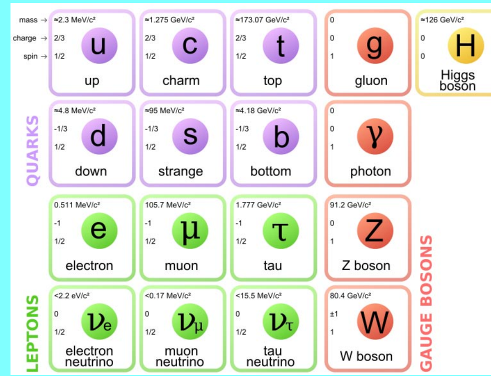
8.286 Lecture 22
December 3, 2018

GRAND UNIFIED THEORIES AND THE MAGNETIC MONOPOLE PROBLEM, CONTINUED

The Standard Model of Particle Physics (Review)

Particle Content:



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Summary of Lecture 21

Gauge Theories: Electromagnetic Example

Fields and potentials: $\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$, $\vec{B} = \vec{\nabla} \times \vec{A}$.

Four-vector notation: $A_\mu = (-\phi, A_i)$.

Gauge transformations:

$$\phi'(t,\vec{x}) = \phi(t,\vec{x}) - \frac{1}{c}\frac{\partial \Lambda(t,\vec{x})}{\partial t} \ , \quad \vec{A}'(t,\vec{x}) = \vec{A}(t,\vec{x}) + \vec{\nabla}\Lambda(t,\vec{x}) \ ,$$

or in four-vector notation,

$$A'_\mu(x) = A_\mu(x) + \frac{\partial \Lambda}{\partial x^\mu} \ , \text{ where } x \equiv (t, \vec{x}).$$

$\vec{E}$ and $\vec{B}$ are gauge-invariant, so A_μ and A'_μ both satisfy the equations of motion, and describe the SAME physical situation.

Gauge transformations can be combined, forming a group:

$$\Lambda_3(x) = \Lambda_1(x) + \Lambda_2(x) \ .$$

Gauge symmetry ≡ local symmetry [$\Lambda \equiv \Lambda(x)$].

Summary of Lecture 21

Electromagnetism as a $U(1)$ Gauge Theory

$\Lambda(x)$ is an element of the real numbers.

But if we included an electron field $\psi(x)$, it would transform as

$$\psi'(x) = e^{ie_0\Lambda(x)}\psi(x) \ ,$$

where e_0 is the charge of a proton and $e = 2.71828\ldots$. So we might think of $u(x) \equiv e^{ie_0\Lambda(x)}$ as describing the gauge transformation. u contains LESS information than Λ, since it defines Λ only mod $2\pi/e_0$.

But u is enough to define the gauge transformation, since

$$\frac{\partial \Lambda}{\partial x^\mu} = \frac{1}{ie_0} e^{-ie_0\Lambda(x)} \frac{\partial}{\partial x^\mu} e^{ie_0\Lambda(x)} \ .$$

u is an element of the group $U(1)$, the group of complex phases $u = e^{i\chi}$, where χ is real. So E&M is a $U(1)$ gauge theory.

Gauge Groups of the Standard Model

$U(1)$ is abelian (commutative), but Yang and Mills showed in 1954 how to construct a nonabelian gauge theory. The standard model contains the following gauge symmetries:

$SU(3)$: This is the group of 3×3 complex matrices that are

S $\equiv$ Special: they have determinant 1.

U $\equiv$ Unitary: they obey $u^\dagger u = 1$, which means that when they multiply a 1×3 column vector v, they preserve the norm $|v| \equiv \sqrt{v_i^* v_i}$.

$SU(2)$: The group of 2×2 complex matrices that are special (S) and unitary (U). As you may have learned in quantum mechanics, $SU(2)$ is almost the same as the rotation group in 3D, with a 2:1 group-preserving mapping between $SU(2)$ and the rotation group.

$U(1)$: The group of complex phases. The $U(1)$ of the standard model is not the $U(1)$ of E&M; instead $U(1)_{\text{E\&M}}$ is a linear combination of the $U(1)$ of the standard model and a rotation about one fixed direction in $SU(2)$.

