8.286 Lecture 11

October 17, 2018

NON-EUCLIDEAN SPACES: THE GEODESIC EQUATION

Announcements

★ Problem Set 5 is due Friday, 5:00 pm. Remember that the extra credit Problem 5, from Problem Set 4, should be handed in with Problem Set 5, if you choose to do it.

Implications of General Relativity

- $ds^2 = R^2 \left[d\psi^2 + \sin^2\psi\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$, where R is radius of curvature.
- According to GR, matter causes space to curve.
- R cannot be arbitrary. Instead, $R^2(t) = \dfrac{a^2(t)}{k}$.
- Finally,

$$\boxed{\mathrm{d}s^2 = a^2(t)\left\{\frac{\mathrm{d}r^2}{1-kr^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right\}\ ,}$$

where $r = \dfrac{\sin\psi}{\sqrt{k}}$. Called the Robertson-Walker metric.

From Space to Spacetime

In special relativity,

$$s_{AB}^2 \equiv (x_A - x_B)^2 + (y_A - y_B)^2 + (z_A - z_B)^2 - c^2 (t_A - t_B)^2 \ .$$

s_{AB}^2 is Lorentz-invariant — it has the same value for all inertial reference frames.

Meaning of s_{AB}^2:

If positive, it is the distance between the two events in the inertial frame in which they are simultaneous. (Spacelike.)

If negative, it is the time interval between the two events in the inertial frame in which they occur at the same place. (Timelike.)

If zero, it implies that a light pulse could travel from the earlier to the later event.

Adding Time to the Robertson-Walker Metric

$$\mathrm{d}s^2 = -c^2\,\mathrm{d}t^2 + a^2(t)\left\{\frac{\mathrm{d}r^2}{1-kr^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right\} .$$

Why does $\mathrm{d}t^2$ term look like it does:

- The coefficient of $\mathrm{d}t^2$ term must be independent of position, due to homogeneity.
- Terms such as $\mathrm{d}t\,\mathrm{d}r$ or $\mathrm{d}t\,\mathrm{d}\phi$ cannot appear, due to isotropy. That is, a term $\mathrm{d}t\,\mathrm{d}r$ would behave differently for $\mathrm{d}r > 0$ and $\mathrm{d}r < 0$, creating an asymmetry between the $+r$ and -r directions.
- The coefficient must be negative, to match the sign in Minkowski space for a locally free-falling coordinate system.

Adding Time to the Robertson-Walker Metric

$$ds^2 = -c^2\,dt^2 + a^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\} .$$

Meaning:

- If $ds^2 > 0$, it is the square of the spatial separation measured by a local free-falling observer for whom the two events happen at the same time.
- If $ds^2 < 0$, it is $-c^2$ times the square of the time separation measured by a local free-falling observer for whom the two events happen at the same location.
- If $ds^2 = 0$, then the two events can be joined by a light pulse.