8.286 Lecture 1
September 2, 2020

WELCOME TO 8.286!

OVERVIEW:

INFLATIONARY COSMOLOGY — IS OUR UNIVERSE PART OF A MULTIVERSE?

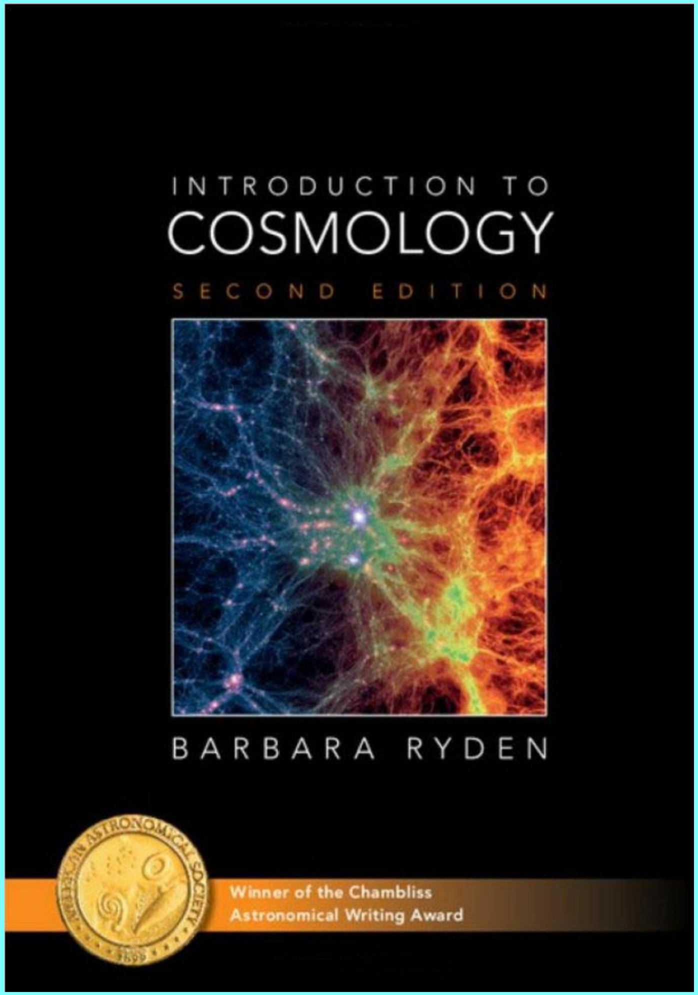
Course Staff

Lecturer: Me (Alan Guth), guth@ctp.mit.edu.

Teaching Assistant: Bruno Scheihing, bscheihi@mit.edu.

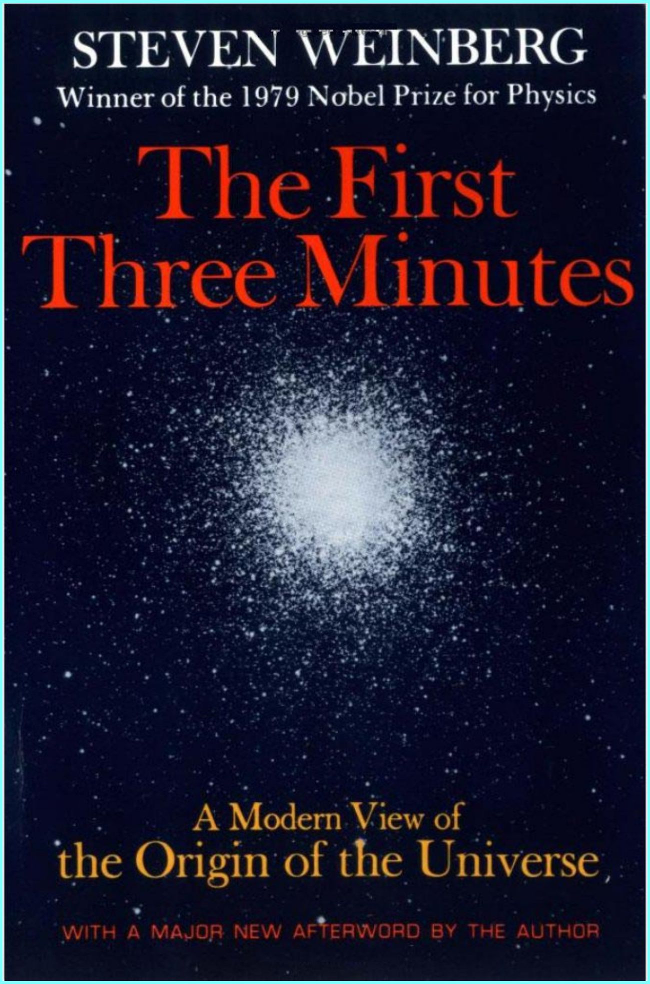
Required Textbook #1

Introduction to Cosmology, Second Edition (Cambridge University Press, 2016), by Barbara Ryden.



Required Textbook #2

The First Three Minutes, Second paperback edition, (Basic Books, 1993), by Steven Weinberg.



Grading

Three "In-Class" Quizzes: 66%

The quizzes will tentatively be on the following dates:

Wednesday, September 30, 2020
Wednesday, October 28, 2020
Wednesday, December 2, 2020

If you have a problem of any kind with any of these dates, you should email me (Alan Guth) as soon as possible.

Problem Sets: 34% No final exam.

Alternative Grading Scheme: In previous years, the division was 75% on quizzes, 25% on homework. If your average is higher by the previous weighting, then that will be your grade.

Problem Sets

About 1 problem set per week, nine altogether, mostly due on Fridays at 5:00 pm Boston time.

The problem sets will not all be worth the same number of points. Your grade will be the total number of points you earn, compared to the maximum possible. That is, problem sets with more points will count a little more than the others.

Lowest Problem Set Will Be Dropped

The lowest* problem set will be dropped, but I strongly encourage you to do all of them.

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However: I understand that you are all have very overfilled lives. That's what MIT students are like! So, to encourage you to do all the problem sets, I will be very generous with extensions. If you are having an unusually busy week (or if you are sick, have a family crisis, or have a fight with your brother), just send me an email describing the situation, and ask me for an extension.

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* Since the problem sets will have unequal weights, the one that will be dropped will be the one that increases your grade the most.

If the solutions are posted before you turn in your problem set, you are on your honor not to look at the solutions, or discuss the problems with anyone who has (other than me or the other course staff), until you have turned it in.

Extra Credit

Some of the the problem sets will offer additional problems for extra credit. We will keep track of the extra credit grades separately.

At the end of the course I will consult with Bruno Scheihing to set grade cuts based solely on the regular coursework. We will try to make sure that the grade cuts are reasonable with respect to this data set.

Then the extra credit grades will be added, allowing the grades to change upwards accordingly.

Finally, we will look at each student's grades individually, and we might decide to give a higher grade to some students who are slightly below a borderline. Students whose grades have improved significantly during the term, and students whose average has been pushed down by single low grade, will be the ones most likely to be boosted.

Homework Policy

I regard the problem sets primarily as an educational experience, rather than a mechanism of evaluation.

Your are encouraged — even strongly encouraged — to work on the homework in groups. I will be setting up a Class Contact webpage to help you find each other. But, you are each expected to **write up** your own solutions, even if you found those solutions as a group project.

8.286 Problem Solutions from previous years are strictly off limits, but other sources — textbooks, webpages — are okay, as long as you rewrite the solution in your own words.

A homework problem that appears to be copied from another student, from a previous year's solution, or copied from some other source without rewording might be given zero credit. Except in blatant cases, the first time you will be given a chance to redo it.

Remember that this homework policy does not apply to other classes.

Expect a Questionnaire

By tomorrow morning, I hope to have emailed a short questionnaire to each of you. Please fill it out within 24 hours and email it back to me.

One question will be about your time zone, and your available times for office hours.

Another will be about the class contact list.

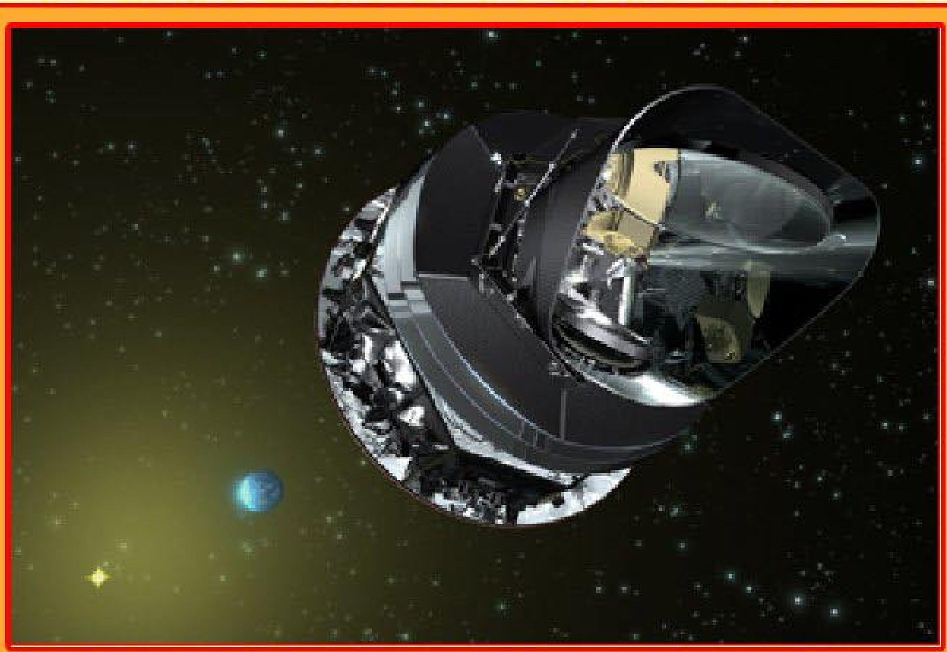
Another will be about your working conditions at your present location: a place to work, quietness, internet access, etc.

I will also ask if there is anything else that I should know to understand your situation.

INFLATIONARY COSMOLOGY:

IS OUR UNIVERSE PART OF A MULTIVERSE?

PART 1



— Alan Guth —

Massachusetts Institute of Technology

8.286 Opening Lecture
September 2, 2020

The Standard Big Bang

What it is:

- ☆ Theory that the universe as we know it began 13-14 billion years ago. (Latest estimate: 13.80 ± 0.02 billion years, from the Planck satellite collaboration, 2018.)
- ☆ Initial state was a hot, dense, uniform soup of particles that filled space uniformly, and was expanding rapidly.

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What it describes:

- ☆ How the early universe expanded and cooled
- ☆ How the light chemical elements formed
- ☆ How the matter congealed to form stars, galaxies, and clusters of galaxies

What it doesn't describe:

- ☆ What caused the expansion? (The big bang theory describes only the **aftermath** of the bang.)
- ☆ Where did the matter come from? (The theory assumes that **all matter** existed from the very beginning.)

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Gravitational Repulsion.

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(a) was never taught to me when I was a student; and

(b) is so far-reaching in its consequences that it can change our picture of the universe.

Miracle of Physics # 1: Gravitational Repulsion

- Since the advent of general relativity, physicists have known that gravity can act repulsively.
- In GR, pressures can create gravitational fields, and negative pressures create repulsive gravitational fields.
- Einstein used this possibility, in the form of the "cosmological constant," to build a static mathematical model of the universe, with repulsive gravity preventing its collapse.
- Modern particle physics suggests that at superhigh energies there should be many states with negative pressures, creating repulsive gravity.

★ Inflation proposes that a patch of repulsive gravity material existed in the early universe — for inflation at the grand unified theory scale ($\sim 10^{16}$ GeV), the patch needs to be only as large as 10^{-28} cm. (Since any such patch is enlarged fantastically by inflation, the initial density or probability of such patches can be very low.)

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- ☆ The gravitational repulsion created by this material was the driving force behind the big bang. The repulsion drove it into exponential expansion, doubling in size every 10^{-37} second or so!

- ☆ The patch expanded exponentially by a factor of at least 10^{28} ($\sim$ 100 doublings), but it could have expanded much more. Inflation lasted maybe 10^{-35} second, and at the end, the region destined to become the presently observed universe was about the size of a marble.

- ☆ The repulsive-gravity material is unstable, so it decayed like a radioactive substance, ending inflation. The decay released energy which produced ordinary particles, forming a hot, dense "primordial soup." Standard cosmology began.

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  Caveat: The decay happens almost everywhere, but not everywhere — we will come back to this subtlety, which is the origin of eternal inflation.

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- Although more and more mass/energy appeared as the repulsive-gravity material expanded, total energy was conserved! HOW????

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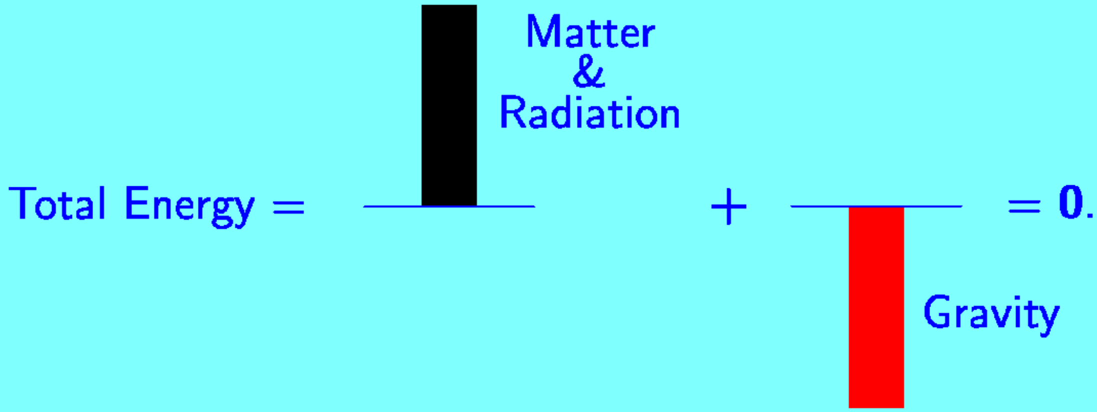
- ☆ The energy of a gravitational field is negative (both in Newtonian gravity and in general relativity).
- ☆ The negative energy of gravity canceled the positive energy of matter, so the total energy was constant and possibly zero.
- ☆ The total energy of the universe today is consistent with zero. Schematically,



Alan Guth
Massachusetts Institute of Technology

Evidence for Inflation

1) **Large scale uniformity.** The cosmic background radiation is uniform in temperature to one part in 100,000. It was released when the universe was about 400,000 years old. In standard cosmology without inflation, a mechanism to establish this uniformity would need to transmit energy and information at about 100 times the speed of light.

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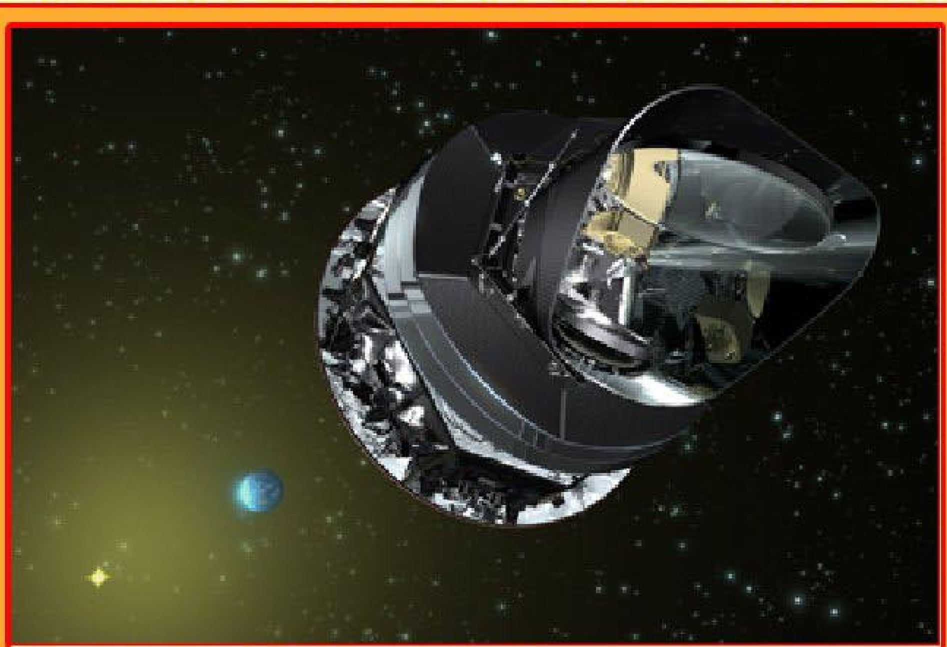
Inflationary Solution: In inflationary models, the universe begins so small that uniformity is easily established — just like the air in the lecture hall spreading to fill it uniformly. Then inflation stretches the region to be large enough to include the visible universe.

INFLATIONARY COSMOLOGY:

IS OUR UNIVERSE PART OF A MULTIVERSE?

PART 2



— Alan Guth —

Massachusetts Institute of Technology

8.286 Lecture 3
September 14, 2020

SUMMARY OF LAST LECTURE

The Conventional Big Bang Theory (i.e., without inflation): Really describes only the aftermath of a bang: It says nothing about what banged, why it banged, or what happened before it banged. The description begins with a hot dense uniform soup of particles filling an expanding space.

Cosmic Inflation: The prequel, describes how **repulsive gravity** — a consequence of **negative pressure** — could have driven a tiny patch of the early universe into exponential expansion. The total energy would be very small or maybe zero, with the **negative energy of the cosmic gravitational field** canceling the energy of matter.

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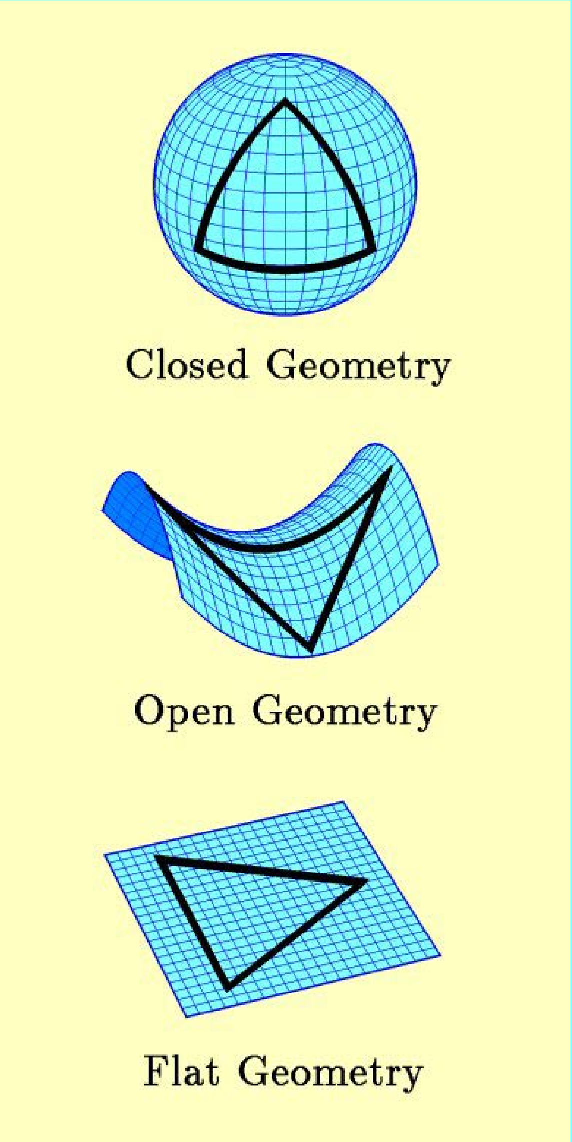

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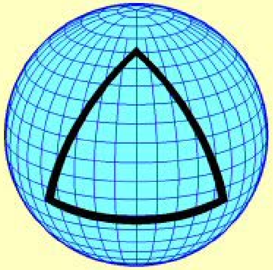
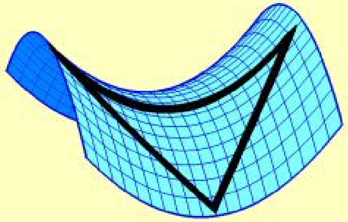
- ☆ According to general relativity, the flatness of the universe is related to its mass density:

$$\Omega(Omega) = \frac{\text{actual mass density}}{\text{critical mass density}} \; ,$$

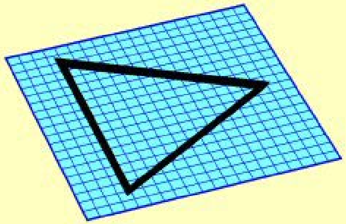
where the "critical density" depends on the expansion rate. $\Omega = 1$ is flat, $\Omega > 1$ is closed, $\Omega < 1$ is open.



Closed Geometry



Open Geometry



Flat Geometry

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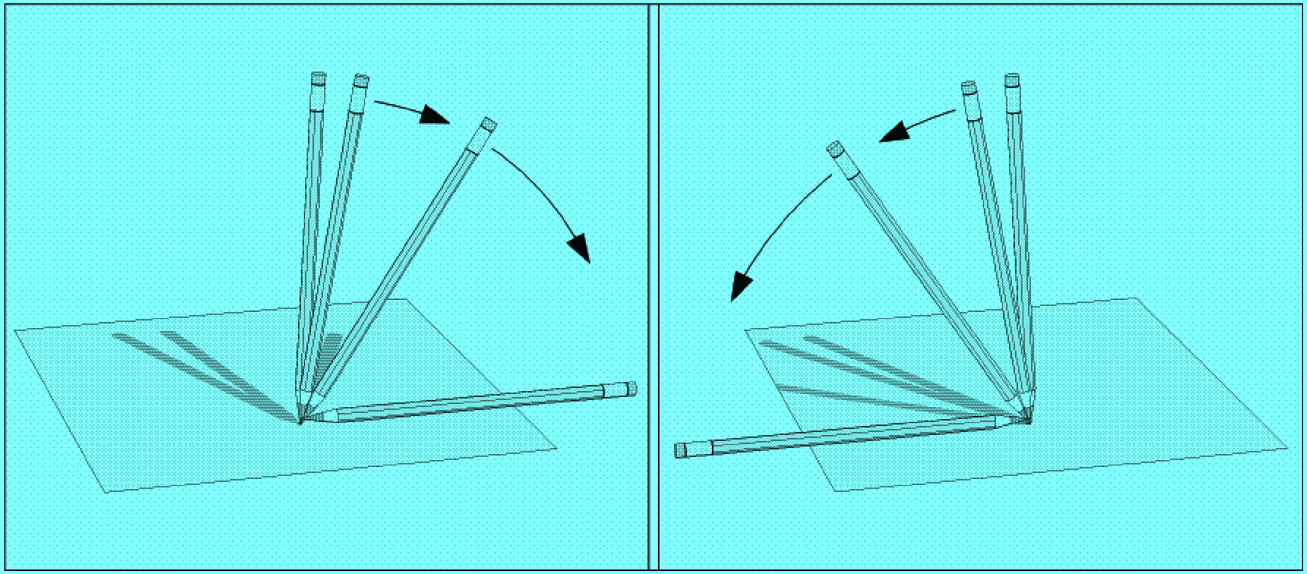

- If Ω in the early universe was slightly below 1, it would rapidly fall to zero — and no galaxies would form.
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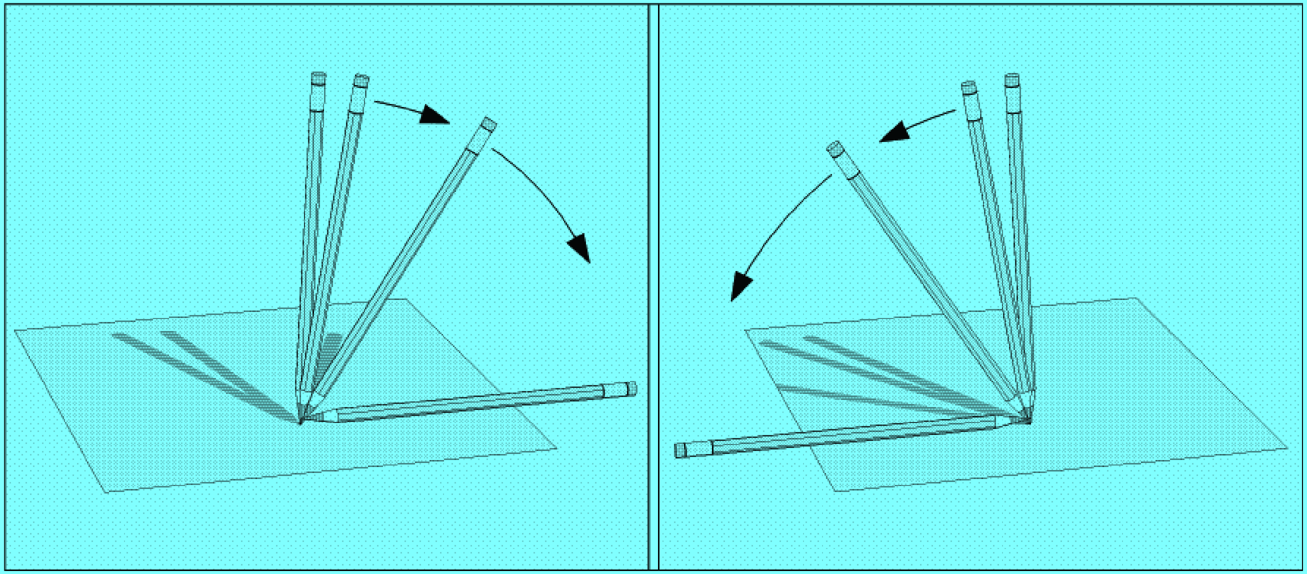

- If Ω in the early universe was slightly below 1, it would rapidly fall to zero — and no galaxies would form.
- If Ω was slightly greater than 1, it would rapidly rise to infinity, the universe would recollapse, and no galaxies would form.
- To be even within a factor of 10 of the critical density today (which is what we knew in 1980), at one second after the big bang, Ω must have been equal to one **to 15 decimal places!**

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- ☆ New ingredient: Dark Energy. In 1998 it was discovered that the expansion of the universe has been accelerating for about the last 5 billion years. The "Dark Energy" is the energy causing this to happen.