Combining the groups: the gauge symmetry group of the standard model is usually described as $SU(3)\times SU(2)\times U(1)$. An element of this group is an ordered triplet (u_3, u_2, u_1), where $u_3 \in SU(3)$, $u_2 \in SU(2)$, and $u_1 \in U(1)$, so $SU(3)\times SU(2)\times U(1)$ is really no different from thinking of the 3 symmetries separately.

$SU(3)$ describes the strong interactions, and $SU(2)\times U(1)$ together describe the electromagnetic and weak interactions in a unified way, call the electroweak interactions.

$SU(3)$ acts on the quark fields by rotating the 3 "colors" into each other. Thus the strong interactions of the quarks are entirely due to their "colors", which act like charges.

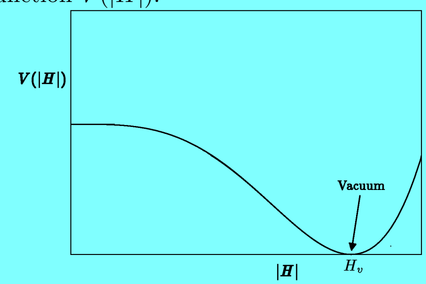
The Higgs Field and Spontaneous Symmetry Breaking

The Higgs field is a complex doublet:

$$H(x) \equiv \begin{pmatrix} h_1(x) \\ h_2(x) \end{pmatrix} .$$

Under $SU(2)$ transformations, $H'(x) = u_2(x)H(x)$, where $u_2(x)$ is the complex 2×2 matrix that defines the $SU(2)$ gauge transformation at x. Since the gauge symmetry implies that the potential energy density of the Higgs field $V(H)$ must be gauge-invariant, V can depend only on $|H| \equiv \sqrt{|h_1|^2 + |h_2|^2}$, which is unchanged by $SU(2)$ transformations.

Potential energy function $V(|H|)$:



The minimum is not at $|H| = 0$, but instead at $|H| = H_v$.

$|H| = 0$ is $SU(2)$ gauge-invariant, but $|H| = H_v$ is not. H randomly picks out some direction in the space of 2D complex vectors.

Spontaneous Symmetry Breaking: Whenever the ground state of a system has less symmetry than the underlying laws, it is called spontaneous symmetry breaking. Examples: crystals, ferromagnetism.

Summary of Lecture 21

Higgs Fields Give Mass to Other Particles

When $H = 0$, all the fundamental particles of the standard model are massless. (Protons, however, are not massless.) Furthermore, there is no distinction between the electron e and the electron neutrino ν_e, or between μ and ν_μ, or between τ and ν_τ.

For $|H| \neq 0$, H randomly picks out a direction in space of 2D complex vectors. Since all directions are otherwise equivalent, we can assume that in the vacuum,

$$H = \begin{pmatrix} H_v \\ 0 \end{pmatrix} .$$

Components of other fields that interact with $\mathrm{Re}(h_1)$ then start to behave differently from fields that interact with other components of H.

Summary of Lecture 21

Mass: mc^2 of a particle is the state of lowest energy above the ground state. In a field theory, this corresponds to a homogeneous oscillation of the field, which in turn corrsponds to a particle with zero momentum.

In the free field limit, the field acts exactly like a harmonic oscillator. The first excited state has energy $h\nu = \hbar\omega$ above the ground state. So, $mc^2 = \hbar\omega$.

ω is determined by inertia and the restoring force. When $H = 0$, the standard model interactions provide no restoring forces. Any such restoring force would break gauge invariance.

When $H = \begin{pmatrix} H_v \\ 0 \end{pmatrix}$, the interactions with H creates a restoring force for some components of other fields, giving them a mass. This "Higgs mechanism" creates the distinction between electrons and neutrinos.

Beyond the Standard Model

With neutrino masses added, the standard model is spectacularly successful: it agrees with all reliable particle experiments.

Nonetheless, most physicists regard it as incomplete, for at least two types of reasons:

1) It does not include gravity, and it does not include any particle to account for the dark matter. (Maybe black holes can do it, but that requires a mass distribution that we cannot explain.)

