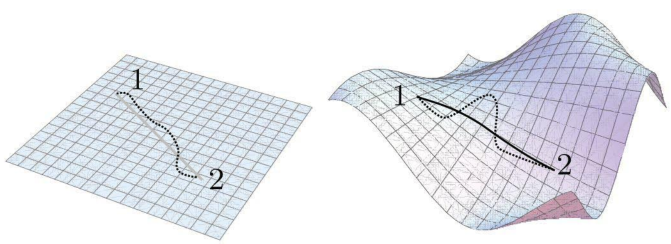
geodesics



a geodesic is a curve along which the distance between two given points is extremised.

note: important!

–22–

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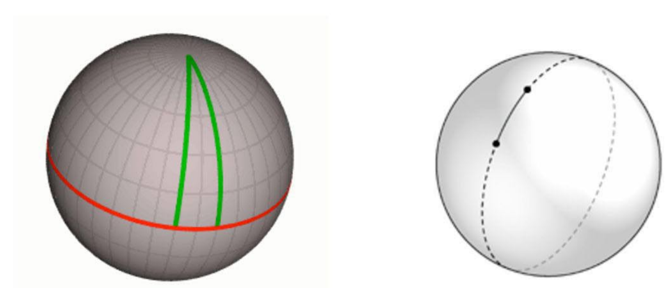
sphere: geodesics



longitudes: yes
latitudes: no

–23–

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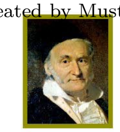
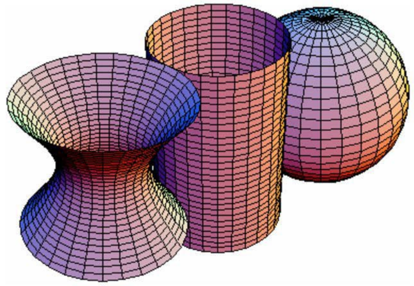
curved ?





Above image:http://en.wikipedia.org/wiki/Gaussian_curvature

$ds^2 = dx^2 + dy^2$ $\qquad$ $ds^2 = g_{xx}(x,y)dx^2 + 2g_{xy}(x,y)dxdy + g_{yy}(x,y)dy^2$

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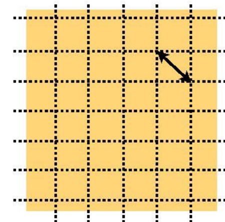
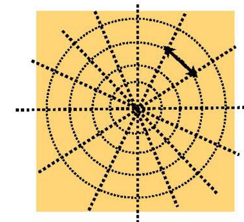
metric and co-ordinates





$ds^2 = dx^2 + dy^2$ $\qquad$ $ds^2 = dr^2 + r^2 d\theta^2$

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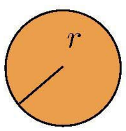
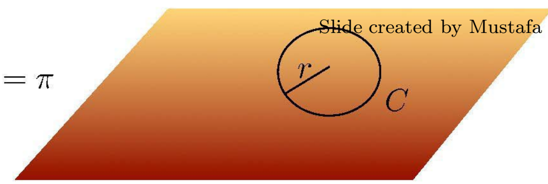
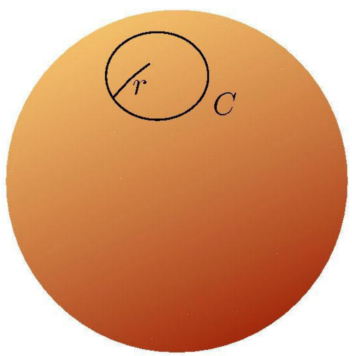
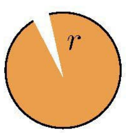


$\frac{C}{2r} = \pi$







$\frac{C}{2r} < \pi$

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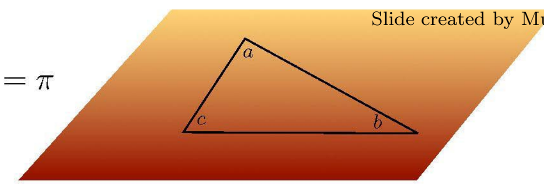
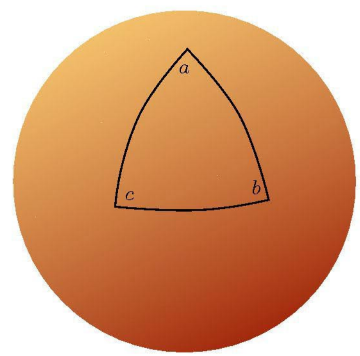
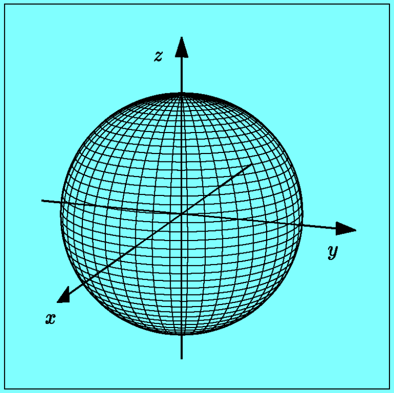
$a + b + c = \pi$