3) **Small scale nonuniformity:** Can be measured in the cosmic background radiation. The intensity is almost uniform across the sky, but there are small ripples. Although these ripples are only at the level of 1 part in 100,000, these nonuniformities are now detectable! Where do they come from?

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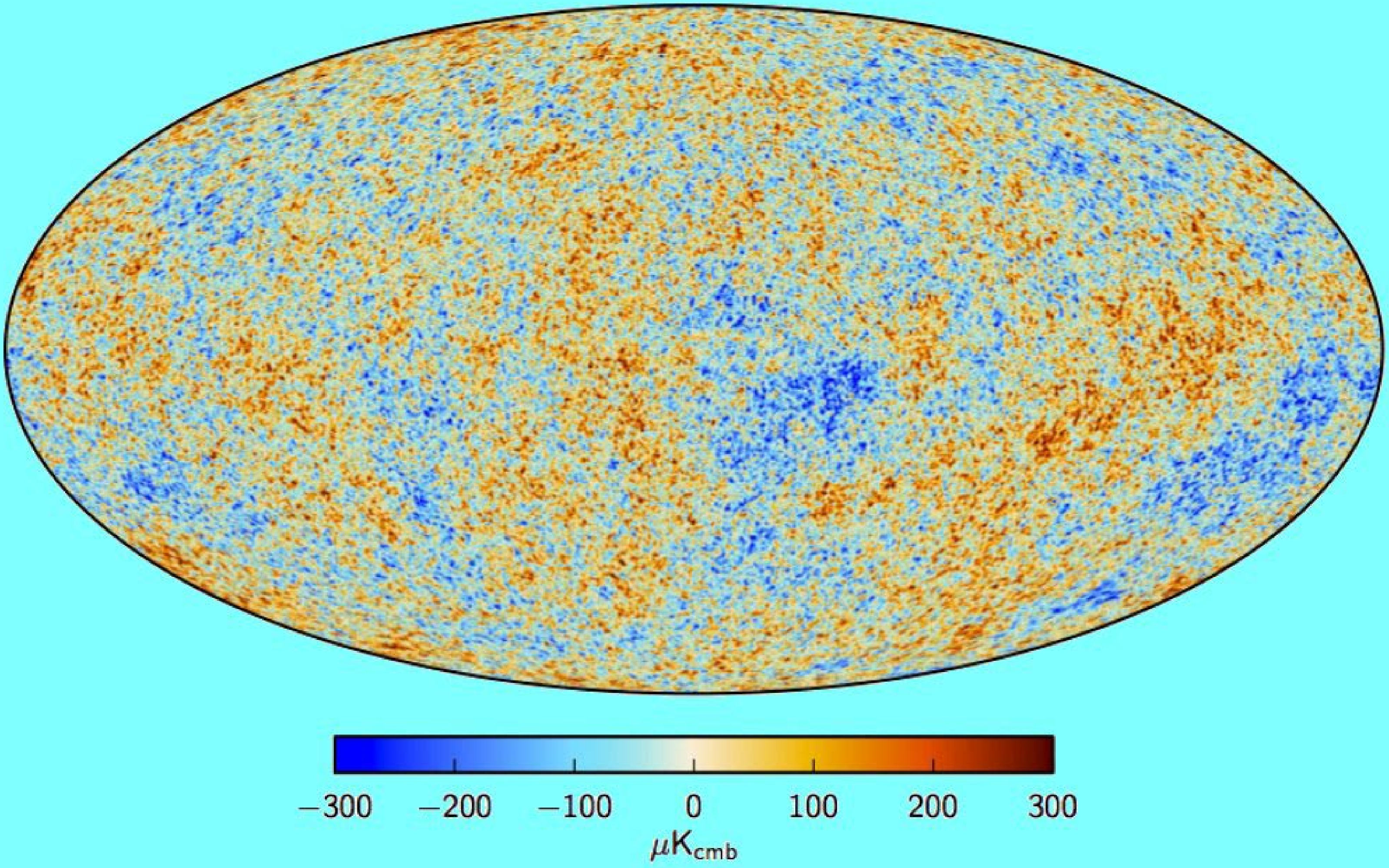
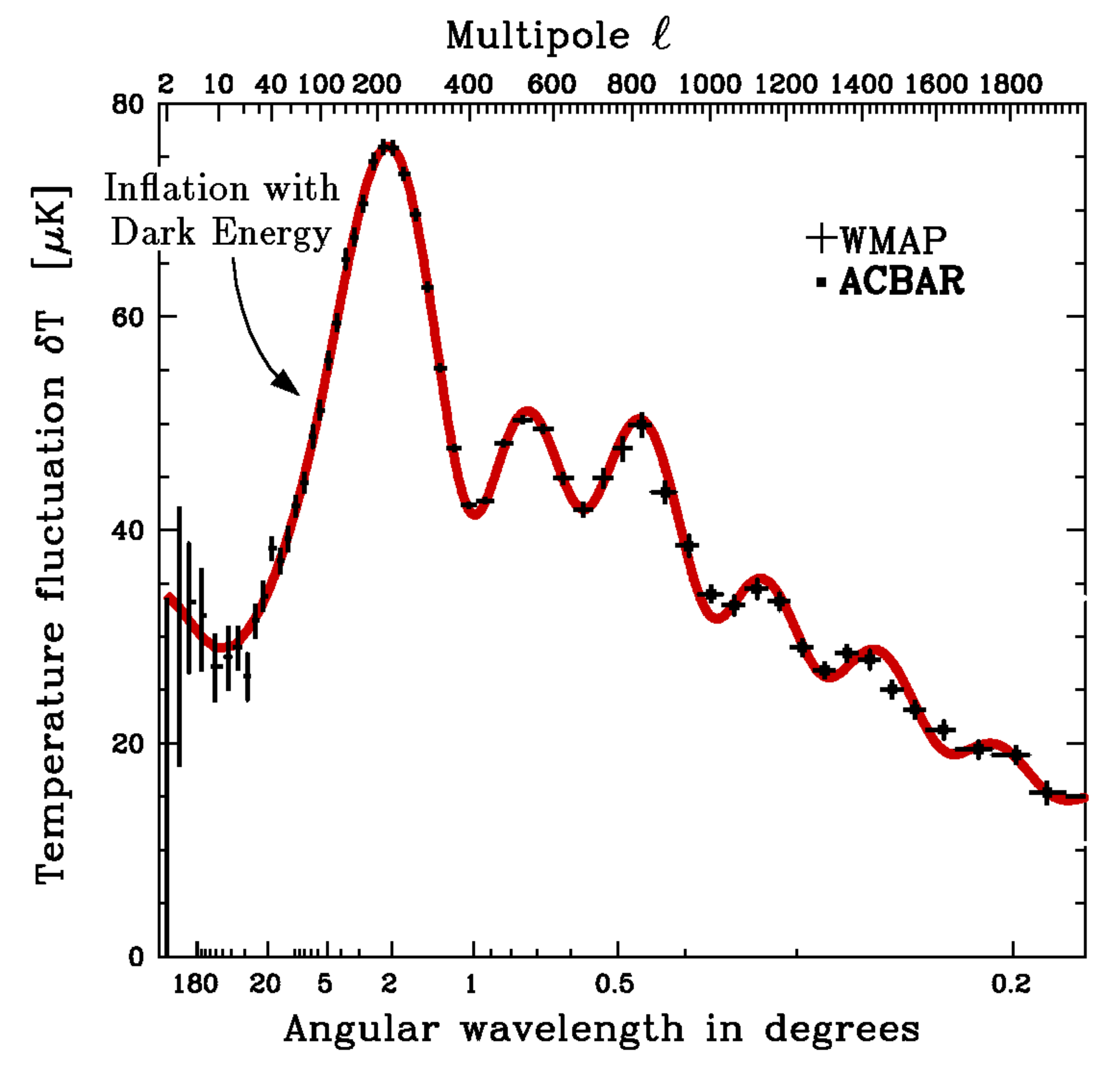
Inflationary Solution: Inflation attributes these ripples to *quantum fluctuations.* Inflation makes generic predictions for the spectrum of these ripples (i.e., how the intensity varies with wavelength). The data measured so far agree beautifully with inflation.

Ripples in the Cosmic Microwave Background

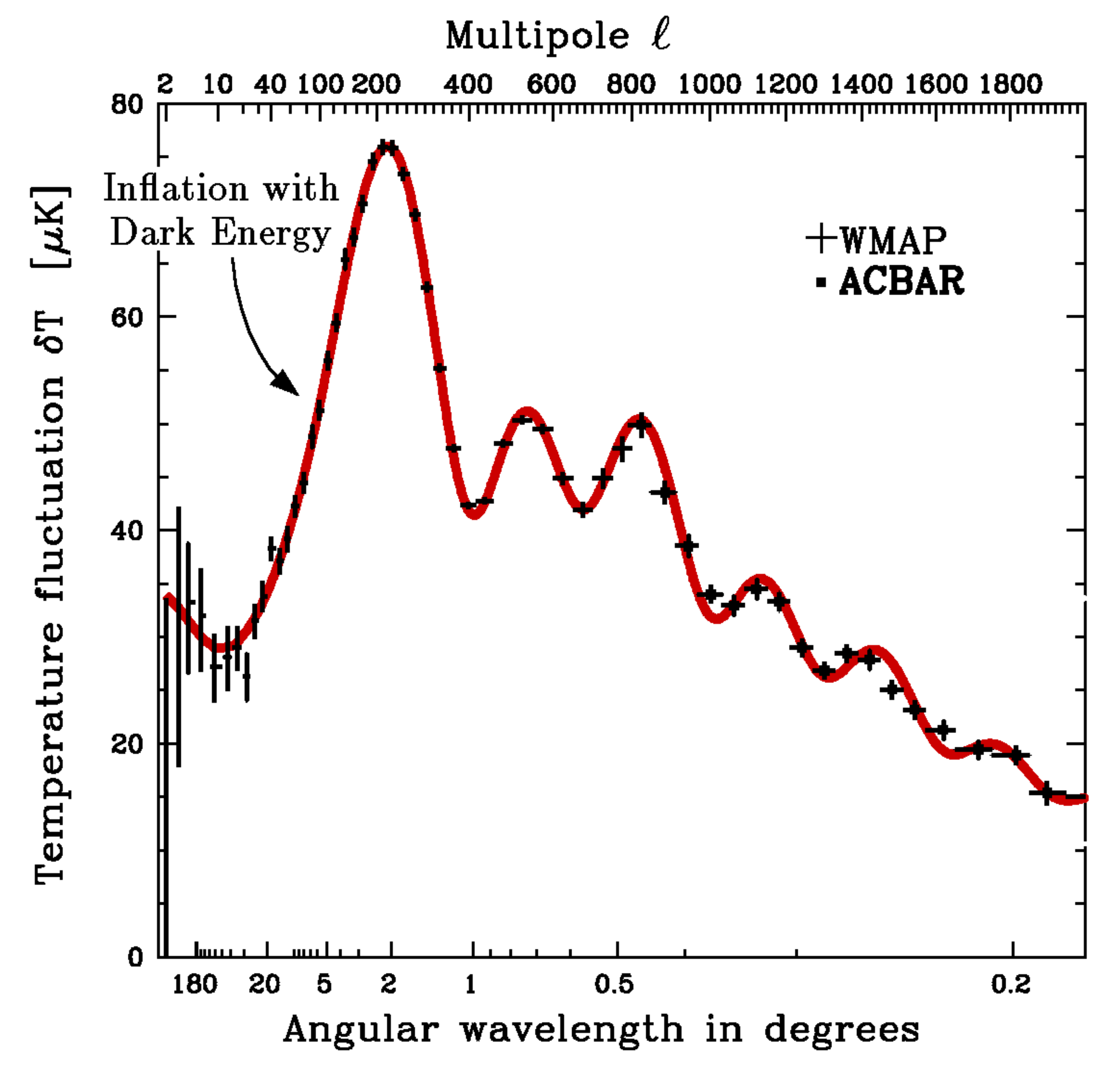
Planck Collaboration: The *Planck* mission



CMB: Comparison of Theory and Experiment



Graph by Max Tegmark, for A. Guth & D. Kaiser, *Science* **307, 884** (Feb 11, 2005), updated to include WMAP 7-year data.



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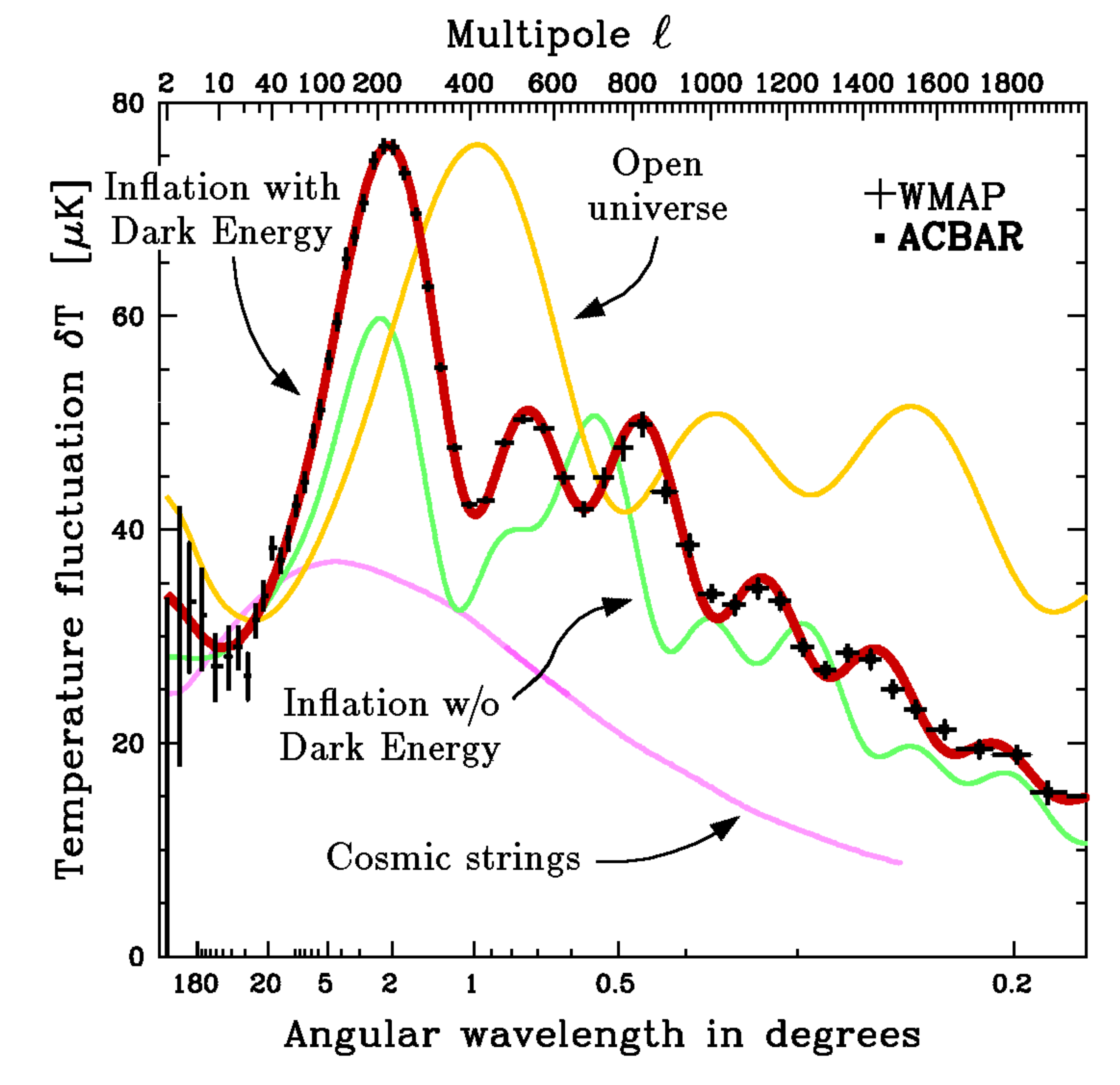

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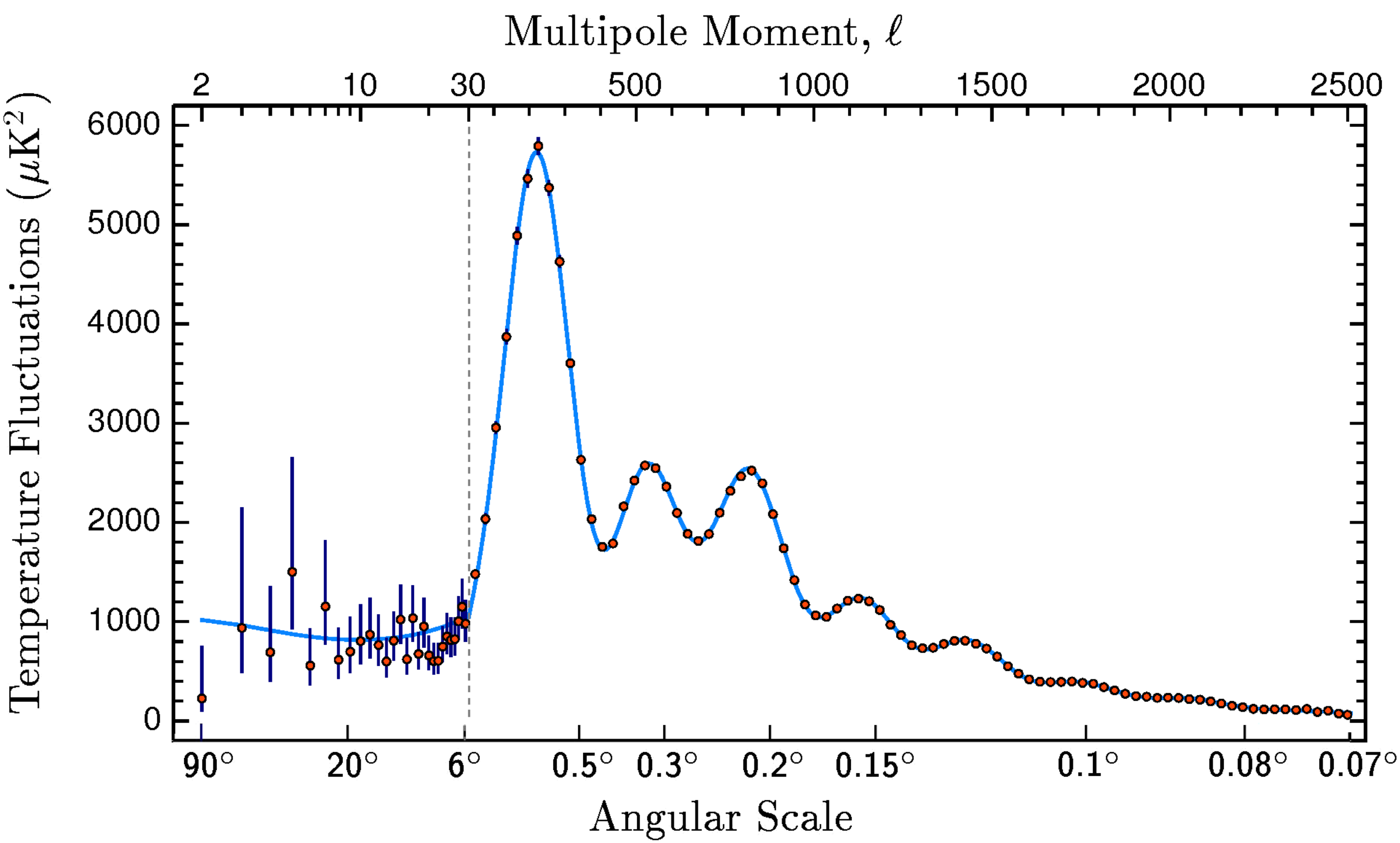
Spectrum of CMB Ripples



Planck Collaboration, 2018

Gravitational Waves:

Gravitational Waves: Came

Gravitational Waves: Came and Went

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April 14, 2015: *A Joint Analysis of BICEP2/Keck Array and Planck Data*: "We find strong evidence for dust and no statistically significant evidence for tensor modes."

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Current limit: $r < 0.07$. Future sensitivity: if $r > 0.001$, it can be found by about 2024 (CMB Stage 4).

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If B-modes are not found, that is not evidence against inflation: many inflationary models predict a B-mode intensity much smaller than 0.001. In 2018 I was involved in a paper about an inflationary model that gave $r \sim 10^{-29}$!

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After one half-life, half of the inflating material has become normal, noninflating matter, but the half that remains has continued to expand exponentially. It is vastly larger than it was at the beginning.

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We would be living in one of the infinity of pocket universes.

The Cosmological Constant Problem

- In 1998, two groups of astronomers discovered that for the past 5–6 billion years, the expansion of the universe has been accelerating.
- According to GR, this requires a repulsive gravity material (i.e., a negative pressure material), which is dubbed "Dark Energy".
- Simplest explanation: dark energy is vacuum energy — the energy density of empty space. The physicist's vacuum is far from empty, so a nonzero energy density is expected.

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It is larger by 120 orders of magnitude!