2) The theory appears too inelegant the final theory. It contains more arbitrary features and free parameters than one would hope for in a final theory. Why $SU(3)\times SU(2)\times U(1)$? Why three generations of fermions? The original theory had 19 free parameters, with more needed for neutrino masses and even more if *supersymmetry* is added.

Result: BSM (Beyond the Standard Model) particle physics has become a major industry.

Grand Unified Theories

Goal: Unify $SU(3)\times SU(2)\times U(1)$ by embedding all three into a single, larger group.

The breaking of the symmetry to $SU(3)\times SU(2)\times U(1)$ is accomplished by introducing new Higgs fields to spontaneously break the symmetry.

In the fundamental theory, before spontaneous symmetry breaking, there is no distinction between an electron, an electron neutrino, or an up or down quark.

The SU(5) Grand Unified Theory

In 1974, Howard Georgi and Sheldon Glashow of Harvard proposed the original grand unified theory, based on $SU(5)$. They pointed out that $SU(3)\times SU(2)\times U(1)$ fits elegantly into $SU(5)$.

To start, let the $SU(3)$ subgroup be matrices of the form

$$u_3 = \begin{pmatrix} x & x & x & 0 & 0 \\ x & x & x & 0 & 0 \\ x & x & x & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} ,$$

where the 3×3 block of x's represents an arbitrary SU(3) matrix.

Similarly let the $SU(2)$ subgroup be matrices of the form

$$u_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & x & x \end{pmatrix} ,$$

where this time the 2×2 block of x's represents an arbitrary SU(2) matrix.

Note that u_3 and u_2 commute, since each acts like the identity matrix in the space in which the other is nontrivial.

Finally, the $U(1)$ subgroup can be written as

$$u_1 = \begin{pmatrix} e^{2i\theta} & 0 & 0 & 0 & 0 \\ 0 & e^{2i\theta} & 0 & 0 & 0 \\ 0 & 0 & e^{2i\theta} & 0 & 0 \\ 0 & 0 & 0 & e^{-3i\theta} & 0 \\ 0 & 0 & 0 & 0 & e^{-3i\theta} \end{pmatrix} ,$$

where the factors of 2 and 3 in the exponents were chosen so that the determinant — in this case the product of the diagonal entries — is equal to 1.

Repeating, the $U(1)$ subgroup can be written as

$$u_1 = \begin{pmatrix} e^{2i\theta} & 0 & 0 & 0 & 0 \\ 0 & e^{2i\theta} & 0 & 0 & 0 \\ 0 & 0 & e^{2i\theta} & 0 & 0 \\ 0 & 0 & 0 & e^{-3i\theta} & 0 \\ 0 & 0 & 0 & 0 & e^{-3i\theta} \end{pmatrix} .$$

u_1 commutes with any matrix of the form of u_2 or u_3, since within either the upper 3×3 block or within the lower 2×2 block, u_1 is proportional to the identity matrix.

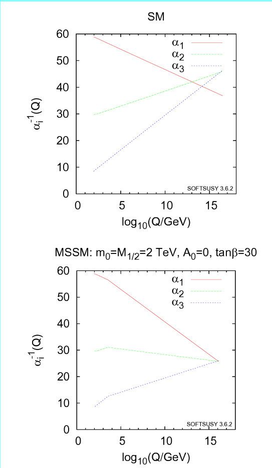
Thus, any element (u_3, u_2, u_1) of $SU(3)\times SU(2)\times U(1)$ can be written as an element u_5 of $SU(5)$, just by setting $u_5 = u_3 u_2 u_1$.

How Can Three Different Types of Interaction Look Like One?

In the standard model, each type of gauge interaction — $SU(3)$, $SU(2)$, and $U(1)$ — has its own interaction strength, described by "coupling constants" g_3, g_2, and g_1. Their values of are different from each other! How can they be one interaction?

BUT: the interaction strength varies with energy in a calculable way. When the calculations are extended to superhigh energies, of the order of 10^{16} GeV, the three interaction strengths become about equal!