$a + b + c > \pi$



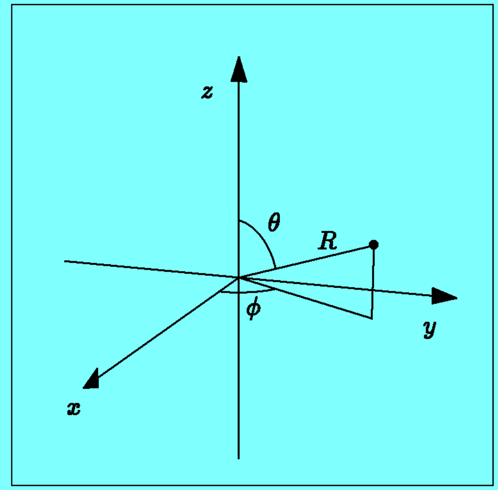
Non-Euclidean Geometry: The Surface of a Sphere



$$x^2 + y^2 + z^2 = R^2 \ .$$

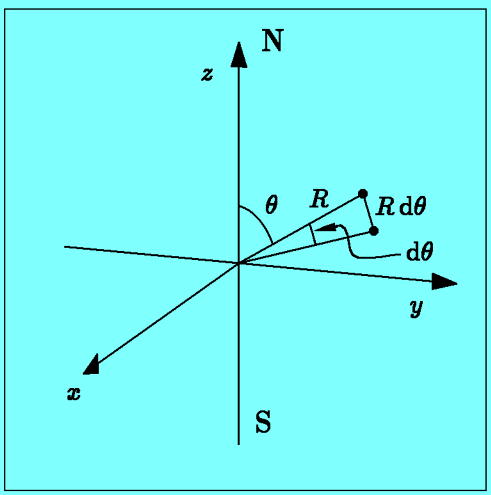
Polar Coordinates:

$$x = R \sin\theta \cos\phi$$
$$y = R \sin\theta \sin\phi$$
$$z = R \cos\theta \ ,$$



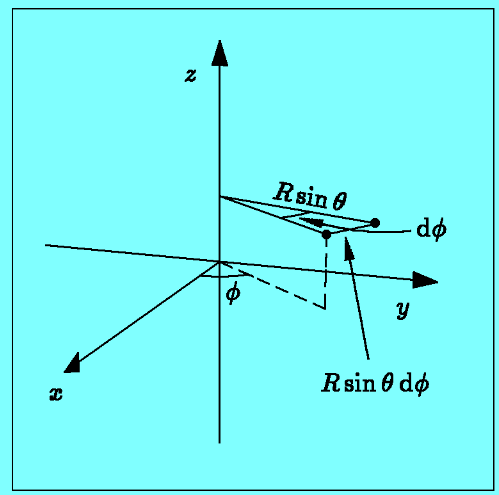
Varying θ:

$$ds = R\, d\theta$$



Varying ϕ:

$$ds = R \sin\theta\, d\phi$$



Varying θ and ϕ

Varying θ: $ds = R\,d\theta$

Varying ϕ: $ds = R\sin\theta\,d\phi$

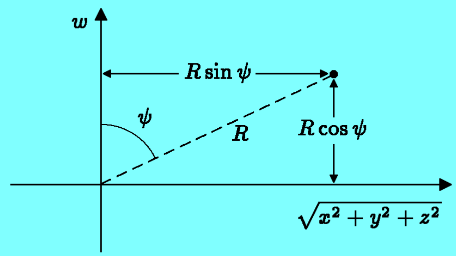
$$ds^2 = R^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)$$

A Closed Three-Dimensional Space

$$x^2 + y^2 + z^2 + w^2 = R^2$$

$$\begin{aligned} x &= R\sin\psi\sin\theta\cos\phi \\ y &= R\sin\psi\sin\theta\sin\phi \\ z &= R\sin\psi\cos\theta \\ w &= R\cos\psi\ , \end{aligned}$$

$$ds = R\,d\psi$$



Metric for the Closed 3D Space

Varying ψ: $ds = R\,d\psi$

Varying θ or ϕ: $ds^2 = R^2\sin^2\psi(d\theta^2 + \sin^2\theta\,d\phi^2)$

If the variations are orthogonal to each other, then

$$ds^2 = R^2\left[d\psi^2 + \sin^2\psi\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right]$$

Proof of Orthogonality of Variations

Let $d\vec{r}_\psi$ = displacement of point when ψ is changed to $\psi + d\psi$.

Let $d\vec{r}_\theta$ = displacement of point when θ is changed to $\theta + d\theta$.

- ☆ $d\vec{r}_\theta$ has no w-component $\implies$ $d\vec{r}_\psi\cdot d\vec{r}_\theta = d\vec{r}_\psi^{\,(3)}\cdot d\vec{r}_\theta^{\,(3)}$, where (3) denotes the projection into the x-y-z subspace.
- ☆ $d\vec{r}_\psi^{\,(3)}$ is radial; $d\vec{r}_\theta^{\,(3)}$ is tangential

$$\implies \quad d\vec{r}_\psi^{\,(3)}\cdot d\vec{r}_\theta^{\,(3)} = 0$$

Implications of General Relativity

- $ds^2 = R^2 \left[d\psi^2 + \sin^2\psi\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$, where R is radius of curvature.
- According to GR, matter causes space to curve.
- R cannot be arbitrary. Instead, $R^2(t) = \dfrac{a^2(t)}{k}$.
- Finally,

$$\boxed{\mathrm{d}s^2 = a^2(t)\left\{\frac{\mathrm{d}r^2}{1-kr^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right\}\ ,}$$

where $r = \dfrac{\sin\psi}{\sqrt{k}}$. Called the Robertson-Walker metric.