The Multiverse and the Cosmological Constant Problem

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- But why should we find ourselves in such an extremely rare kind of vacuum?
- The vacuum energy affects cosmic evolution: if it is too large and positive, the universe flies apart too fast for galaxies to form. If too large and negative, the universe implodes.
- It is therefore plausible that life only forms in those pocket universes with incredibly small vacuum energies, so all living beings would observe a small vacuum energy. (Anthropic principle, or observational selection effect.)

SUMMARY

The Inflationary Paradigm is in Great Shape!

- ☆ Explains large scale uniformity.
- ☆ Predicts the mass density of the universe to better than 1% accuracy.
- ☆ Explains the ripples we see in the cosmic background radiation as the result of quantum fluctuations.

Three Strong Winds Blowing Us Towards the Multiverse — a diverse multiverse where selection effects play an important role

1) Theoretical Cosmology: Almost all inflationary models are eternal into the future. Once inflation starts, it never stops, but goes on forever producing pocket universes.

2) Observational Astronomy: Astronomers have discovered that the universe is accelerating, which probably indicates a vacuum energy that is nonzero, but incredibly much smaller than we can understand. Why should this happen?

3) String Theory: String theorists mostly agree that string theory has no unique vacuum, but instead a landscape of perhaps 10^{500} or more long-lived metastable states, any of which could serve as the substrate for a pocket universe, including our own. This situation allows an "anthropic" argument: perhaps we see an incredibly small vacuum energy density because conscious beings only form in those parts of the multiverse where the vacuum energy density is incredibly small.

8.286 Lecture 3
September 14, 2020

THE KINEMATICS of a HOMOGENEOUSLY EXPANDING UNIVERSE

Hubble's Law

$$v = Hr \ .$$

Here

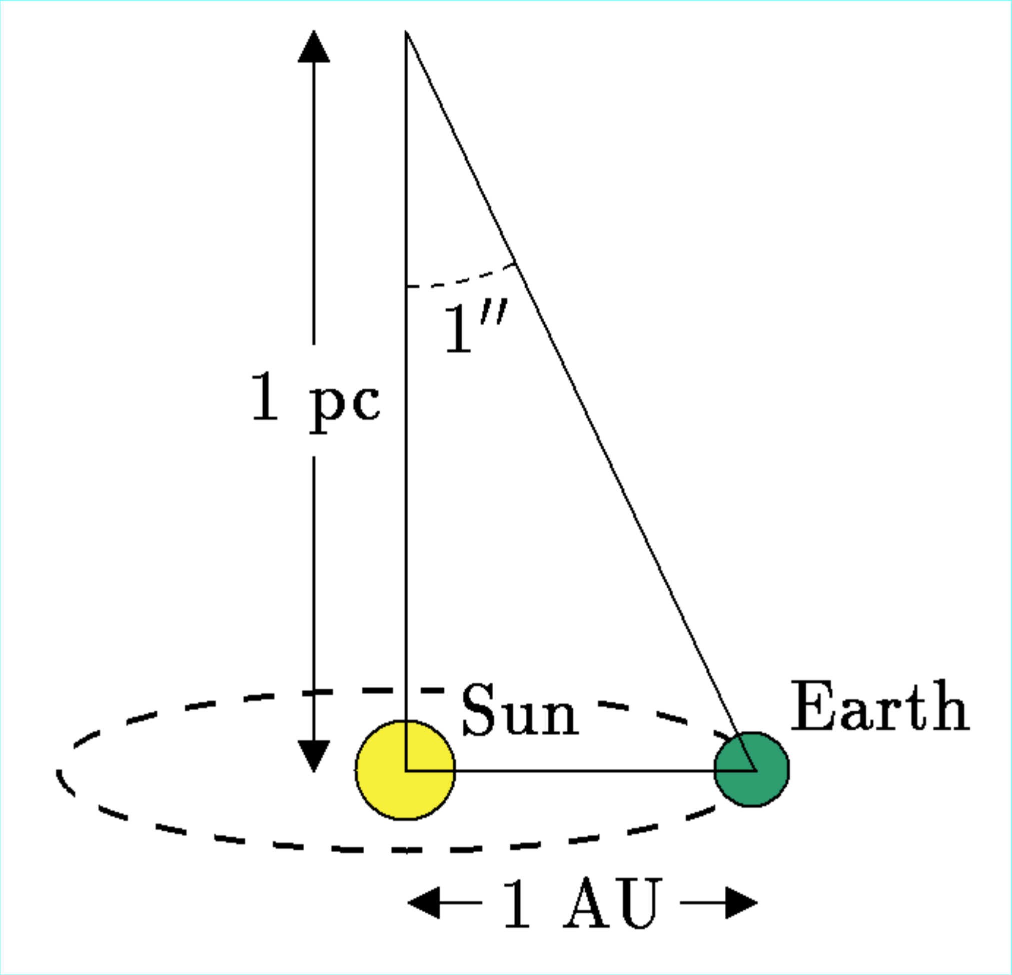
$$v \equiv \text{recession velocity} \ ,$$

$$H \equiv \text{Hubble expansion rate} \ ,$$

and

$$r \equiv \text{distance to galaxy} \ .$$

The Parsec



Units for the Hubble Expansion Rate

$v = Hr \quad \Longrightarrow \quad [H] = [v]/[r] = (L/T)/L = 1/T.$

Astronomers invariably think in terms of velocity/distance, which they measure in km-s^{-1}-Mpc^{-1}.

1 pc = 3.2616 light-yr

Relation to inverse time:

$$\frac{1}{10^{10}\ \text{yr}} = 97.8\ \text{km-s}^{-1}\text{-Mpc}^{-1}.$$

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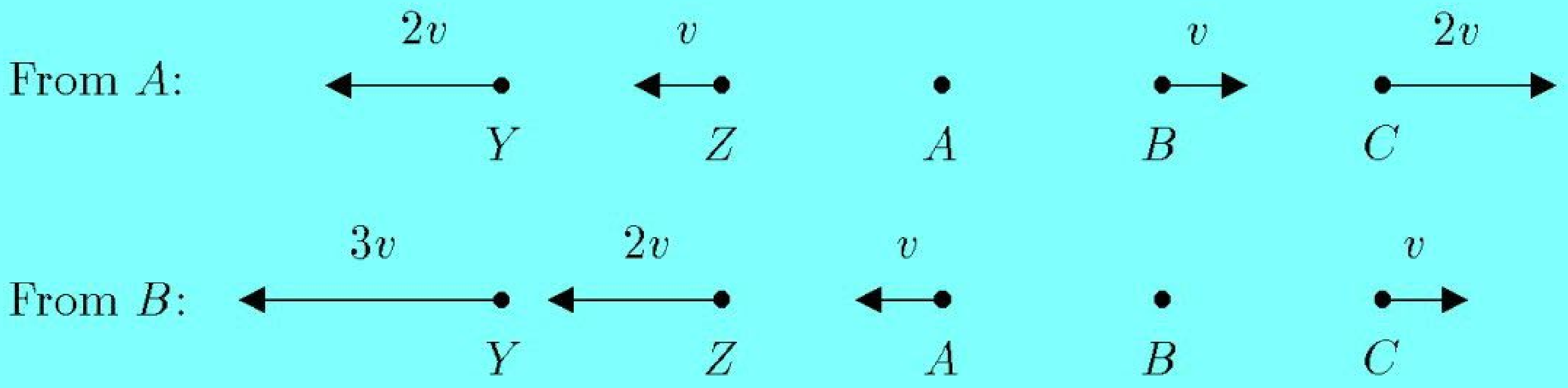

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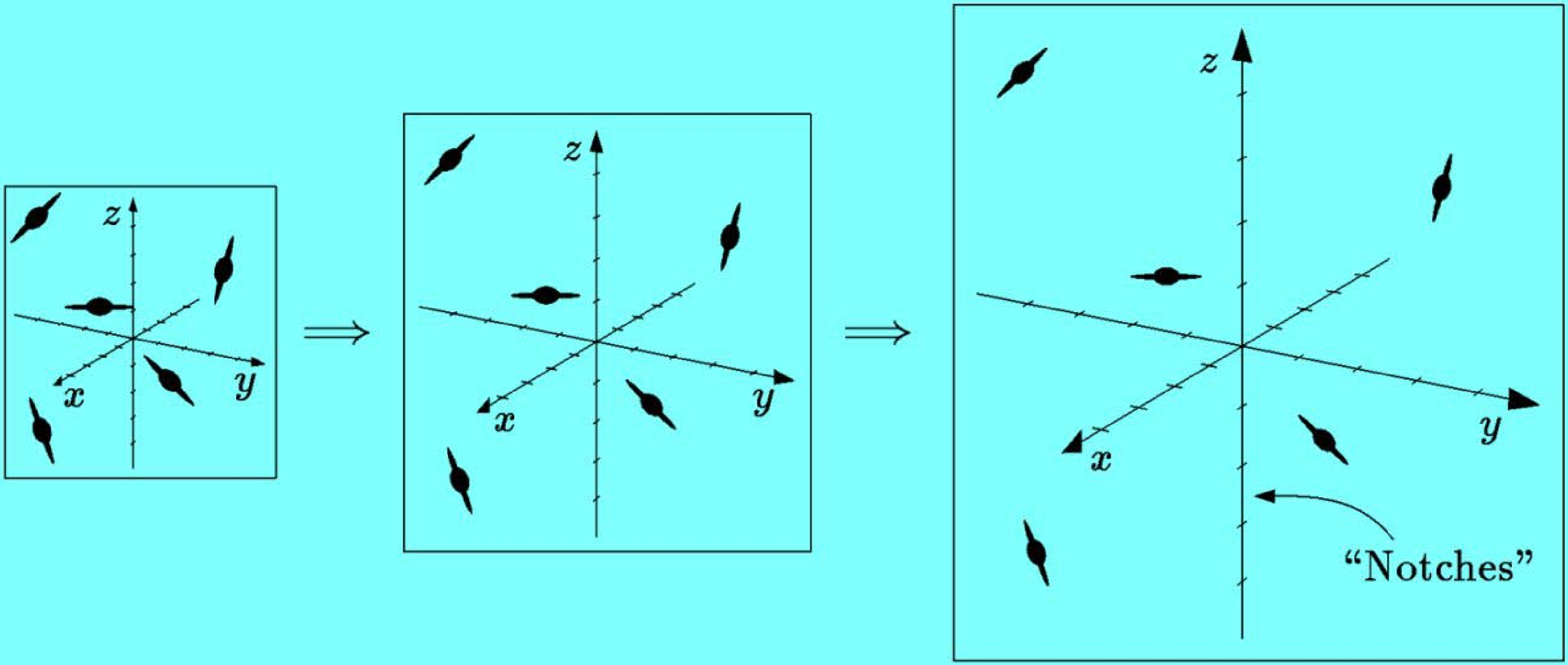
- We imagine a fixed 3D map of the universe, with distances marked in some arbitrary unit: I call them "notches," to make it clear that they have no fixed meaning in terms of any standard units of length. The "scale," or scale factor, is denoted by $a(t)$, where $a(t)$ is measured in notches per meter (or light-year, or Mpc, or whatever). The relation is then

$$\ell_p(t) = a(t)\, \ell_c \ ,$$

where $\ell_p(t)$ is the **physical** distance, measured in meters (or light-years, etc.), and ℓ_c is the **coordinate** distance, measured in notches.


z
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Hubble's Law as a Consequence of Uniform Expansion

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So how fast does $\ell_p(t)$ change?

$$v = \frac{\mathrm{d}\ell_p}{\mathrm{d}t} = \frac{\mathrm{d}a}{\mathrm{d}t}\ell_c = \left[\frac{1}{a(t)}\,\frac{\mathrm{d}a(t)}{\mathrm{d}t}\right] a(t)\ell_c\ .$$

Note that this can be rewritten as

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8.286 Lecture 4
September 16, 2020

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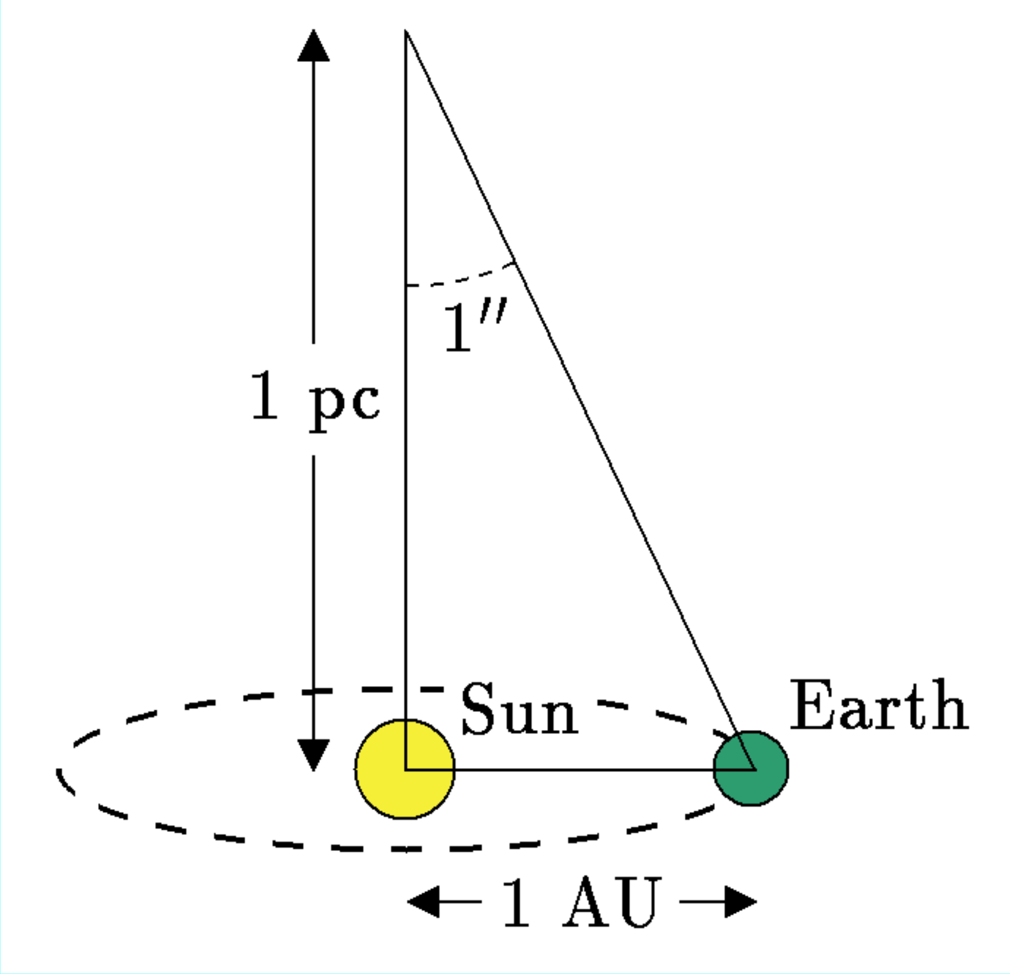

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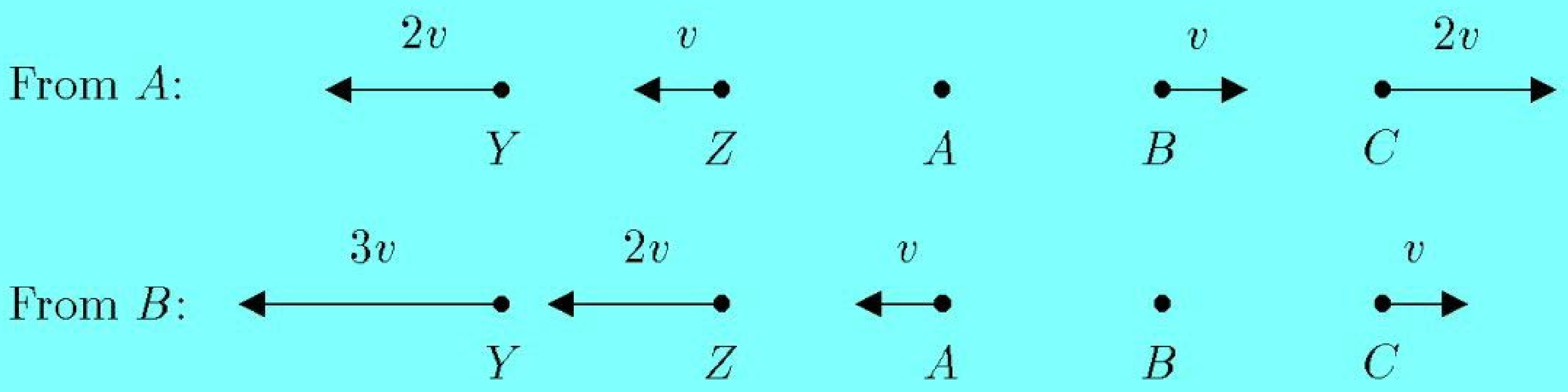

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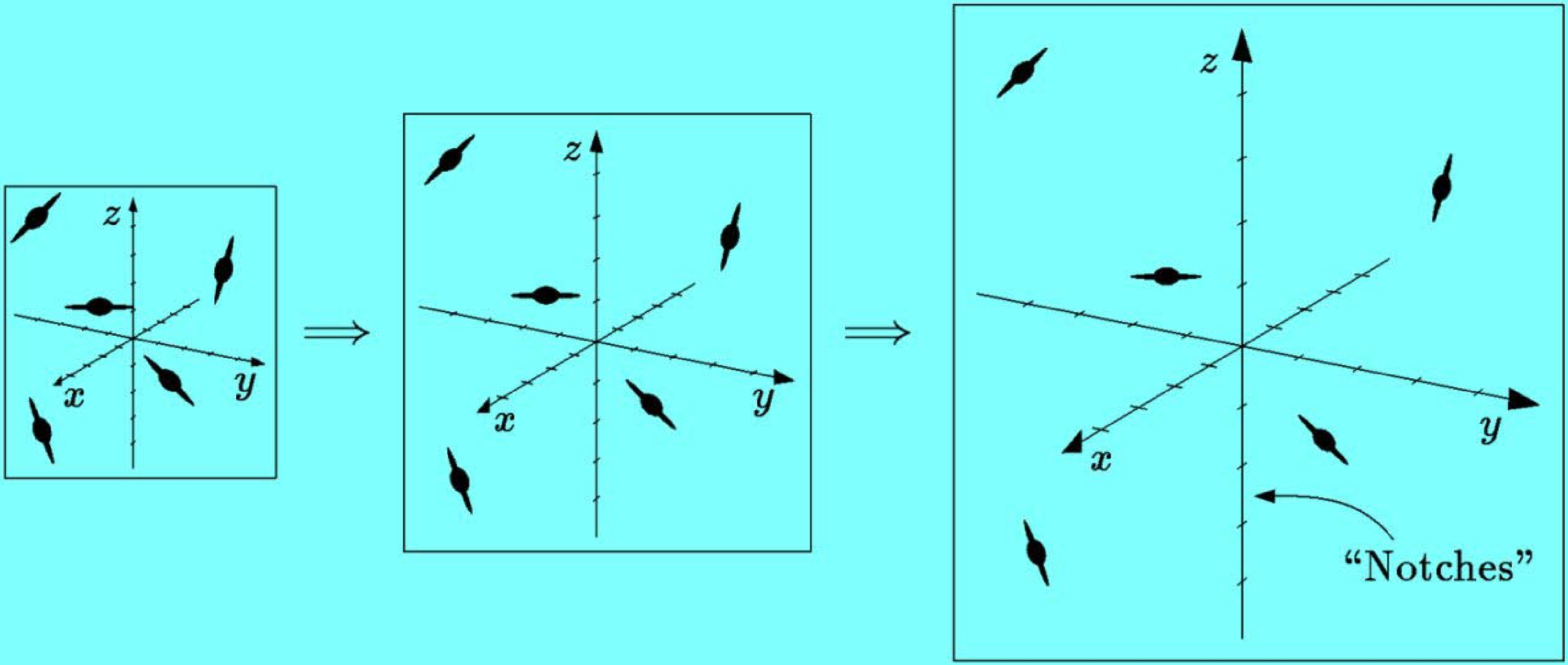
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Light Rays in an Expanding Universe

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- ☆ How do we describe light rays in the comoving coordinate system?
- ☆ The answer is simple:
  Light rays travel on a straight line, with a speed that would be measured by each local observer, as the light ray passes, at the standard value $c = 299,792,458$ m/s.
- ☆ Consider a light pulse moving along the x-axis. If the speed of light in m/s is c, and the number of meters per notch is $a(t)$, then the speed in notches per second is given by

$$\frac{dx}{dt} = \frac{c}{a(t)} \, .$$

- ☆ Justification: the above formula can be derived in general relativity by considering hypothetical point particles that travel at the speed of light, or by incorporating Maxwell's equations into general relativity.

Importance of Comoving Coordinates

Any problem involving an expanding (homogeneous and isotropic) universe should be described in **comoving coordinates.**

Why?

Because the paths of light rays are simple in comoving coordinates.

If instead you tried to use coordinates that directly measure physical distances, the path of a light ray would be **complicated!**

Cosmic Time and the Synchronization of Clocks

- In special relativity, clocks can be synchronized by sending time signals from a central clock. Other clocks, when using these time signals, use their distances from the central clock to take into account the light travel time.

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- ☆ In special relativity, clocks can be synchronized by sending time signals from a central clock. Other clocks, when using these time signals, use their distances from the central clock to take into account the light travel time.

- ☆ In an expanding universe, this does not work!

  1) Because the clocks are moving relative to each other, time dilation would have to be taken into account.

  2) Because the distances are changing with time, one can't know the distance until one knows the time, so the light travel time cannot be taken into account.

- In cosmology, we can imagine that "cosmic time" t is measured locally, on comoving clocks that tick in seconds defined by atomic standards. But they need to be synchronized somehow. Instead of using a central clock, one needs to find a clock that is available everywhere.

- In a simple model of the universe, there are three possibilities:

  1) The Hubble expansion rate H. It can be measured anywhere, so can be used to define the $t = 0$ of cosmic time.

  2) The temperature T of the cosmic background radiation.

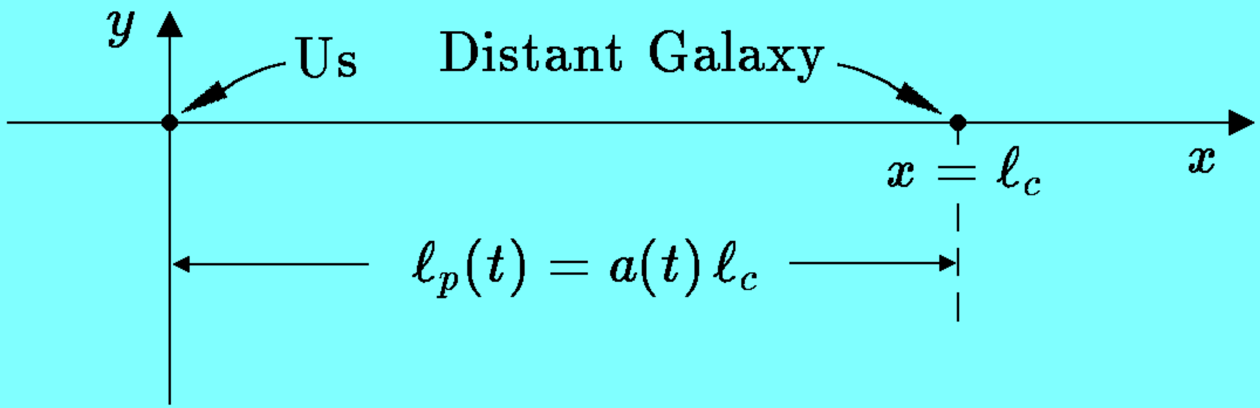
  3) If the universe starts with the scale factor $a = 0$, this starting time can be taken as $t = 0$.