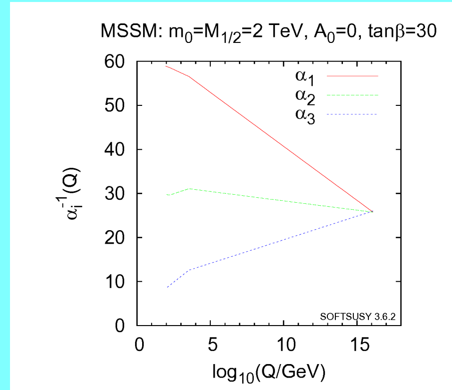
Running Coupling Constants



The top graph shows the running of coupling constants in the standard model, showing that the three coupling constants do not quite meet. The bottom graph shows the running of coupling constants in the MSSM — the Minimal Supersymmetric Standard Model, in which the meeting of the couplings is almost perfect. $\alpha_i = g_i^2/4\pi$.

Source: Particle Data Group 2016 Review of Particle Physics, Chapter 16, *Grand Unified Theories*, Revised January 2016 by A. Hebecker and J. Hisano.

Running Couplings
Minimal Supersymmetric Standard Model



Bottom line: An $SU(5)$ grand unified theory can be constructed by introducing a Higgs field that breaks the $SU(5)$ symmetry to $SU(3) \times SU(2) \times U(1)$ at an energy of about 10^{16} GeV. At energies above 10^{16} GeV, the theory behaves like a fully unified $SU(5)$ gauge theory. At lower energies, it behaves like the standard model. The gauge particles that are part of $SU(5)$ but not part of $SU(3) \times SU(2) \times U(1)$ acquire masses of order 10^{16} GeV.

GUTs (Grand Unified Theories) allow two unique phenomena at low energies, neither of which have been seen:

1) Proton decay. The superheavy gauge particles can mediate proton decay. The minimal $SU(5)$ model — with the simplest conceivable particle content — predicts a proton lifetime of about 10^{31} years, which is ruled out by experiments, which imply a lifetime $\gtrsim 10^{34}$ years.

2) Magnetic monopoles. All grand unified theories imply that magnetic monopoles should be a possible kind of particle. None have been seen.

The absence of evidence does not imply that GUTs are wrong, but we don't know.

The Grand Unified Theory Phase Transition

When $kT \gg 10^{16}$ GeV, the Higgs fields of the GUT undergo large fluctuations, and average to zero. The GUT symmetry is unbroken, and the theory behaves as an SU(5) gauge theory.

As kT falls to about 10^{16} GeV, the matter filling the universe would go through a phase transition, in which some of the components of the GUT Higgs field acquire nonzero values in the thermal equilibrium state, breaking the GUT symmetry. The breaking to $SU(3) \times SU(2) \times U(1)$ might occur in one phase transition, or in a series of phase transitions. We'll assume a single phase transition.

The Higgs fields start to randomly acquire nonzero values, but the nonzero values that form in one region may not align with those in another.

The expression for the energy density contains a term proportional to $|\nabla\Phi|^2$, so the fields tend to fall into low energy states with small gradients. But sometimes the fields in one region acquire a pattern that cannot be smoothly joined with the pattern in a neighboring region, so the smoothing is imperfect, leaving "defects".

Topological Defects

There are three types of defects:

1) Domain walls. For example, imagine a single real scalar field ϕ for which the potential energy function has two local minima, at ϕ_1 and ϕ_2. Then, as the system cools, some regions will have $\phi \approx \phi_1$ and others will have $\phi \approx \phi_2$. The boundaries between these regions will be surfaces of high energy density: domain walls. Some GUTS allow domain walls, others do not. The energy density of a domain wall is so high that none can exist in the visible universe.
2) Cosmic strings. Linelike defects, which exist in some GUTs but not all.
3) Magnetic monopoles: Pointlike defects, which exist in all GUTs.

Magnetic Monopoles

We'll consider the simplest theory in which monopoles arise. It has a 3-component (real) Higgs field, ϕ_a, where $a = 1, 2$ or 3. Gauge symmetry acting on ϕ_a has the same mathematical form as the rotations of an ordinary Cartesian 3-vector.

To be gauge-invariant, the energy density function can depend only on

$$|\phi| \equiv \sqrt{\phi_1^2 + \phi_2^2 + \phi_3^2}\ ,$$

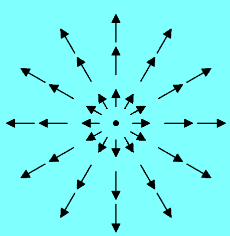
and we assume that it looks qualitatively like the graph for the standard model, with a minimum at H_v.

Now consider the following static configuration,

$$\phi_a(\vec{r}) = f(r)\hat{r}_a\ ,$$

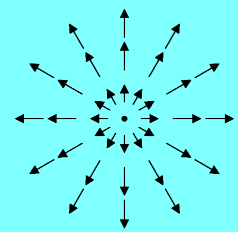
where $r \equiv |\vec{r}|$, $\hat{r}_a$ denotes the a-component of the unit vector $\hat{r} = \vec{r}/r$, and $f(r)$ is a function which vanishes when $r = 0$ and approaches H_v as $r \to \infty$.

Pictorially,



where the 3 components of the arrow at each point describe the 3 Higgs field components.

Repeating,



where the 3 components of the arrow at each point describe the 3 Higgs field components.

The directions in gauge space ϕ_a really have nothing to do with directions in physical space, but there is nothing that prevents the fields from existing in this configuration.

The configuration is topologically stable in the following sense: if the boundary conditions at infinity are fixed, and the fields are continuous, then there is always at least one point where $\phi_1 = \phi_2 = \phi_3 = 0$.

Thus, the monopoles are topologically stable knots in the Higgs field.

Why Are These Things Magnetic Monopoles?

Definition: A magnetic monopole is an object with a net magnetic charge, north or south, with a radial magnetic field of the same form as the electric field of a point charge.

Known magnets are always dipoles, with a north end and a south end. If such a magnet is cut in half, one gets two dipoles, each with a north and south end.

Energy of the configuration: the energy density contains a term $\sum_a \vec{\nabla}\phi_a \cdot \vec{\nabla}\phi_a$, but the changing direction of ϕ_a (always radially outward) implies $|\nabla\phi_a| \propto 1/r$. The total energy in a sphere of radius R is proportional to

$$4\pi \int^R r^2 \mathrm{d}r \left(\frac{1}{r}\right)^2 ,$$

which diverges as R for large R.

With the vector gauge fields, however, the energy density is more complicated. It can be made finite only if the gauge field configuration corresponds to a net magnetic charge.

Prediction of Magnetic Charge

The magnetic charge is uniquely determined, and is equal to $1/(2\alpha)$ times the electric charge of an electron, where $\alpha \simeq 1/137$ (α = fine-structure constant $= e^2/\hbar c$ in Gaussian units, or $e^2/(4\pi\epsilon_0\hbar c)$ in SI.)

The force between two monopoles is therefore $(68.5)^2$ times as large as the force between two electrons at the same distance. I.e., large!

Kibble Estimate of Magnetic Monopole Production

Since magnetic monopoles are knots in the GUT Higgs fields, they form at the GUT phase transition, when the Higgs fields acquire nonzero mean values. (“Mean” = average over time, not space.)

The density of these knots will be related to the misalignment of the Higgs field in different regions.

Define a correlation length ξ, crudely, as the minimum distance such that the Higgs field at point is almost uncorrelated with the Higgs field a distance ξ away.

T.W.B. Kibble of Imperial College (London) proposed that the number density of magnetic monopoles (and antimonopoles) can be estimated as

$$n_M \approx 1/\xi^3 .$$

Estimate of Correlation Length ξ

In the context of conventional (non-inflationary) cosmology, we can assume

1) that the Higgs field well before the GUT phase transition is in a thermal state, with no long-range correlations.
2) that the universe before the phase transition is well-approximated by a flat radiation-dominated Friedman-Robertson-Walker description.
3) phase transition happens promptly when the temperature of the GUT phase transition is reached, at $kT \approx 10^{16}$ GeV.

Under these assumptions, we are confident that the correlation length ξ must be less than or equal to the horizon length at the time of the phase transition. This seemingly mild limit turns out to have huge implications.

On Problem Set 10, you will calculate the contribution to Ω today, from the monopoles. I won’t give away the answer, but you should find that it is greater than 10^{20}.