- Will these three methods agree?

- Yes, they must, by the assumption of homogeneity. Homogeneity implies that the relation between H and T must be the same everywhere. So if you and I, in far away galaxies, measure the same value of H, we must also measure the same value of T.

Cosmological Redshift

☆ Use comoving coordinates!

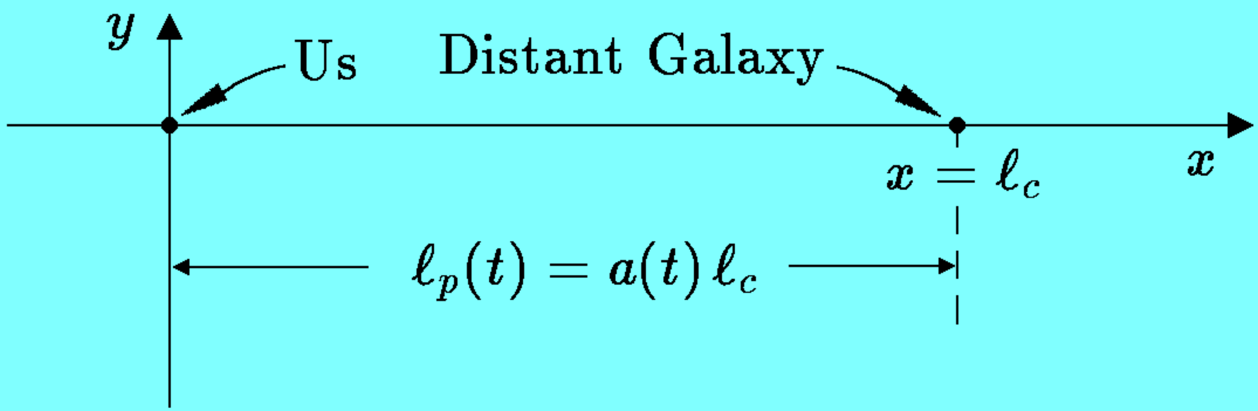


☆ Let Δt_S be the time between wave crests, as measured at the source.

☆ Since cosmic time t is measured on local clocks, Δt_S is the separation in cosmic time between the emission of crests.

☆ The physical wavelength at the source is $\lambda_S = c\Delta t_S$. When the 2nd crest is emitted, the first crest will be a physical distance λ_S from the source.

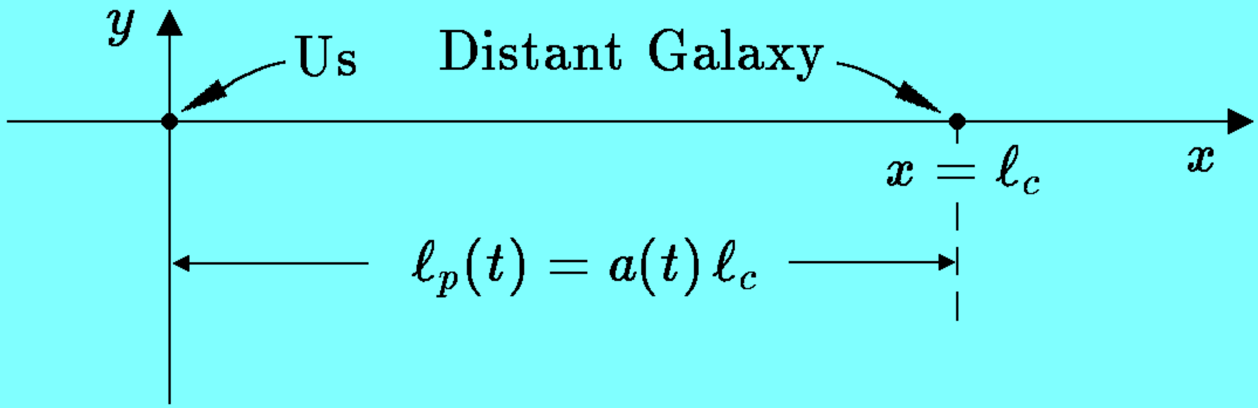
☆ When the 2nd crest is emitted, the first crest will be a coordinate distance $\Delta x = \lambda_S / a(t_S)$ from the source.



- When the 2nd crest is emitted, the first crest will be a coordinate distance $\Delta x = \lambda_S / a(t_S)$ from the source.
- As the first and second crests travel from source to us, they both travel at coordinate speed

$$\frac{dx}{dt} = \frac{c}{a(t)} .$$

- The speed depends on time, but not position: so the crests remain the same coordinate distance apart.



☆ When the crests reach us, at cosmic time t_O, they still have a coordinate separation $\Delta x = \lambda_S / a(t_S)$. The physical distance at the observer (us) is therefore

$$\lambda_O = a(t_O)\,\Delta x = \left[\frac{a(t_O)}{a(t_S)}\right] \lambda_S \ ,$$

so the wavelength is simply stretched with the expansion of the universe.

☆ The period of a light wave is proportional to its wavelength, so

$$1 + z \equiv \frac{\Delta t_O}{\Delta t_S} = \frac{\lambda_O}{\lambda_S} = \frac{a(t_O)}{a(t_S)} \ .$$

THE DYNAMICS OF NEWTONIAN COSMOLOGY, PART 1

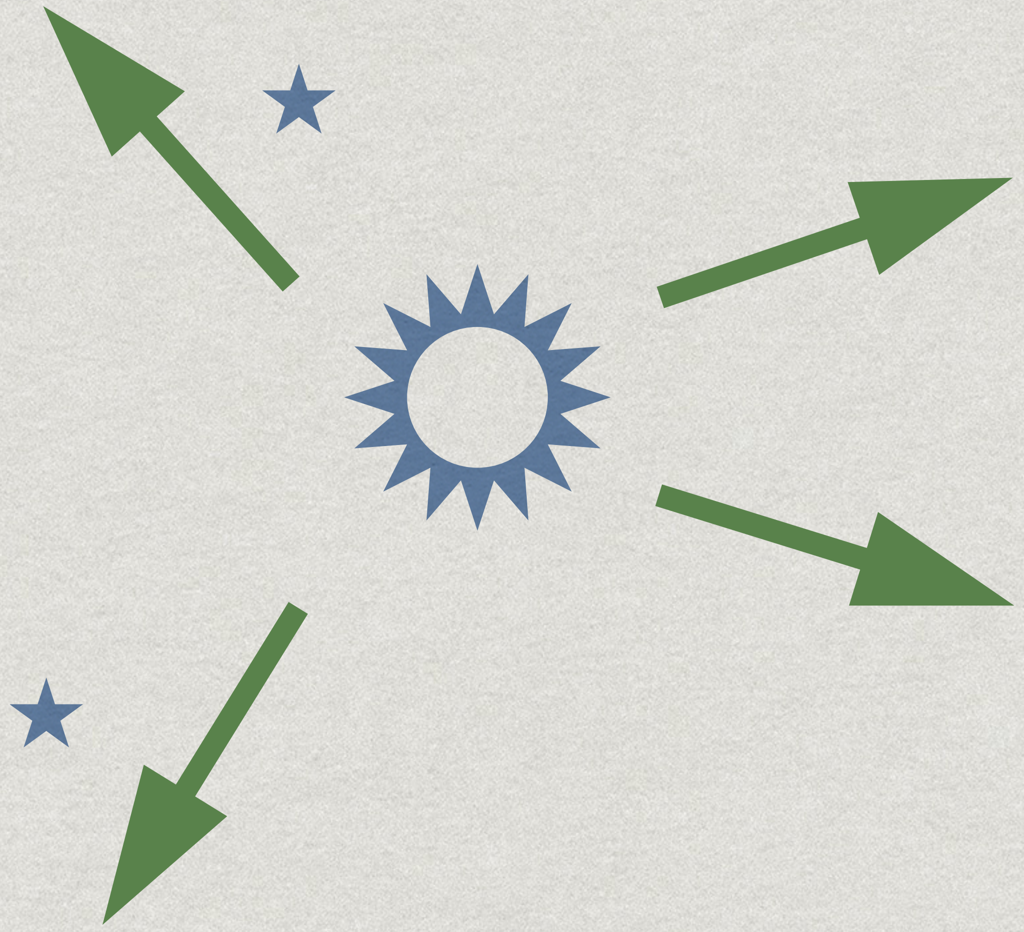
8.286 LECTURE 5

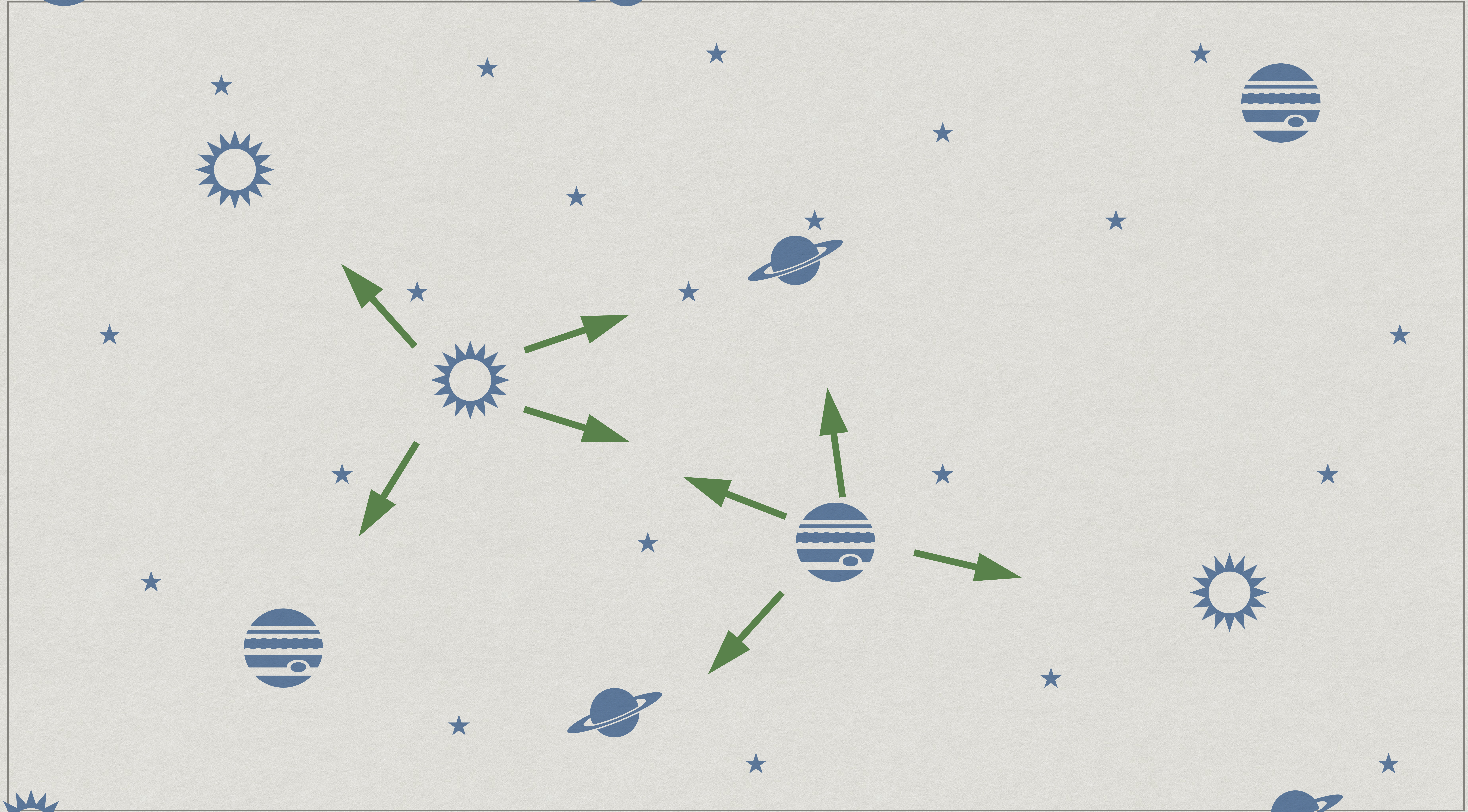
The Dynamics of a Homogeneous Mass Distribution

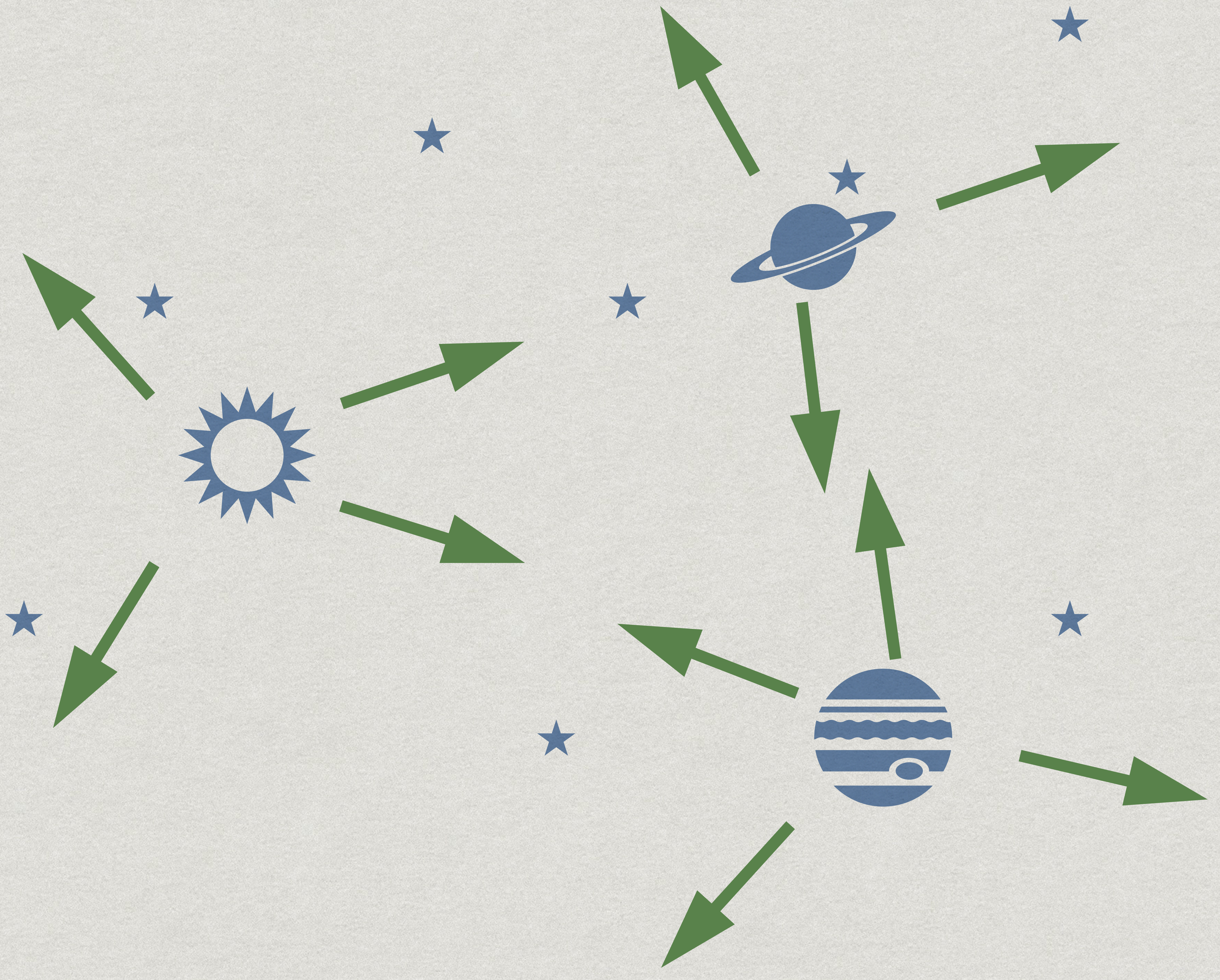
"As to your first query, it seems to me that if the matter of our sun and planets and all the matter of the universe were evenly scattered throughout all the heavens, and every particle had an innate gravity toward all the rest, and the whole space throughout which this matter was scattered was but finite, the matter on the outside of this space would, by its gravity, tend toward all the matter on the inside and, by consequence, fall down into the middle of the whole space and there compose one great spherical mass. But if the matter was evenly disposed throughout an infinite space, it could never convene into one mass; but some of it would convene into one mass and some into another, so as to make an infinite number of great masses, scattered at great distances from one to another throughout all that infinite space. And thus might the sun and fixed stars be formed, supposing the matter were of a lucid nature." (Isaac Newton to Richard Bentley, December 10, 1692)

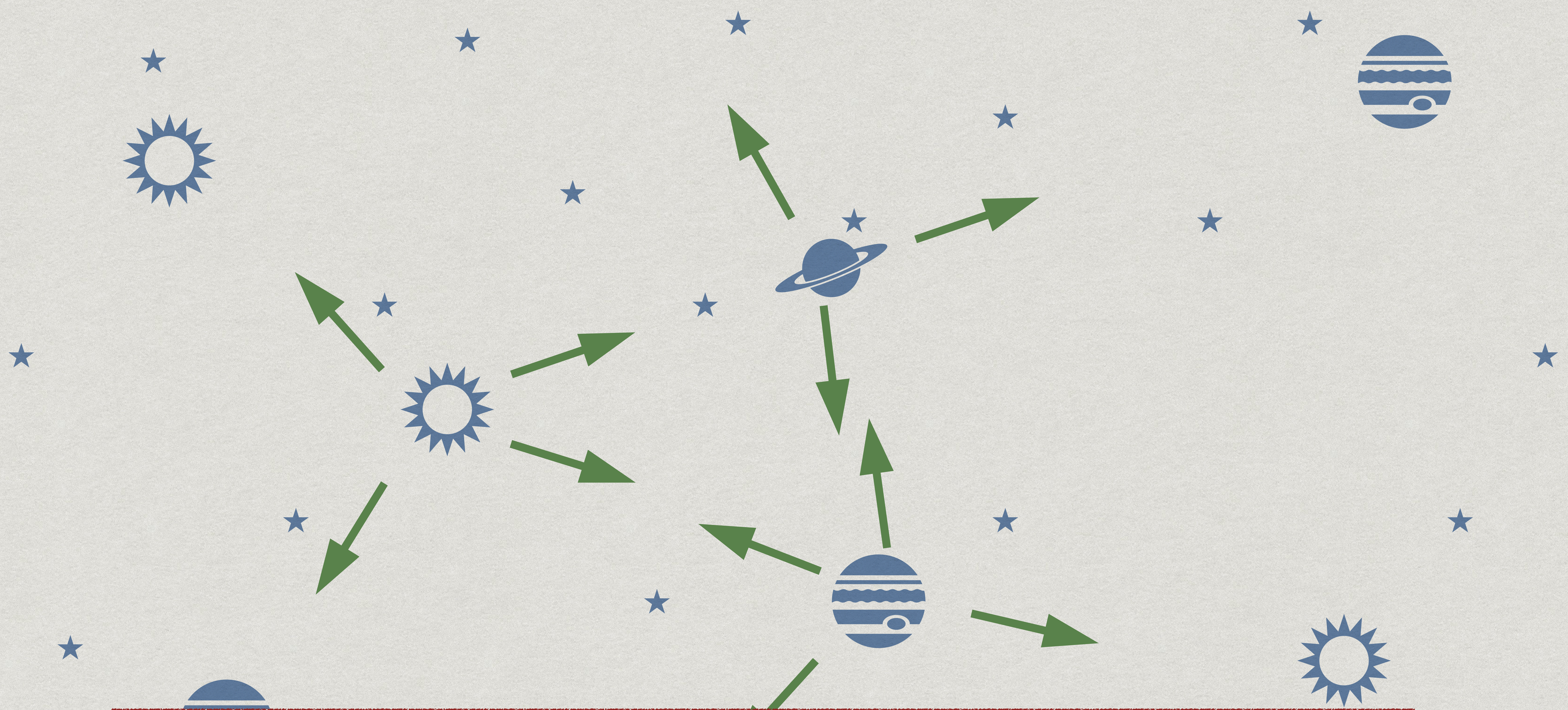
Web references: http://books.google.com/books?id=8DkCAAAAQAAJ&pg=PA201

http://www.newtonproject.sussex.ac.uk/view/texts/normalized/THEM00254

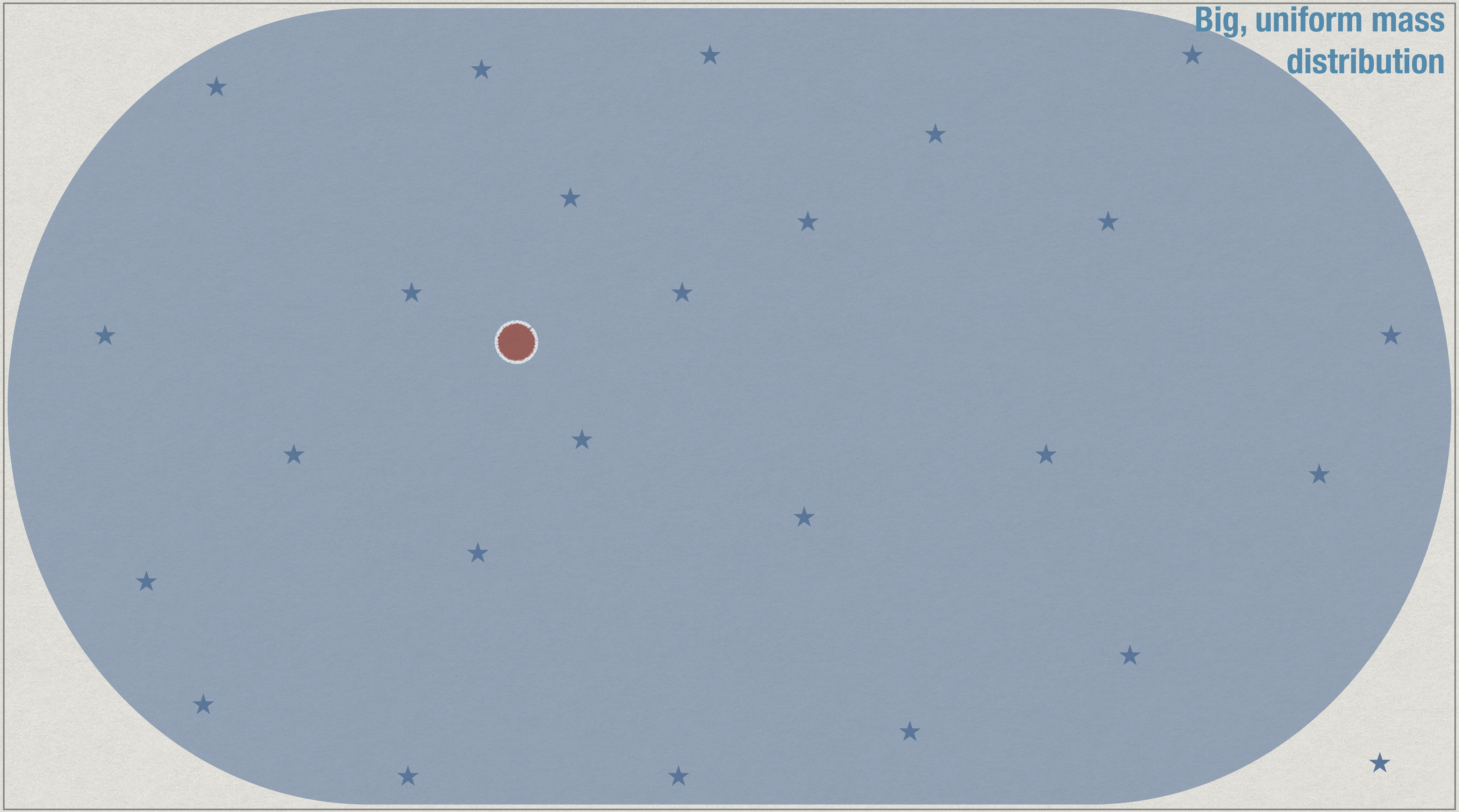




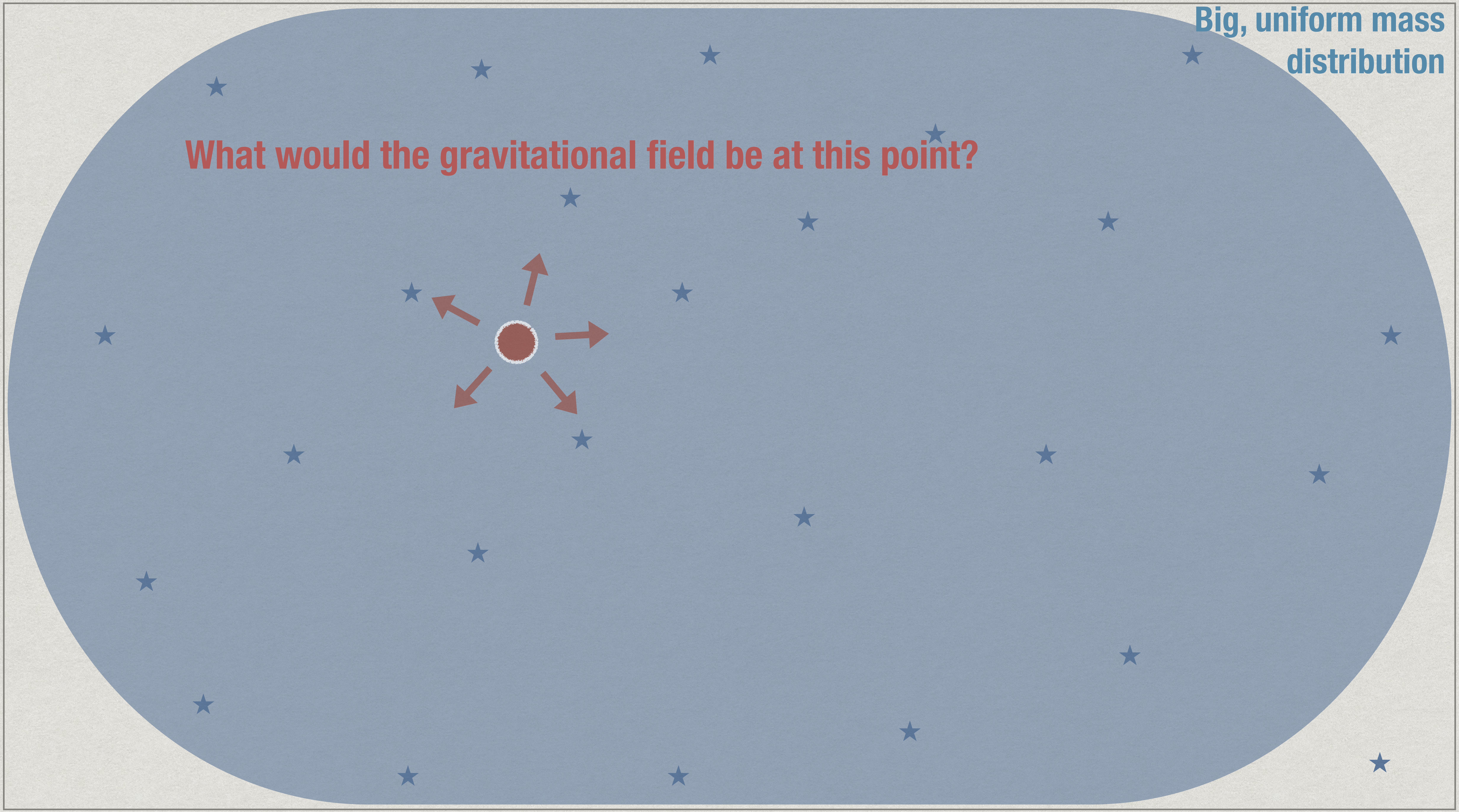


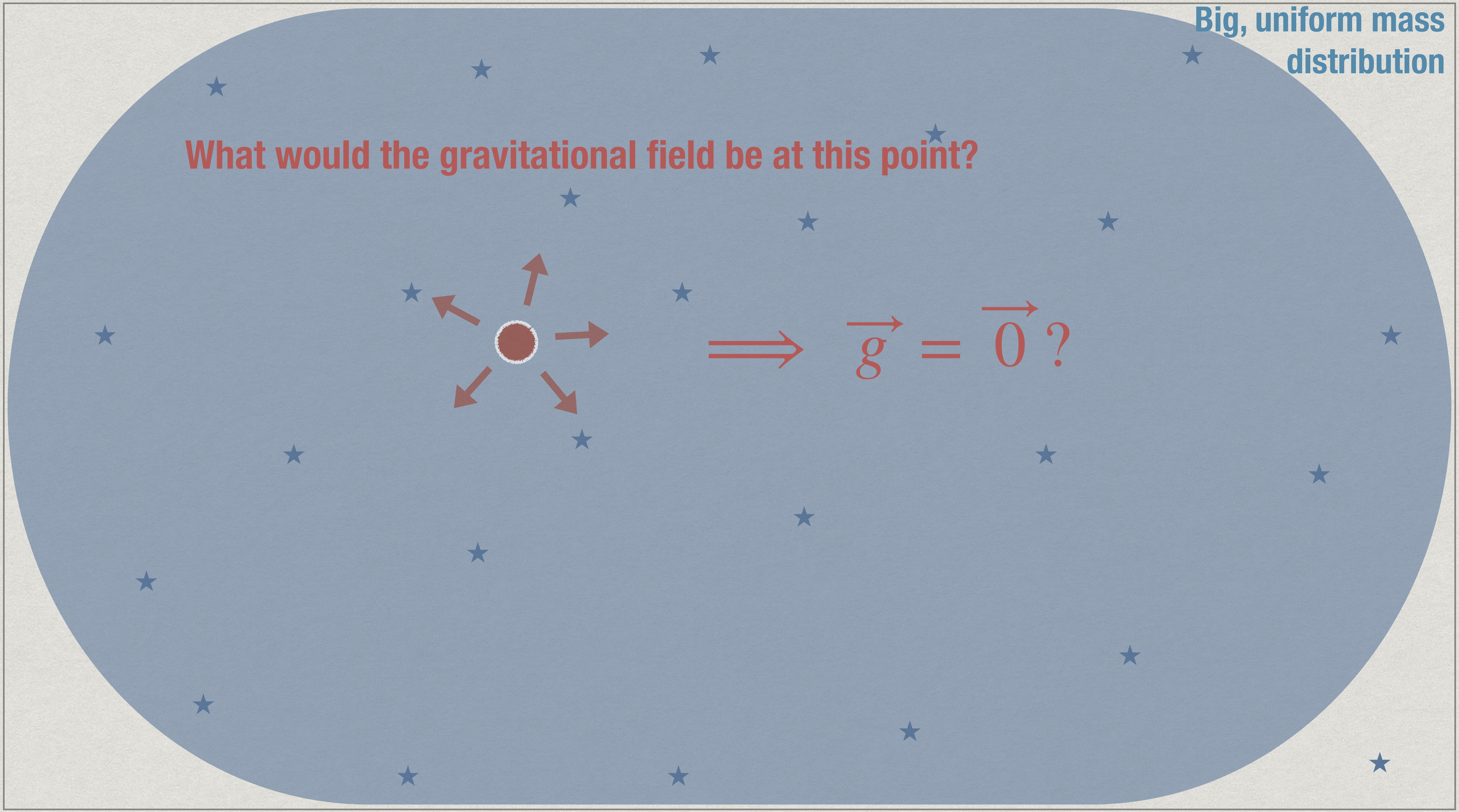


From this point of view, if the mass distribution were homogeneous, it seems reasonable to think that all forces cancel each other out, leading to a static universe.


Big, uniform mass
distribution


Big, uniform mass distribution
What would the gravitational field be at this point?


Big, uniform mass distribution
What would the gravitational field be at this point?



$$\implies \vec{g} = \vec{0}\,?$$

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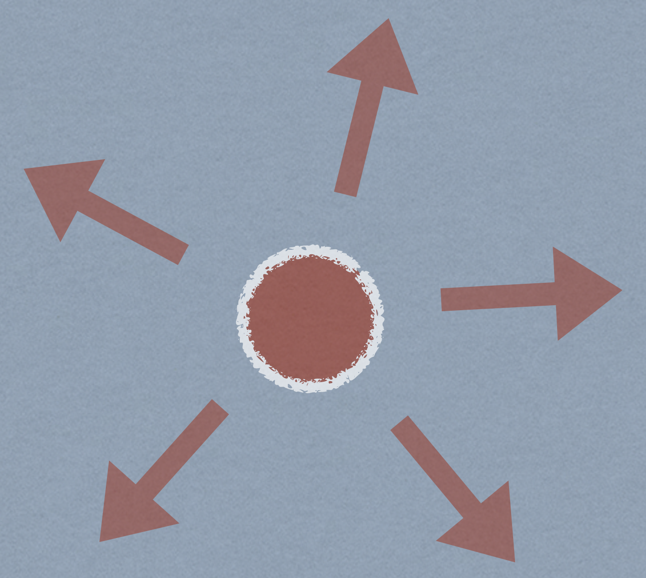

$$\Longrightarrow \quad \vec{g} = \vec{0}\,?$$

The gravitational field can't be zero everywhere!

Gauss' law:

- Electromagnetism:

$$\vec{E} = \frac{q}{r^2}\hat{r} \iff \oint \vec{E} \cdot d\vec{A} = 4\pi q_{\text{enclosed}}$$

- Gravity:

$$\vec{g} = -\frac{GM}{r^2}\hat{r} \iff \oint \vec{g} \cdot d\vec{A} = -4\pi G M_{\text{enclosed}}$$

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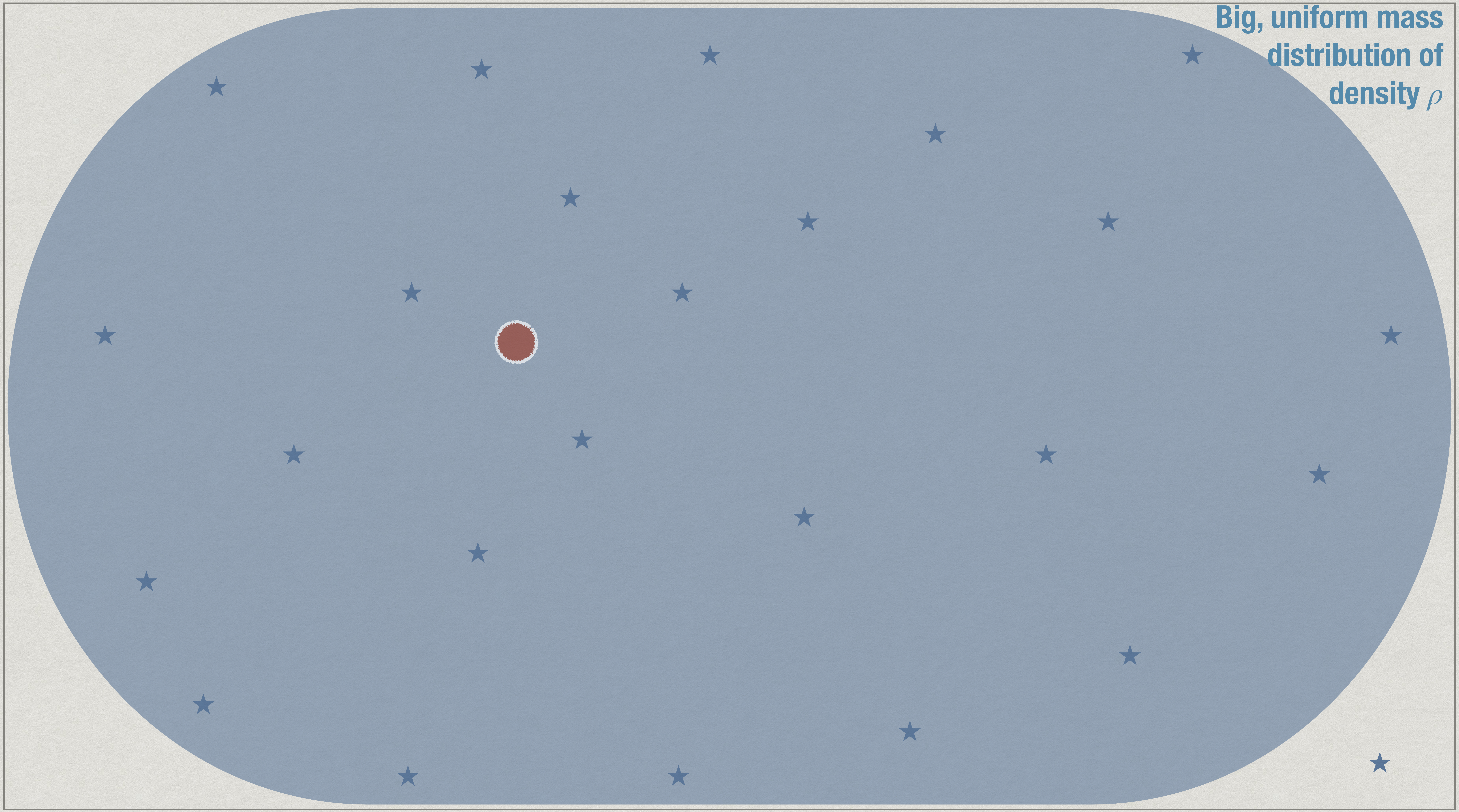
If $\overrightarrow{g} = 0$ were true, Gauss' law would imply that inside any Gaussian surface that we choose, the enclosed mass would be zero.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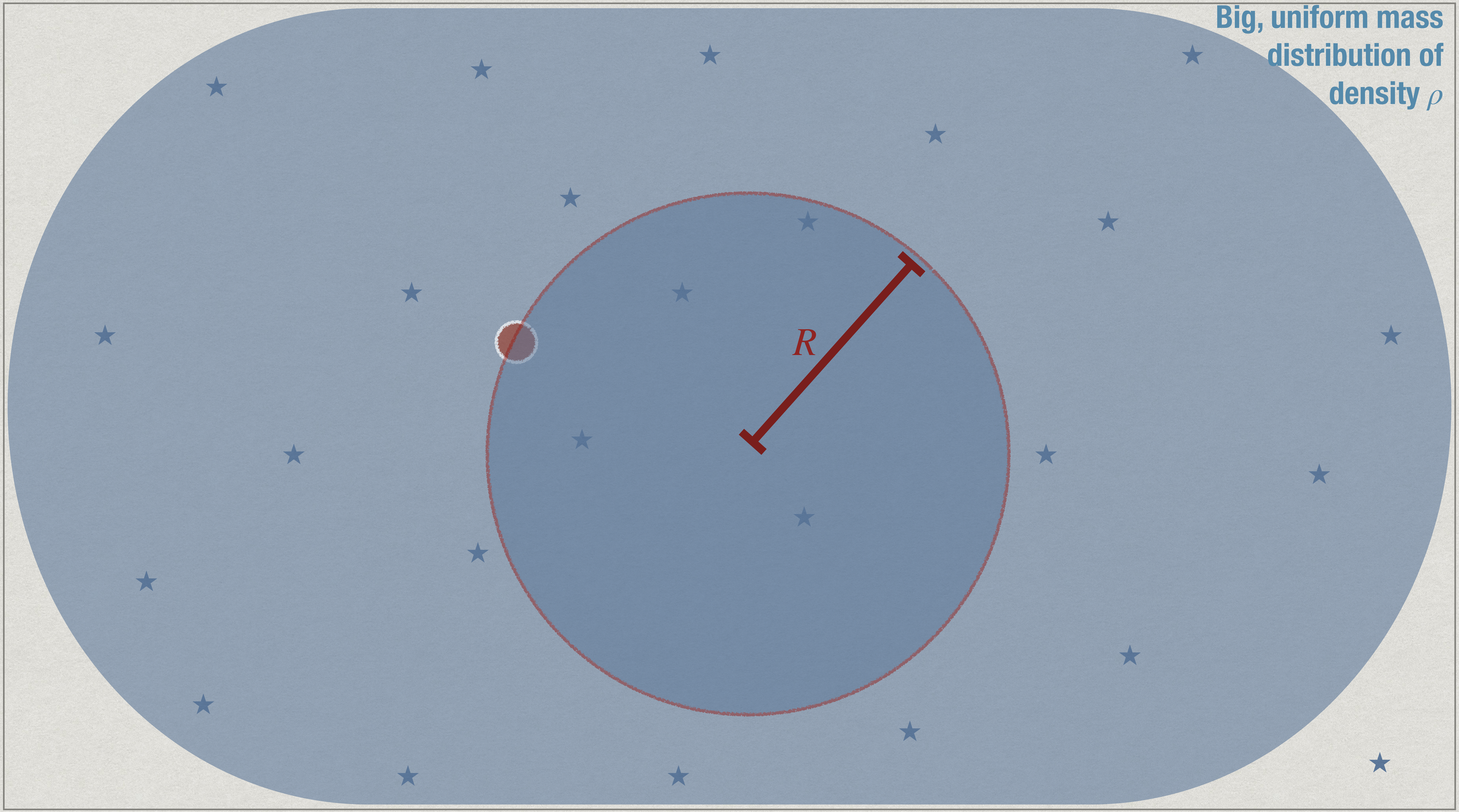

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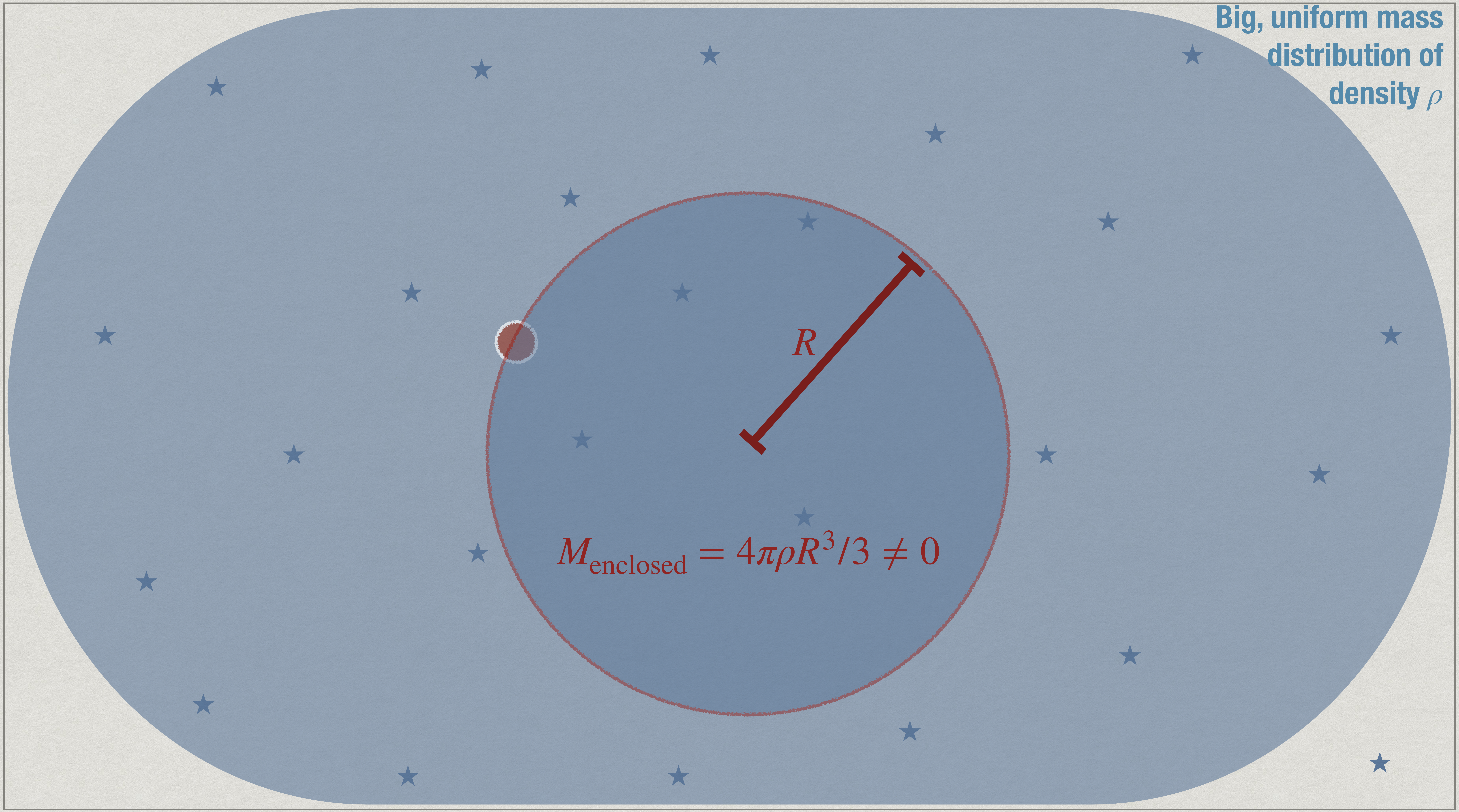
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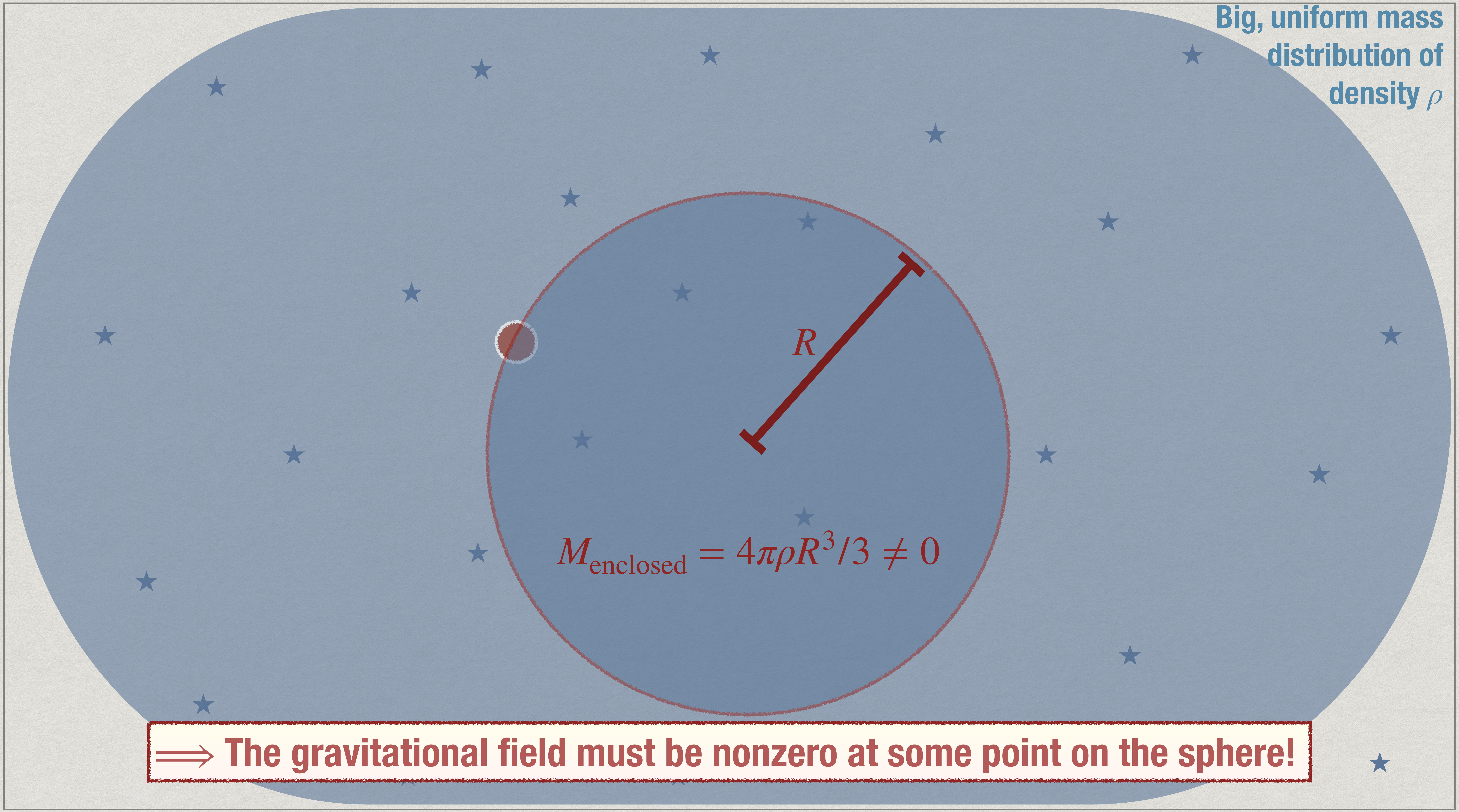
$M_{\text{enclosed}} = 4\pi\rho R^3/3 \neq 0$


Big, uniform mass distribution of density ρ
R
$M_{\text{enclosed}} = 4\pi\rho R^3/3 \neq 0$
$\Longrightarrow$ The gravitational field must be nonzero at some point on the sphere!

Poisson's equation

- We can also formulate Newtonian gravity in terms of a differential equation:

$$\nabla^2 \phi = 4\pi G\rho \qquad \text{where} \qquad \nabla \phi = -\overrightarrow{g}\,.$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2},\ \nabla = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

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Therefore, if $\overrightarrow{g}$ were zero, then ϕ would be constant, and consequently $\rho = 0$.

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What is the gravitational field then?

Conditionally convergent integrals

- One of the problems with calculating the gravitational field as the sum of infinitely many point-like particles is that the sum (integration) is ambiguous: let's say we want to calculate $\overrightarrow{g}$ at some point P

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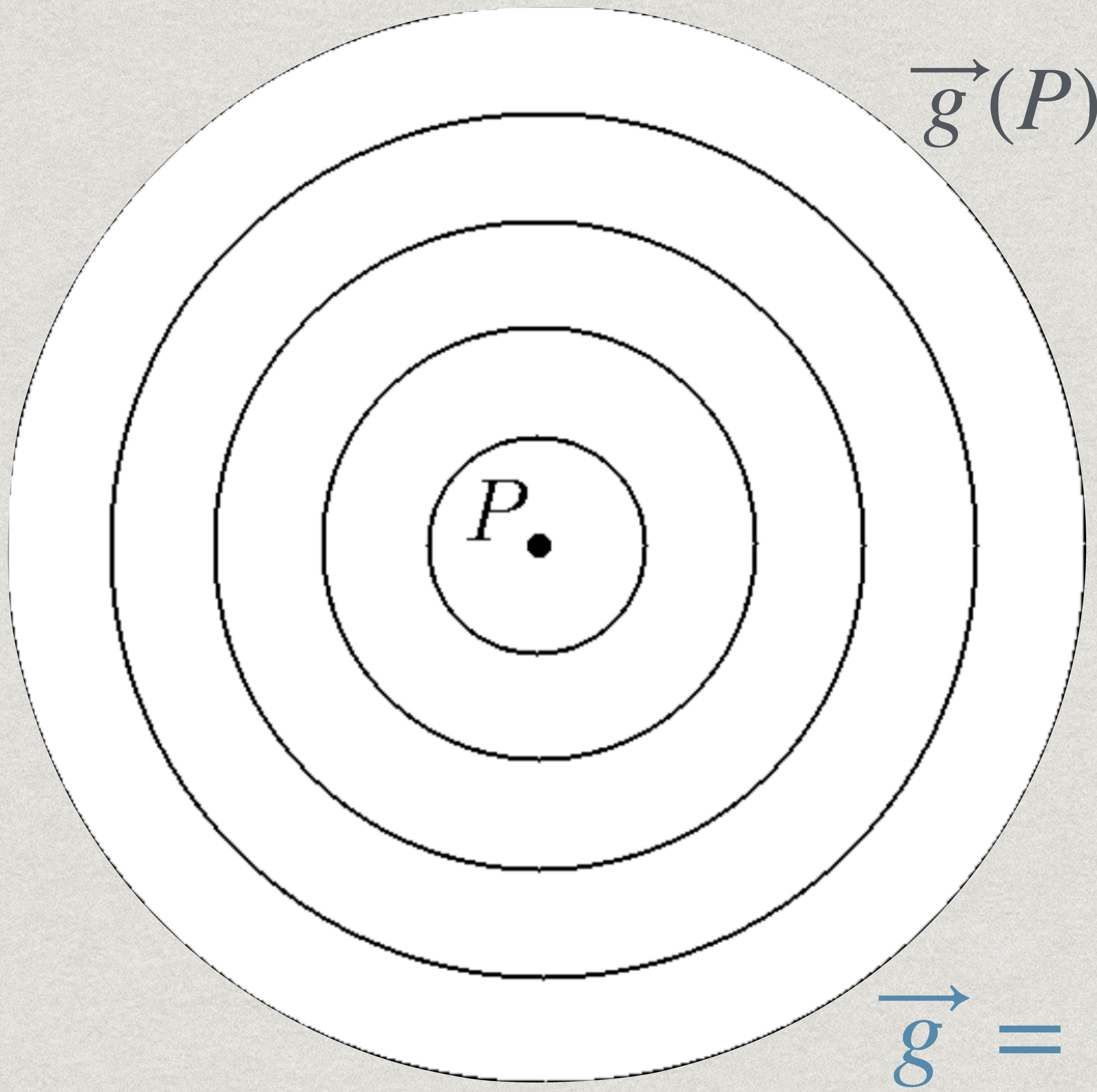

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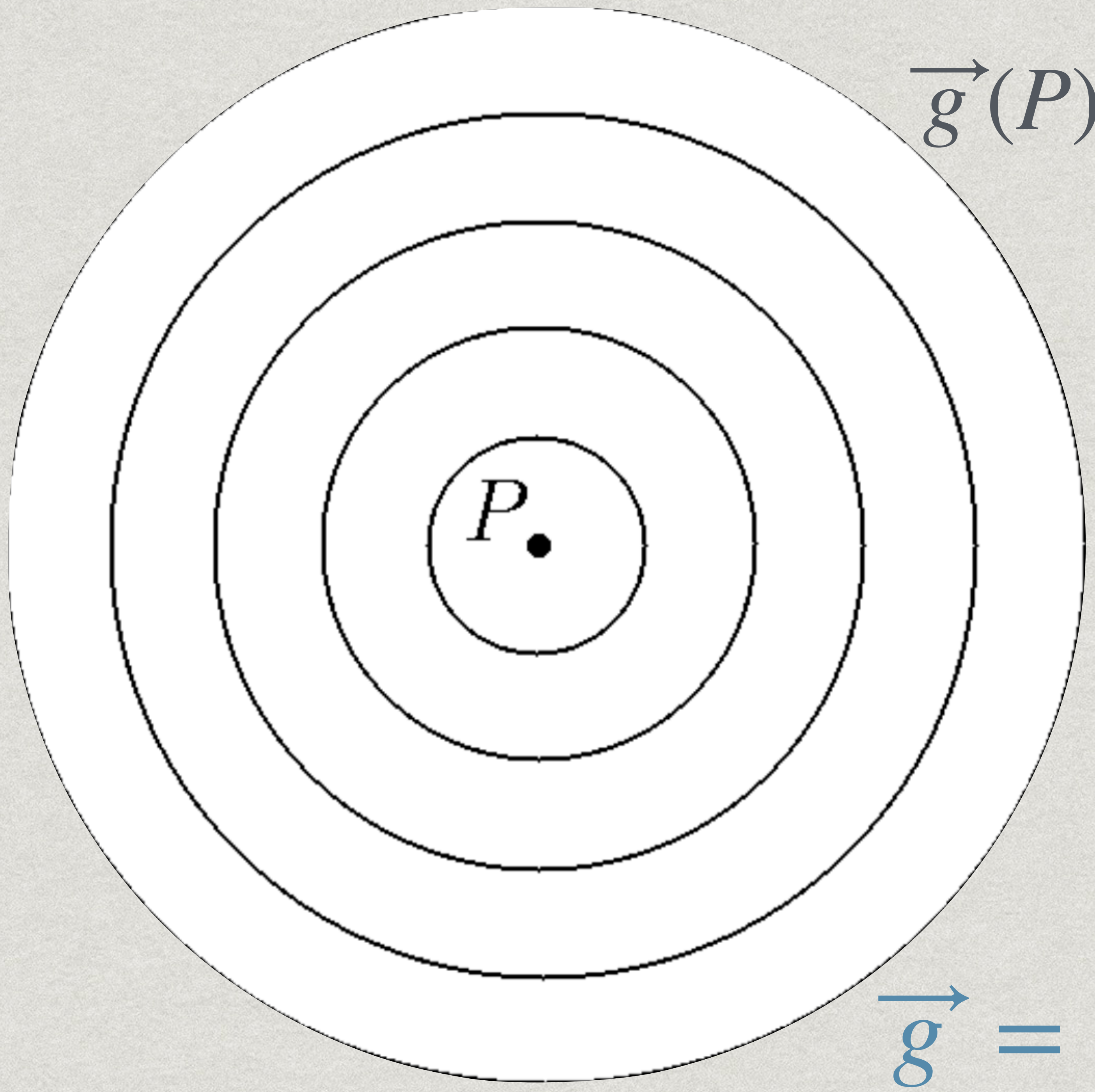

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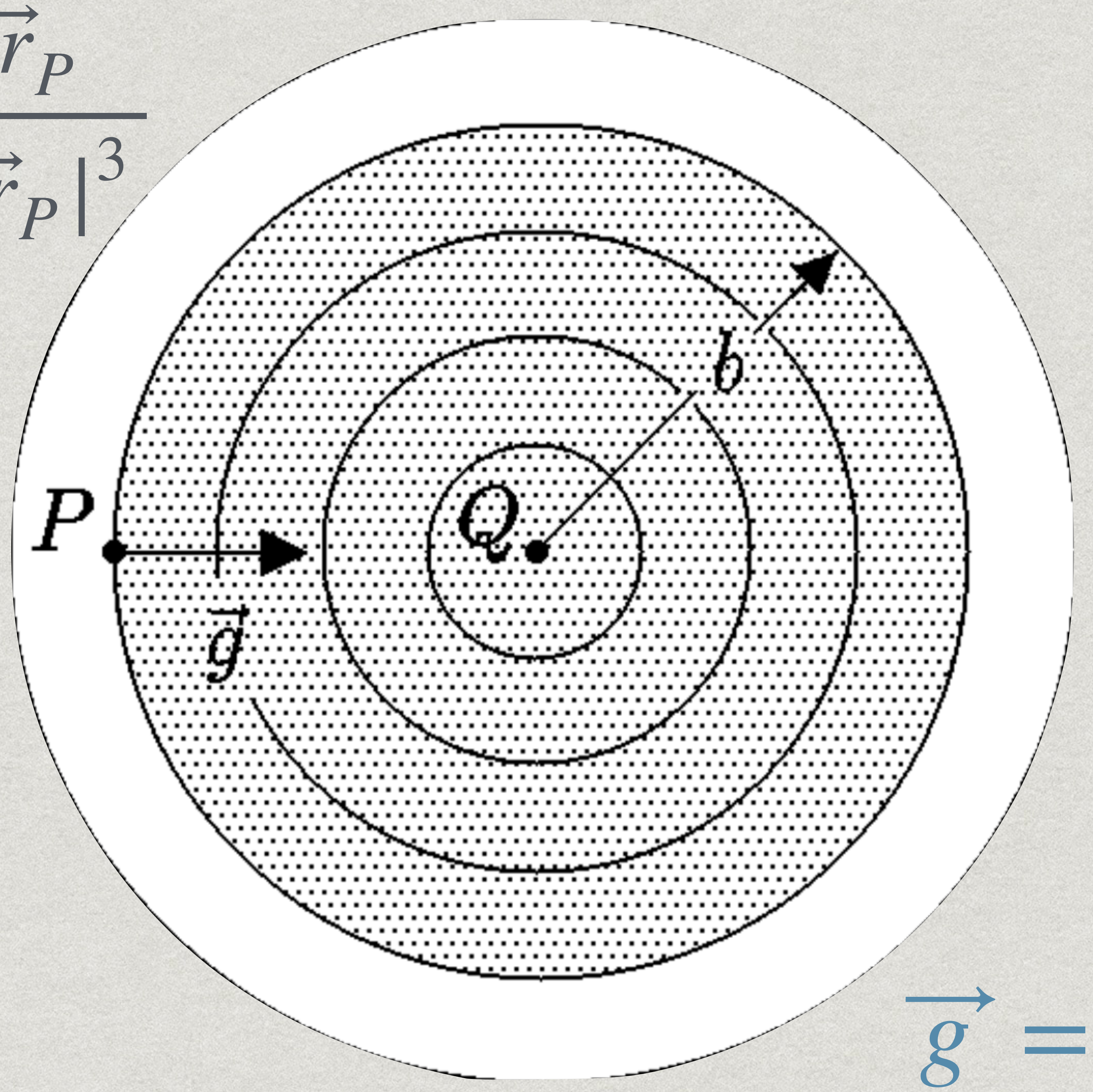

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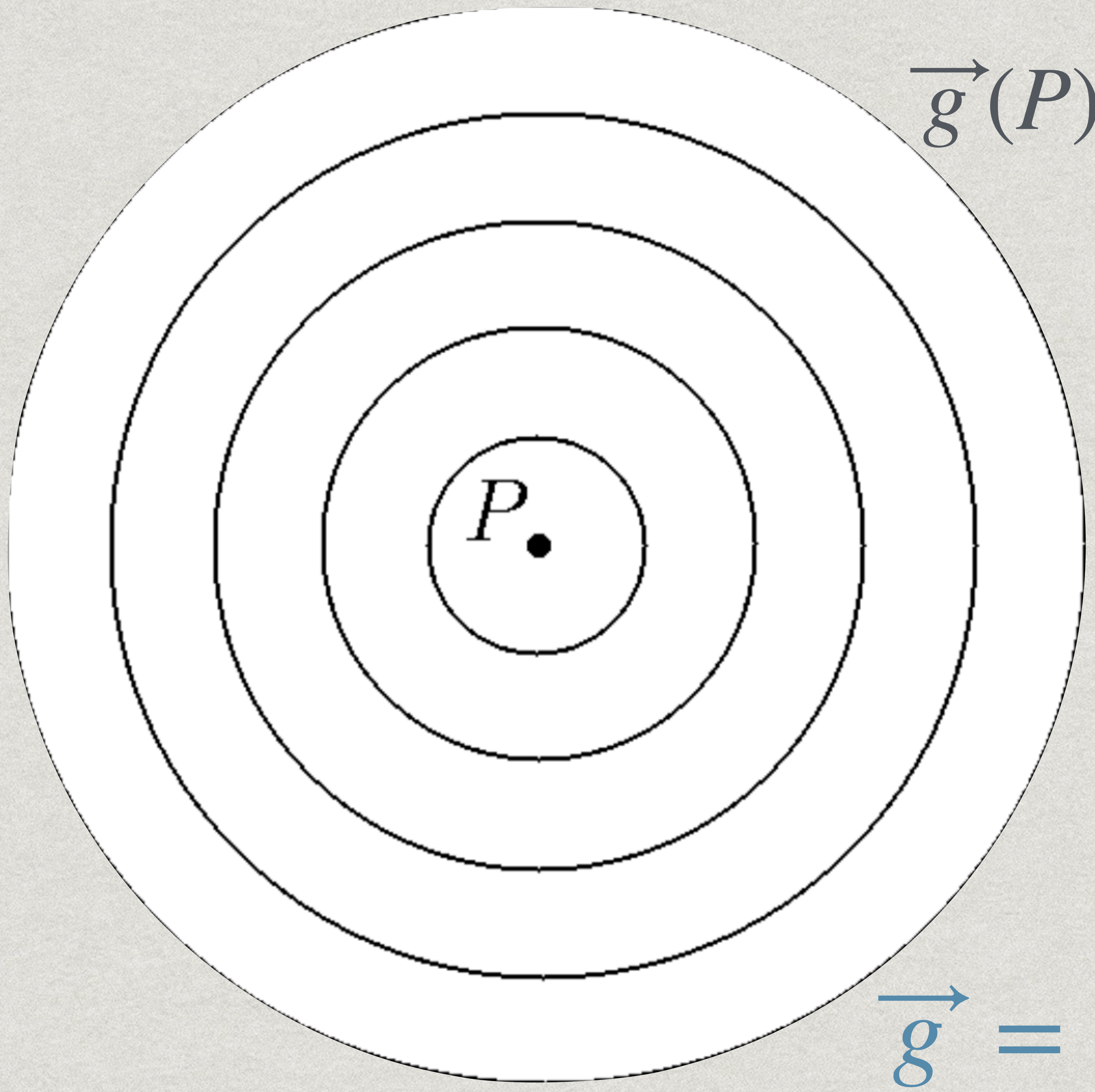

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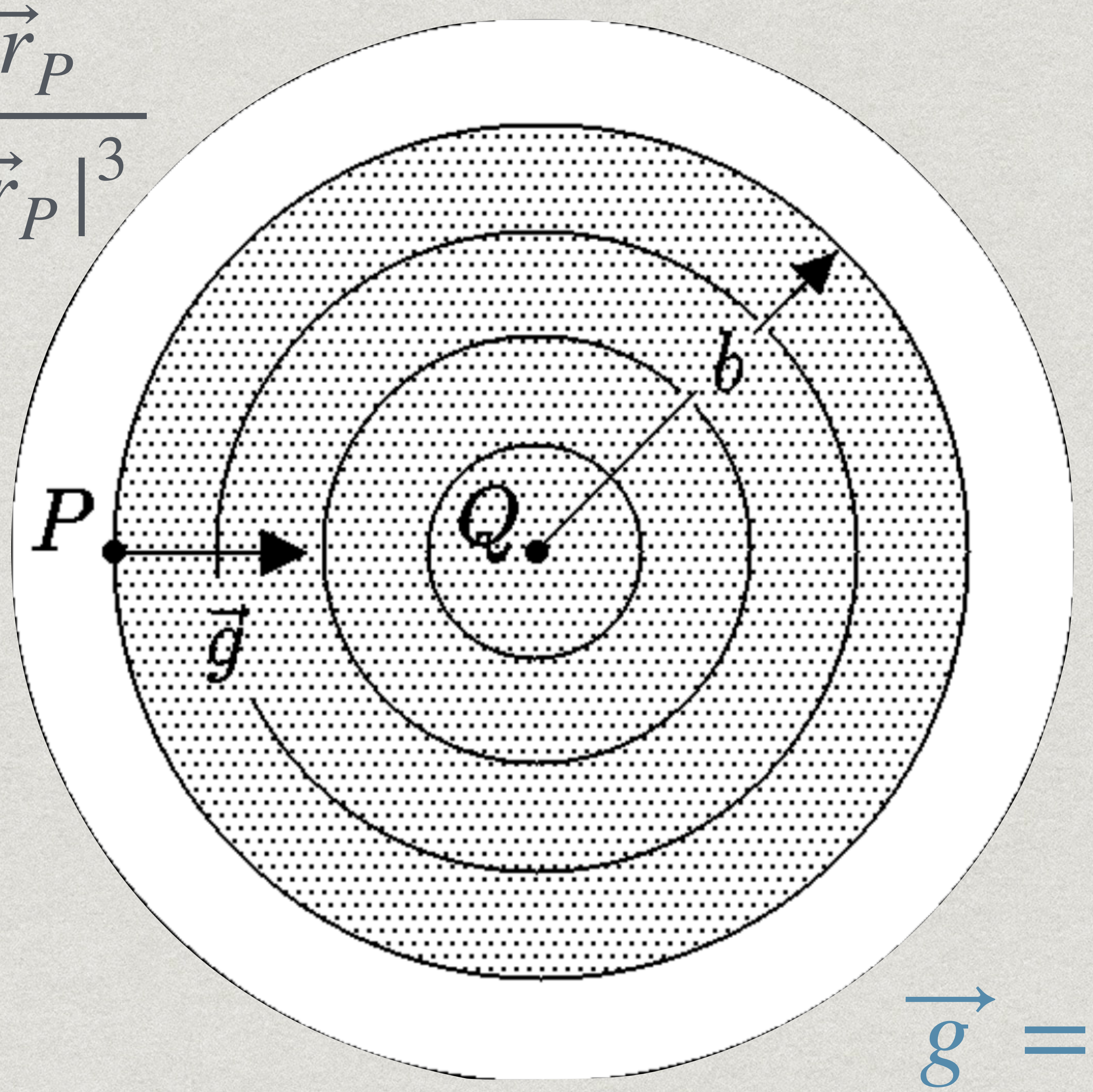

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- In this case, we say that integration is *conditionally convergent.*

What about symmetry?

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- Since the whole universe is filled with (gravitating) matter, such an inertial frame does not exist. Therefore accelerations, as well as velocities, are <u>relative</u>.

$\Longrightarrow$ Each observer can consider itself as non-accelerating.

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- Integration is conditionally convergent because the mass density has an infinite spatial extension.
- Therefore, to be on the safe side, we must define $\vec{g}$ as a limit of finite quantities.
- Symmetry compels us to use spheres around the reference point of our choice.

A brief aside: History of the universe

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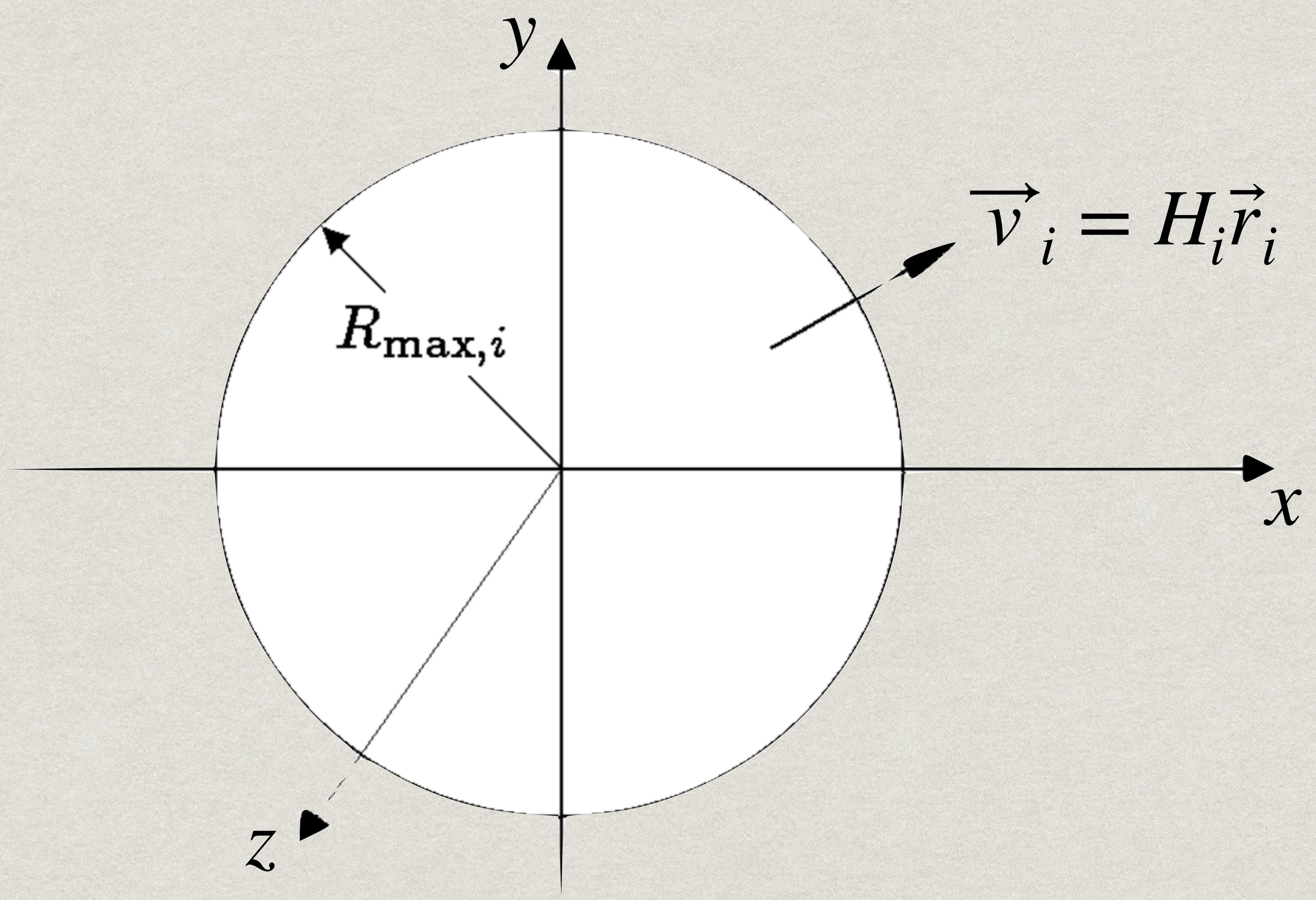
- For the next 9 billion years, the energy density of the universe was dominated by dust-like matter. **Newtonian gravity applies here!**
- There is astronomical evidence that for the last 5 billion years, the energy density of our universe has been dominated by a mysterious "Dark Energy."

“Initial” conditions of our model



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$$\overrightarrow{v}_i = H_i \vec{r}_i$$

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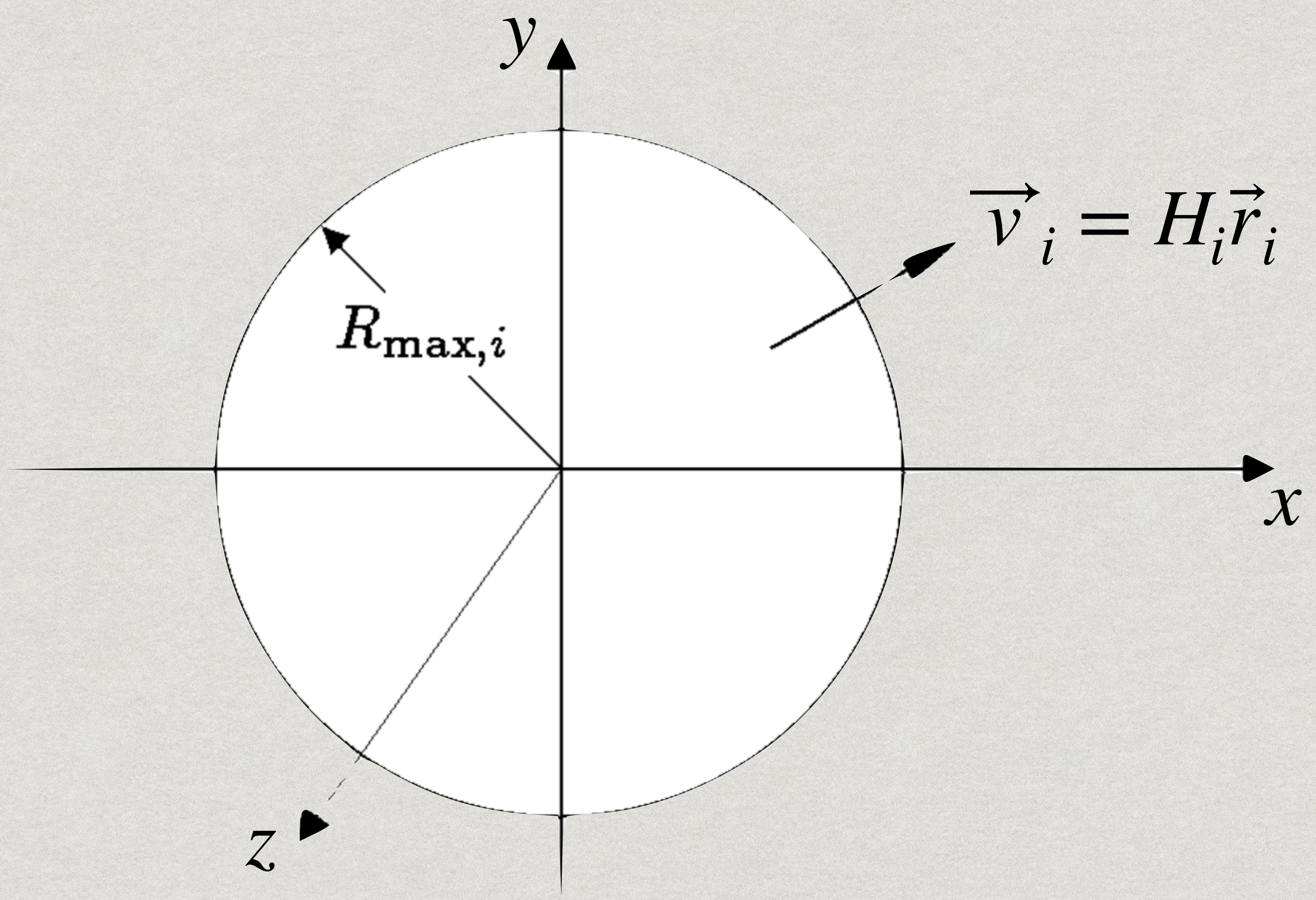

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To match Hubble's law

Solving the equations

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- We label each spherical shell by their initial radius r_i, and follow their trajectories $r(r_i, t)$.
- We will assume that the trajectories of two shells at different initial radii do not cross, and verify this condition *a posteriori*.

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- On the other hand, the (conserved) mass inside each shell initially at r_i is given by

$$M(r_i) = \frac{4\pi}{3} r_i^3 \rho_i.$$

- Therefore, the gravitational field experienced by particles on a mass shell that was originally at r_i is

$$\overrightarrow{g} = -\frac{GM(r_i)}{r^2(r_i, t)} \hat{r}.$$

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The dependence on r_i has disappeared!

$$\implies u(r_i, t) = u(r_{\text{any}}, t) \equiv a(t)$$

The scale factor $a(t)$

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- Therefore, we have shown that

$$r(r_i, t) = a(t) r_i.$$

$\Longrightarrow$ **The properties of the Universe are encoded in the "scale factor" $a(t)$.**

- Furthermore,

$$\rho(t) = \frac{M(r_i)}{\frac{4\pi}{3} r^3} = \frac{\rho_i}{a^3(t)} \qquad \text{and} \qquad \ddot{a} = -\frac{4\pi}{3} G \rho(t) a$$

Some remarks:

- No shell crossings occur as their radii satisfy $r = a(t) r_i$
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- Any observer inside the sphere $(r_i \ll R_{\max,i})$ will see all of its neighbors recede following Hubble's law:

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- Only somebody outside the sphere (or close to the edge) would know the difference.

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$$\dot{a}\left\{\ddot{a} + \frac{4\pi G\rho_i}{3a^2}\right\} = 0 \implies \frac{1}{2}\dot{a}^2 - \frac{4\pi G\rho_i}{3a} = E\,,$$

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For reasons that will become clear later in the course, we define $E \equiv -kc^2$

Summary: Equations

- We wanted to determine $r(r_i, t) \equiv$ radius at t of shell initially at r_i, and found $r(r_i, t) = a(t)r_i$, where

Friedmann Equations

$$\begin{cases} \ddot{a} = -\dfrac{4\pi}{3}G\rho(t)a \\ H^2 = \left(\dfrac{\dot{a}}{a}\right)^2 = \dfrac{8\pi}{3}G\rho - \dfrac{kc^2}{a^2} \end{cases}$$

and $\rho(t) \propto a^{-3}(t)$, or equivalently, $\rho(t) = \left[\dfrac{a(t_1)}{a(t)}\right]^3 \rho(t_1)$ for any t_1.

Units, conventions, and $a(t)$

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- A common convention (e.g., Ryden's) is to take $a(t_0) = 1$ m/notch (where t_0 = now).
- Many other books normalize a so that $k = \pm 1$ if $k \neq 0$ (i.e., k $k = \pm 1$/notch^2).

Solutions to the Friedmann equations

$$\dot{a}^2 = \frac{8\pi G}{3}\frac{\rho(t_1)a^3(t_1)}{a(t)} - kc^2$$

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- $k > 0$ $(E < 0)$: In this case $\dot{a} = 0$ when

$$a = a_{\max} \equiv \frac{8\pi G}{3}\frac{\rho(t_1)a^3(t_1)}{kc^2}$$

and then (because $\ddot{a} < 0$) the universe contracts to a Big Crunch. **Closed Universe.**

- $k = 0$ ($E = 0$):

$$H^2 = \frac{8\pi G}{3}\rho \implies \rho = \rho_c \equiv \frac{3H^2}{8\pi G}$$

where ρ_c is defined as the critical mass density, the mass density which puts the universe on the borderline between eternal expansion and eventual collapse.

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$$\implies \frac{2}{3}a^{3/2} = (\text{const})\, t + c' \implies a(t) \propto t^{2/3},$$

where we chose $t = 0$ such that $c' = 0$.

Age of a matter-dominated universe

Note that

$$a(t) \propto t^{2/3} \implies \frac{\dot{a}}{a} = H = \frac{2}{3t} \implies t = \frac{2}{3}H^{-1},$$

and today we have $H = H_0 = 67.7 \pm 0.5$ km-s^{-1}-Mpc^{-1}

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$$t_0 = \text{9.56 - 9.70 billion years.}$$

Since some stars are older than this, we can conclude that our universe is not (or hasn't always been) matter-dominated.

QUESTIONS?

8.286 Lecture 6
September 23, 2020

THE DYNAMICS OF NEWTONIAN COSMOLOGY, PART 2

(Corrected 9/25/20: on pp. 9–11, H was changed to H_i)

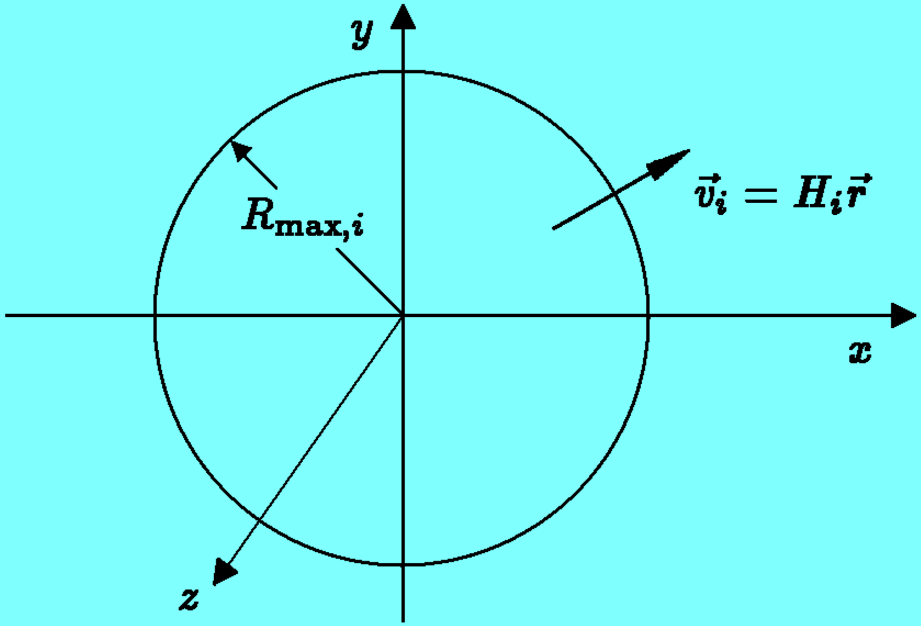
Announcements

- Problem Set 3 is due this Friday at 5 pm EDT.
- Quiz 1 will take place a week from Wednesday, on 9/30/2020. Full details about the quiz are on the class website, and are on the *Review Problems for Quiz 1.* One problem on the quiz will be taken verbatim, or at least almost verbatim, from the problem sets or from the starred problems on the *Review Problems*.
- Review session for the quiz, by Bruno Scheihing: Sunday, 9/27/2020, at 1:00 pm EDT. Same Zoom ID as our classes. Will be recorded.

Mathematical Model of a Uniformly Expanding Universe

- Desired properties: homogeneity, isotropy, and Hubble's law.
- The model should be finite, to avoid the conditional convergence problems discussed last time. At the end we will take the limit as the size approaches infinity.
- Newtonian dynamics: we choose the initial conditions, and then Newton's laws of motion will determine how it will evolve.
- To impose isotropy, we model the initial state as a solid sphere, of some radius $R_{\max,i}$.
- To impose homogeneity, we take the initial mass density to be constant, ρ_i. The matter is treated as a gas, that can thin as the universe expands. Think of a gas of very low speed particles, so the pressure is negligible.
- We take the initial velocities according to Hubble's law, with some initial expansion rate H_i

Mathematical Model of a Uniformly Expanding Universe



$$t_i \equiv \text{time of initial picture}$$

$$R_{\max,i} \equiv \text{initial maximum radius}$$

$$\rho_i \equiv \text{initial mass density}$$

$$\vec{v}_i = H_i\vec{r}\ .$$

Description of Evolution

- As the model universe evolves, the spherical symmetry will be preserved: each gas particle will continue on a radial trajectory, since there are no forces that might pull it tangentially.

- Spherical symmetry $\Longrightarrow$ all particles that start at the same initial radius will behave the same way. So, a particle that begins at radius r_i will be found at a later time t at some radius

$$r = r(r_i, t) \ .$$

- Our goal is to figure out what determines $r(r_i, t)$

- The only relevant force is gravity. Gravity and electromagnetism are the only (known) long-range forces. The universe appears to be electrically neutral, so long-range electric forces are not present.

Reminder: the Gravitational Field of a Shell of Matter

- For points outside the shell, the gravitational force is the same as if the total mass of the shell were concentrated at the center.
- For points inside the shell, the gravitational field is **zero.**
- Newton figured this out by integration. For us, Gauss's law makes it obvious.

Shell Crossings?

Can shells cross? I.e., can two shells that start at different r_i ever cross each other?

The answer is no, but we don't know that when we start.

But we do know that Hubble's law implies that any two shells are initially moving apart. Therefore there must be at least some interval before any shell crossings can happen.

We will write equations that are valid assuming no shell crossings.

These equations will be valid until any possible shell crossing.

If there was a shell crossing, these equations would have to show two shells becoming arbitrarily close.

We will find, however, that the equations imply uniform expansion, so no shell crossings ever happen in this system.

Equations of Motion

- Newtonian gravity of a shell:

  Inside: $\vec{g} = 0$.

  Outside: Same as point mass at center, with same M.

- $r(r_i, t) \equiv$ radius at t of shell initially at r_i.

- Let $M(r_i) \equiv$ mass inside r_i-shell $= \frac{4\pi}{3} r_i^3 \rho_i$ at all times.

- Pressure? When a gas with pressure $p > 0$ expands, it pushes on its surroundings and loses energy. Relativistically, energy = mass (times c^2). By assuming that $M(r_i)$ is constant, we are assuming that $p \simeq 0$.

Equations:

☆ For particles at radius r,

$$\vec{g} = -\frac{GM(r_i)}{r^2}\,\hat{r} \ ,$$

where

$$M(r_i) = \frac{4\pi}{3} r_i^3 \rho_i \ .$$

Since $\vec{g}$ is the acceleration,

$$\ddot{r} = -\frac{GM(r_i)}{r^2} = -\frac{4\pi}{3}\frac{G r_i^3 \rho_i}{r^2} \ , \ \text{where } r \equiv r(r_i, t),$$

where an overdot indicates a derivative with respect to t.

$$\ddot{r} = -\frac{GM(r_i)}{r^2} = -\frac{4\pi}{3}\frac{Gr_i^3\rho_i}{r^2}\ , \text{ where } r \equiv r(r_i, t),$$

☆ For a second order equation like this, the solution is uniquely determined if the initial value of r and $\dot{r}$ are specified:

$$r(r_i, t_i) = r_i\ ,$$

and, by the Hubble law initial condition $\vec{v}_i = H_i\vec{r}_i$,

$$\dot{r}(r_i, t_i) = H_i r_i\ .$$

Miraculous Scaling Relations

$$\ddot{r} = -\frac{4\pi}{3}\frac{Gr_i^3\rho_i}{r^2} \ , \quad r(r_i,t_i) = r_i \ , \quad \dot{r}(r_i,t_i) = H_i r_i \ .$$

☆ Suppose we define

$$u(r_i,t) \equiv \frac{r(r_i,t)}{r_i} \ .$$

Then

$$\ddot{u} = \frac{\ddot{r}}{r_i} = -\frac{4\pi}{3}\frac{G\rho_i}{u^2} \ .$$

There is no r_i-dependence. This "miracle" depended on gravity being a $1/r^2$ force.

$$\ddot{r} = -\frac{4\pi}{3}\frac{G r_i^3 \rho_i}{r^2} \ , \quad r(r_i, t_i) = r_i \ , \quad \dot{r}(r_i, t_i) = H_i r_i \ .$$

$$u(r_i, t) \equiv \frac{r(r_i, t)}{r_i} \quad \Longrightarrow \quad \ddot{u} = -\frac{4\pi}{3}\frac{G\rho_i}{u^2} \ .$$

What about the initial conditions for $u(r_i, t)$?

$$u(r_i, t_i) = \frac{r(r_i, t_i)}{r_i} = 1 \ , \quad \dot{u}(r_i, t_i) = \frac{\dot{r}(r_i, t_i)}{r_i} = H_i \ .$$

Since the differential equation and the intial conditions determine $u(r_i, t)$, it does not depend on r_i. We can rename it

$$u(r_i, t) \equiv a(t) \ , \qquad \text{so} \qquad r(r_i, t) = a(t)\, r_i \ .$$

This describes uniform expansion by a scale factor $a(t)$.

Time Dependence of $\rho(t)$

We know how the mass density depends on time, because we assumed that $M(r_i)$ — the total mass contained inside a shell of particles whose initial radius was r_i — does not change with time. The radius of the shell at time t is $a(t)r_i$. The mass density is just the mass divided by the volume,

$$\rho(t) = \frac{M(r_i)}{\frac{4\pi}{3} a^3(t) r_i^3} = \frac{\frac{4\pi}{3} r_i^3 \rho_i}{\frac{4\pi}{3} a^3(t) r_i^3} = \frac{\rho_i}{a^3(t)} \; .$$

So

$$\ddot{u} = -\frac{4\pi}{3}\frac{G\rho_i}{u^2} \quad \Longrightarrow \quad \ddot{a} = -\frac{4\pi}{3}\frac{G\rho_i}{a^2} \; .$$

$$\Longrightarrow \quad \ddot{a} = -\frac{4\pi}{3} G\rho(t)\, a(t) \; . \qquad \text{Friedmann equation.}$$

Nothing Depends on $R_{\text{max},i}$

- An observer living in this model universe would see uniform expansion all around herself, and would only be aware of the boundary at R_{max} if she was close enough to the boundary to see it.
- Thus, we can take the limit $R_{\text{max},i} \to \infty$ without doing anything, since nothing of interest depends on $R_{\text{max},i}$.

A Conservation Law

☆ The equation for $\ddot{a}$ has the same form as an equation for the motion of a particle with a time-independent potential energy function. So, there is a conservation law:

$$\ddot{a} = -\frac{4\pi}{3}\frac{G\rho_i}{a^2} \quad \Longrightarrow \quad \dot{a}\left\{\ddot{a} + \frac{4\pi}{3}\frac{G\rho_i}{a^2}\right\} = 0 \quad \Longrightarrow \quad \frac{dE}{dt} = 0 ,$$

where

$$E = \frac{1}{2}\dot{a}^2 - \frac{4\pi}{3}\frac{G\rho_i}{a} .$$

Summary: Equations

Want: $r(r_i, t) \equiv$ radius at t of shell initially at r_i

Find: $r(r_i, t) = a(t) r_i$, where

$$\text{Friedmann Equations} \begin{cases} \ddot{a} = -\dfrac{4\pi}{3} G\rho(t) a \\ H^2 = \left(\dfrac{\dot{a}}{a}\right)^2 = \dfrac{8\pi}{3} G\rho - \dfrac{kc^2}{a^2} \quad \text{(Friedmann Eq.)} \end{cases}$$

and

$$\rho(t) \propto \frac{1}{a^3(t)} \; , \; \text{or } \rho(t) = \left[\frac{a(t_1)}{a(t)}\right]^3 \rho(t_1) \text{ for any } t_1 .$$

☆ Note that t_i no longer plays any role. It does not appear on this slide!

The Return of the 'Notch'

- ☆ Definition: $r(r_i, t) = a(t) r_i$.
- ☆ In the previous derivation, r_i was the initial radius of some particle, measured in meters. But when we finished, r_i was being used only as a coordinate to label shells, where the shell corresponding to $r_i = 1$ had a radius of one meter only at time t_i.
- ☆ But t_i no longer appears, and will not be mentioned again! So, the connection between the numerical value of r_i and the length of a meter has disappeared from the formalism.
- ☆ Bottom line: r_i is the radial coordinate in a comoving coordinate system, measured in units that have no particular meaning. I will refer to the units of r_i as "notches," but you should be aware that the term is not standard.

Conventions for the Notch

Us: For us, the notch is an arbitrary unit that we use to mark off intervals on the comoving coordinate system. We are free to use a different definition every time we use the notch.

Ryden: $a(t_0) = 1$ (where t_0 = now). (In our language, Ryden's convention is $a(t_0) = 1$ m/notch.)

Many Other Books: if $k \neq 0$, then $k = \pm 1$.

In our language, this means $k = \pm 1/\text{notch}^2$. To see the units of k, recall that the Friedmann equation is

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{a^2} \; .$$

We will use $[x]$ to mean the units of x, and we will use T and L to denote the units of time and length, respectively. The units of the left-hand side are $1/T^2$, with the units of a canceling. So

$$[k] = \frac{1}{T^2}\left[\frac{a}{c}\right]^2 = \frac{1}{T^2}\left[\frac{L/\text{notch}}{L/T}\right]^2 = \frac{1}{\text{notch}^2} \; .$$

Types of Solutions

$$\dot{a}^2 = \frac{8\pi G}{3}\frac{\rho(t_1)a^3(t_1)}{a(t)} - kc^2 \quad (\text{for any } t_1) \; .$$

For intuition, remember that $k \propto -E$, where E is a measure of the energy of the system.

Types of Solutions:

1) $k < 0$ $(E > 0)$: unbound system. $\dot{a}^2 > (-kc^2) > 0$, so the universe expands forever. **Open Universe.**

2) $k > 0$ $(E < 0)$: bound system. $\dot{a}^2 \geq 0 \quad \Longrightarrow$

$$a_{\max} = \frac{8\pi G}{3}\frac{\rho(t_1)a^3(t_1)}{kc^2} \; .$$

Universe reaches maximum size and then contracts to a Big Crunch. **Closed Universe.**

3) $k = 0$ ($E = 0$): critical mass density.

$$H^2 = \frac{8\pi G}{3}\rho - \underbrace{\frac{kc^2}{a^2}}_{=0} \quad \Longrightarrow \quad \boxed{\rho \equiv \rho_c = \frac{3H^2}{8\pi G}\ .}$$

Flat Universe.

Summary: $\rho > \rho_c \Longleftrightarrow$ closed, $\rho < \rho_c \Longleftrightarrow$ open, $\rho = \rho_c \Longleftrightarrow$ flat.

Numerical value: For $H = 68$ km-s^{-1}-Mpc^{-1} (Planck 2015 plus other experiments),

$$\rho_c = 8.7 \times 10^{-27}\ \text{kg/m}^3 = 8.7 \times 10^{-30}\ \text{g/cm}^3$$

$$\approx 5 \text{ proton masses per m}^3.$$

Definition: $\Omega \equiv \dfrac{\rho}{\rho_c}$.

8.286 Lecture 7
September 28, 2020

THE DYNAMICS OF NEWTONIAN COSMOLOGY, PART 3

Announcements

- Quiz 1 will take place this Wednesday (9/30/2020). Full details about the quiz are on the class website, and are on the *Review Problems for Quiz 1.* One problem on the quiz will be taken verbatim, or at least almost verbatim, from the problem sets or from the starred problems on the *Review Problems*.

- Quiz Logistics: You may start Quiz 1 anytime from 11:05 am Wed to 11:05 am Thurs. The default time is 11:05 am Wed. If you want to take it at a different time, you should email me before midnight on Tues night, telling me the time that you want to start.

- ☆ The quiz will be contained in a PDF file, which I am planning to distribute by email. You will each be expected to spend up to 85 minutes working on it, and then you will upload your answers to Canvas as a PDF file. I won't place any precise time limit on scanning or photographing and uploading, because the time needed for that can vary. If you have questions about the meaning of the questions, I will be available on Zoom during the September 30 class time, and we will arrange for either Bruno or me to be available by email as much as possible during the other quiz times.

- ☆ If you have any special circumstances that might make this procedure difficult, or if you need a postponement beyond the 24-hour window, please let me (guth@ctp.mit.edu) know.

- ☆ The recording of the review session for the quiz, by Bruno Scheihing, is on the website.

- ★ Special office hours this week:

  Bruno: today (Mon 9/28): 6-7 pm.
  Me: tomorrow (Tues 9/29): 5-6 pm.
  No office hours Wed or Thurs.

- ★ Since people will be taking the quiz at different times, you will be on your honor, before you take the quiz, not to discuss it with anyone who has seen it.

Summary: Equations

Want: $r(r_i, t) \equiv$ radius at t of shell initially at r_i

Find: $r(r_i, t) = a(t) r_i$, where

$$\text{Friedmann Equations} \begin{cases} \ddot{a} = -\dfrac{4\pi}{3} G \rho(t) a \\ H^2 = \left(\dfrac{\dot{a}}{a}\right)^2 = \dfrac{8\pi}{3} G\rho - \dfrac{kc^2}{a^2} \quad \text{(Friedmann Eq.)} \end{cases}$$

and

$$\rho(t) \propto \frac{1}{a^3(t)} \ , \ \text{or } \rho(t) = \left[\frac{a(t_1)}{a(t)}\right]^3 \rho(t_1) \text{ for any } t_1 .$$

☆ Note that t_i no longer plays any role. It does not appear on this slide!

Reminder from Lecture 6:

Types of Solutions

$$\dot{a}^2 = \frac{8\pi G}{3} \frac{\rho(t_1) a^3(t_1)}{a(t)} - kc^2 \quad \text{(for any } t_1\text{)} .$$

For intuition, remember that $k = -2E/c^2$, where E is a measure of the energy of the system.

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1) $k < 0$ $(E > 0)$: unbound system. $\dot{a}^2 > (-kc^2) > 0$, so the universe expands forever. **Open Universe.**

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Universe reaches maximum size and then contracts to a Big Crunch. **Closed Universe.**

Reminder from Lecture 6:

3) $k = 0$ $(E = 0)$: critical mass density.

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$$\approx 5 \text{ proton masses per m}^3.$$

Definition: $\Omega \equiv \dfrac{\rho}{\rho_c}$.

Evolution of a Flat Universe

If $k = 0$, then

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho = \frac{\text{const}}{a^3} \quad \Longrightarrow \quad \frac{da}{dt} = \frac{\text{const}}{a^{1/2}}$$

$$\Longrightarrow \quad a^{1/2}\, da = \text{const}\, dt \quad \Longrightarrow \quad \frac{2}{3} a^{3/2} = (\text{const})t + c' \ .$$

Choose the zero of time to make $c' = 0$, and then

$$a(t) \propto t^{2/3} \ .$$

Age of a Flat Matter-Dominated Universe

$$a(t) \propto t^{2/3} \quad \Longrightarrow \quad H = \frac{\dot{a}}{a} = \frac{2}{3t} \quad \Longrightarrow$$

$$t = \frac{2}{3} H^{-1}$$

For $H = 67.7 \pm 0.5$ km-s^{-1}-Mpc^{-1}, age = 9.56 – 9.70 billion years — but stars are older. Conclusion: our universe is nearly flat, but not matter-dominated.

Big Bang Singularity

- $a(0) = 0$, so the mass density ρ at $t = 0$ is infinite.
- This instant of infinite mass density is called a singularity.
- But, as we extrapolate backwards to early t, ρ becomes higher than any mass density that we know about.
- Hence, there is no reason to trust the model back to $t = 0$.

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- Quantum gravity? The singularity is a feature of the *classical* theory, but might be avoided by a quantum gravity treatment — but we don't know.

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- ☆ Hence, there is no reason to trust the model back to $t = 0$.
- ☆ Conclusion: the singularity is a feature of the model, but not necessarily the real universe.
- ☆ Quantum gravity? The singularity is a feature of the *classical* theory, but might be avoided by a quantum gravity treatment — but we don't know.
- ☆ In eternal inflation models, to be discussed near the end of the term, the event 13.8 billion years ago was not a singularity, but rather the decay of the repulsive-gravity material that drove the inflation. There might still have been a singularity deeper in the past.

Horizon Distance

Definition: the horizon distance is the present distance of the furthest particles from which light has had time to reach us.

To find it, use comoving coordinates. The coordinate velocity of light is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{c}{a(t)} \ ,$$

so the maximum coordinate distance that light could have traveled by time t (starting at $t = 0$) is

$$\ell_{c,\mathrm{horizon}}(t) = \int_0^t \frac{c}{a(t')}\, \mathrm{d}t' \ .$$

$$\ell_{c,\mathrm{horizon}}(t) = \int_0^t \frac{c}{a(t')}\,\mathrm{d}t' \ .$$

The horizon distance is the maximum *physical* distance that light could have traveled, so

$$\boxed{\ell_{\mathrm{phys,horizon}}(t) = a(t)\int_0^t \frac{c}{a(t')}\mathrm{d}t' \ .}$$

For a flat, matter-dominated universe, $a(t) \propto t^{2/3}$, so

$$\boxed{\ell_{\mathrm{phys,horizon}}(t) = 3ct = 2cH^{-1} \ ,}$$

since $t = \frac{2}{3}H^{-1}$.

Equations for a Matter-Dominated Universe

("Matter-dominated" = dominated by nonrelativistic matter.)

Friedmann equations:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{a^2} \,,$$

$$\ddot{a} = -\frac{4\pi}{3} G\rho(t) a \,.$$

Matter conservation:

$$\rho(t) \propto \frac{1}{a^3(t)} \,, \text{ or } \rho(t) = \left[\frac{a(t_1)}{a(t)}\right]^3 \rho(t_1) \text{ for any } t_1 .$$

Any two of the above equations can allow us to find the third.

Evolution of a Closed Universe

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{a^2} , \quad \rho(t)a^3(t) = \text{constant} , \; k > 0 .$$

Recall $[a(t)] = \text{meter/notch}$, $[k] = 1/\text{notch}^2$.

Define new variables:

$$\tilde{a}(t) \equiv \frac{a(t)}{\sqrt{k}} , \qquad \tilde{t} \equiv ct \qquad \text{(both with units of distance)}$$

Multiplying Friedmann eq by $a^2/(kc^2)$:

$$\frac{1}{kc^2}\left(\frac{da}{dt}\right)^2 = \frac{8\pi}{3}\frac{G\rho a^2}{kc^2} - 1 .$$

Recalling

$$\tilde{a}(t) \equiv \frac{a(t)}{\sqrt{k}} \ , \qquad \tilde{t} \equiv ct,$$

we find

$$\frac{1}{kc^2}\left(\frac{da}{dt}\right)^2 = \frac{8\pi}{3}\frac{G\rho a^2}{kc^2} - 1$$

$$= \frac{8\pi}{3}\frac{G\rho a^3}{k^{3/2}c^2}\frac{\sqrt{k}}{a} - 1 \ .$$

Rewrite as

$$\left(\frac{d\tilde{a}}{d\tilde{t}}\right)^2 = \frac{2\alpha}{\tilde{a}} - 1 \ ,$$

where

$$\alpha \equiv \frac{4\pi}{3}\frac{G\rho\tilde{a}^3}{c^2} \ .$$

$[\alpha]$ = meter. α is constant, since ρa^3 is constant.

$$\left(\frac{d\tilde{a}}{d\tilde{t}}\right)^2 = \frac{2\alpha}{\tilde{a}} - 1 \quad \Longrightarrow \quad d\tilde{t} = \frac{\tilde{a}\, d\tilde{a}}{\sqrt{2\alpha\tilde{a} - \tilde{a}^2}} \ .$$

Then

$$\tilde{t}_f = \int_0^{\tilde{t}_f} d\tilde{t} = \int_0^{\tilde{a}_f} \frac{\tilde{a}\, d\tilde{a}}{\sqrt{2\alpha\tilde{a} - \tilde{a}^2}} \ ,$$

where $\tilde{t}_f$ is an arbitrary choice for a "final time" for the calculation, and $\tilde{a}_f$ is the value of $\tilde{a}$ at time $\tilde{t}_f$.

To carry out the integral, we first complete the square:

$$\tilde{t}_f = \int_0^{\tilde{a}_f} \frac{\tilde{a}\, d\tilde{a}}{\sqrt{\alpha^2 - (\tilde{a} - \alpha)^2}} \ .$$

Now simplify by defining $x \equiv \tilde{a} - \alpha$, so

$$\tilde{t}_f = \int_{-\alpha}^{\tilde{a}_f - \alpha} \frac{(x + \alpha)\, dx}{\sqrt{\alpha^2 - x^2}} \ .$$

$$\tilde{t}_f = \int_{-\alpha}^{\tilde{a}_f - \alpha} \frac{(x+\alpha)\,dx}{\sqrt{\alpha^2 - x^2}} \ .$$

To simplify $\alpha^2 - x^2$, define θ so that $x = -\alpha\cos\theta$.

(Choice of the minus sign simplifies the final answer. Recall that x represents the scale factor, and θ will be replacing x. The minus sign leads to $\mathrm{d}x/\mathrm{d}\theta = \alpha\sin\theta$, which is positive for small positive θ, so both will be growing at the start of the universe.)

Substituting,

$$\mathrm{d}x = \alpha\sin\theta\,\mathrm{d}\theta \ .$$

$$\sqrt{\alpha^2 - x^2} = \alpha\sqrt{1-\cos^2\theta} = \alpha\sin\theta \ .$$

Then

$$\tilde{t}_f = \alpha\int_0^{\theta_f} (1-\cos\theta)d\theta = \alpha(\theta_f - \sin\theta_f) \ .$$

This equation relates t_f to θ_f, but we really want to relate the scale factor and time. But θ_f is related to the scale factor, if we trace back the definitions: $x_f = -\alpha\cos\theta_f = \tilde{a}_f - \alpha$, so

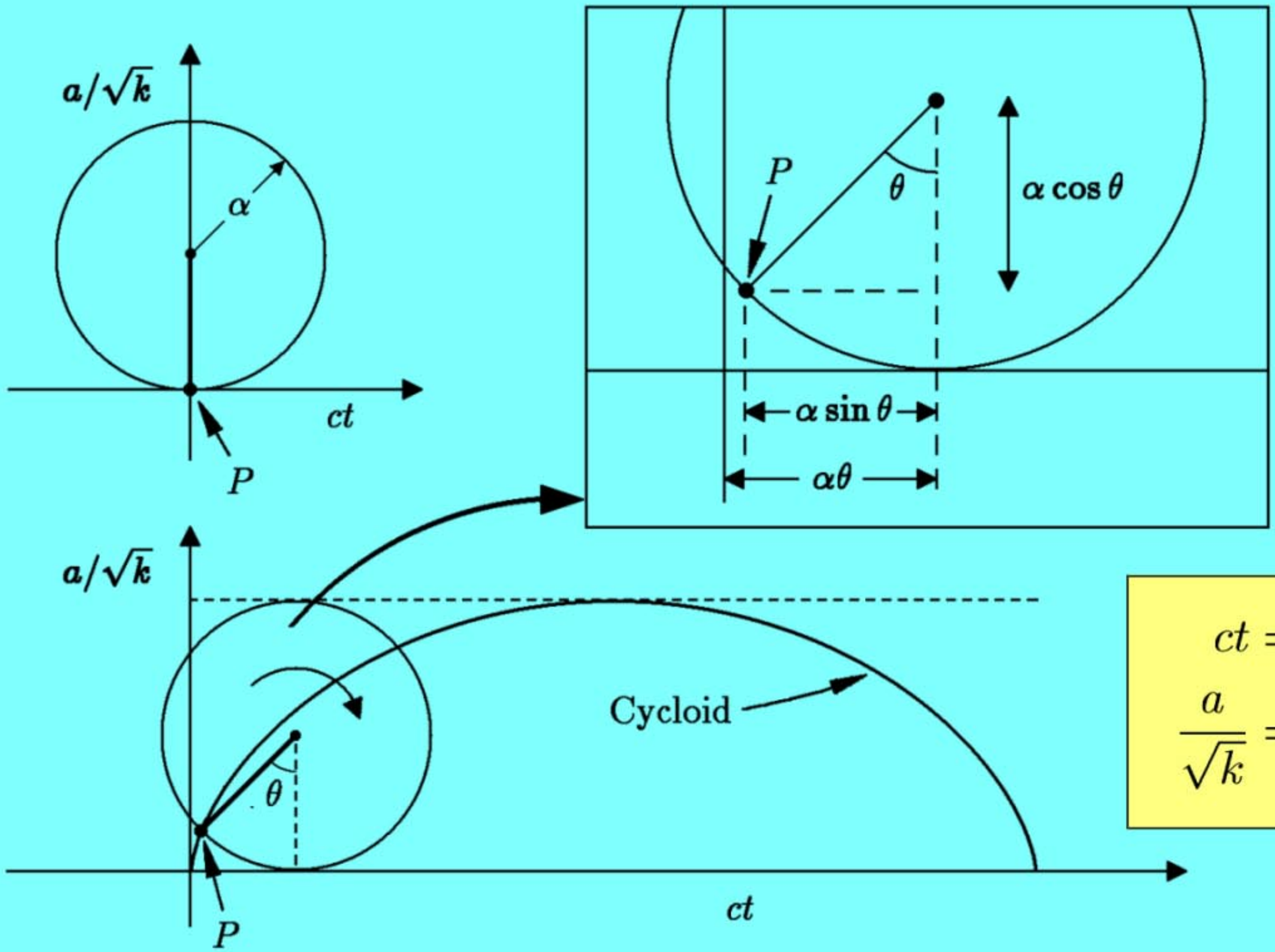
$$\tilde{a}_f = \alpha(1-\cos\theta_f) \ .$$

Parametric Solution for the Evolution of a Closed Matter-Dominated Universe

$$ct = \alpha(\theta - \sin\theta) \; ,$$

$$\frac{a}{\sqrt{k}} = \alpha(1 - \cos\theta) \; .$$

The angle θ is sometimes called the "development angle," because it describes the stage of development of the universe. The universe begins at $\theta = 0$, reaches its maximum expansion at $\theta = \pi$, and then is terminated by a big crunch at $\theta = 2\pi$.



$$ct = \alpha(\theta - \sin\theta) \ ,$$

$$\frac{a}{\sqrt{k}} = \alpha(1 - \cos\theta) \ .$$

Duration and Maximum Size

$$\frac{a}{\sqrt{k}} = \alpha(1 - \cos\theta) \quad \Longrightarrow \quad \boxed{\frac{a_{\rm max}}{\sqrt{k}} = 2\alpha\ ,}$$

where

$$\alpha = \frac{4\pi}{3}\frac{G\rho a^3}{k^{3/2}c^2}\ .$$

Similarly, $ct = \alpha(\theta - \sin\theta)$ implies that the total duration of the universe, from big bang to big crunch is

$$\boxed{t_{\rm total} = \frac{2\pi\alpha}{c} = \frac{\pi a_{\rm max}}{c\sqrt{k}}\ .}$$

Age of a Closed Matter-Dominated Universe

$$ct = \alpha(\theta - \sin\theta)$$

gives the age in terms of α and θ. But astronomers measure H and Ω. So we would like to express the age in terms of H and Ω.

Start with ρ:

$$\rho = \Omega\rho_c = \left(\frac{3H^2}{8\pi G}\right)\Omega \; .$$

The first-order Friedmann equation can then be rewritten as

$$H^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{a^2} \quad \Longrightarrow \quad H^2 = H^2\Omega - \frac{kc^2}{a^2} \; ,$$

so

$$\tilde{a} = \frac{a}{\sqrt{k}} = \frac{c}{|H|\sqrt{\Omega - 1}} \; .$$

$$\tilde{a} = \frac{a}{\sqrt{k}} - \frac{c}{|H|\sqrt{\Omega - 1}} \ .$$

In taking the square root, recall that $a > 0$, $k > 0$, while H changes sign — it is positive during the expansion phase, and negative during the collapse phase. So we need $|H|$, not just H, for the equation to be valid. Then

$$\alpha = \frac{4\pi}{3} \frac{G\rho\tilde{a}^3}{c^2} = \frac{c}{2|H|} \frac{\Omega}{(\Omega - 1)^{3/2}} \ .$$

To find age, we need to express α and θ in terms of H and Ω. To express θ, use expression for $\tilde{a}$ above, and 2nd parametric equation

$$\tilde{a} = \frac{a}{\sqrt{k}} = \alpha(1 - \cos\theta) \ .$$

Then

$$\frac{c}{|H|\sqrt{\Omega - 1}} = \frac{c}{2|H|} \frac{\Omega}{(\Omega - 1)^{3/2}} (1 - \cos\theta) \ ,$$

Then

$$\frac{c}{|H|\sqrt{\Omega-1}} = \frac{c}{2|H|}\,\frac{\Omega}{(\Omega-1)^{3/2}}(1-\cos\theta)\ ,$$

which can be solved for either $\cos\theta$ or for Ω:

$$\boxed{\cos\theta = \frac{2-\Omega}{\Omega}\ , \quad \Omega = \frac{2}{1+\cos\theta}\ .}$$

Evolution of Ω: At $t = 0$, $\theta = 0$, so $\Omega = 1$. Any (matter-dominated) closed universe begins with $\Omega = 1$.

As θ increases from 0 to π, Ω grows from 1 to infinity. At $\theta = \pi$, a reaches its maximum size, and $H = 0$. So $\rho_c = 0$ and $\Omega = \infty$.

During the collapse phase, $\pi < \theta < 2\pi$, Ω falls from ∞ to 1.

What about $\sin\theta$?

$$\sin\theta = \pm\sqrt{1-\cos^2\theta} = \pm\frac{2\sqrt{\Omega-1}}{\Omega} \ .$$

$\sin\theta$ is positive during the expansion phase (while $0 < \theta < \pi$), and negative during the collapse phase (while $\pi < \theta < 2\pi$).

Evolution of a Closed Universe

$$ct = \alpha(\theta - \sin\theta) \ ,$$

$$\frac{a}{\sqrt{k}} = \alpha(1 - \cos\theta) \ .$$

Evolution of a Closed Universe

$$ct = \alpha(\theta - \sin\theta) \ ,$$

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$$t = \frac{\Omega}{2|H|(\Omega - 1)^{3/2}} \left\{ \arcsin\left(\pm \frac{2\sqrt{\Omega - 1}}{\Omega} \right) \mp \frac{2\sqrt{\Omega - 1}}{\Omega} \right\} \ .$$

$$t = \frac{\Omega}{2|H|(\Omega-1)^{3/2}}\left\{\arcsin\left(\pm\frac{2\sqrt{\Omega-1}}{\Omega}\right) \mp \frac{2\sqrt{\Omega-1}}{\Omega}\right\} .$$

Quadrant	Phase	Ω	Sign Choice	$\sin^{-1}()$
1	Expanding	1 to 2	Upper	0 to $\frac{\pi}{2}$
2	Expanding	2 to ∞	Upper	$\frac{\pi}{2}$ to π
3	Contracting	∞ to 2	Lower	π to $\frac{3\pi}{2}$
4	Contracting	2 to 1	Lower	$\frac{3\pi}{2}$ to 2π

8.286 Class 9
October 5, 2020

THE DYNAMICS OF NEWTONIAN COSMOLOGY, PART 4

Announcements

- ☆ Quiz 1 came off smoothly, and the class did extremely well. Class average was 92.3, which is amazing. There were 4 perfect papers, 3 99's, 1 98, 2 97's, and 2 96's. I should have your grades, solutions, and a grade histogram with estimated letter-grade cuts all posted this afternoon.

- ☆ One significant cause for delay was Problem 1(e), "Why is the night sky not uniformly bright?". Bruno and I exchanged many emails about this one. The answer we intended was (iii), referring to

  (C) The universe is not infinitely old.

  (E) The cosmological redshift makes stars look dimmer and dimmer as they are further away from us.

  Actually, we view (E) as the most important factor for our universe. The surface brightness of a star at redshift z falls off as $1/(1+z)^4$. (You've derived the pieces: total radiation flux $\propto 1/(1+z)^2$ – one power from loss of energy of each photon, and one power from rate of arrival of photons. In addition, angular size $\theta \propto (1+z)$, so solid angle $\propto (1+z)^2$.) So stars at high z contribute little to the night sky brightness.

For comparison, the finite age: The finite age means that we don't see any stars further than the horizon distance — the present distance of the most distant objects for which light has had time to reach us. What is the redshift at the horizon?

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Ans: infinite.

For comparison, the finite age: The finite age means that we don't see any stars further than the horizon distance — the present distance of the most distant objects for which light has had time to reach us. What is the redshift at the horizon?

Ans: infinite. Time of emission $t_e = 0$, so $a(t_e) = 0$, and $1 + z = a(t_0)/a(t_e) = \infty$.

For comparison, the finite age: The finite age means that we don't see any stars further than the horizon distance — the present distance of the most distant objects for which light has had time to reach us. What is the redshift at the horizon?

Ans: infinite. Time of emission $t_e = 0$, so $a(t_e) = 0$, and $1 + z = a(t_0)/a(t_e) = \infty$.

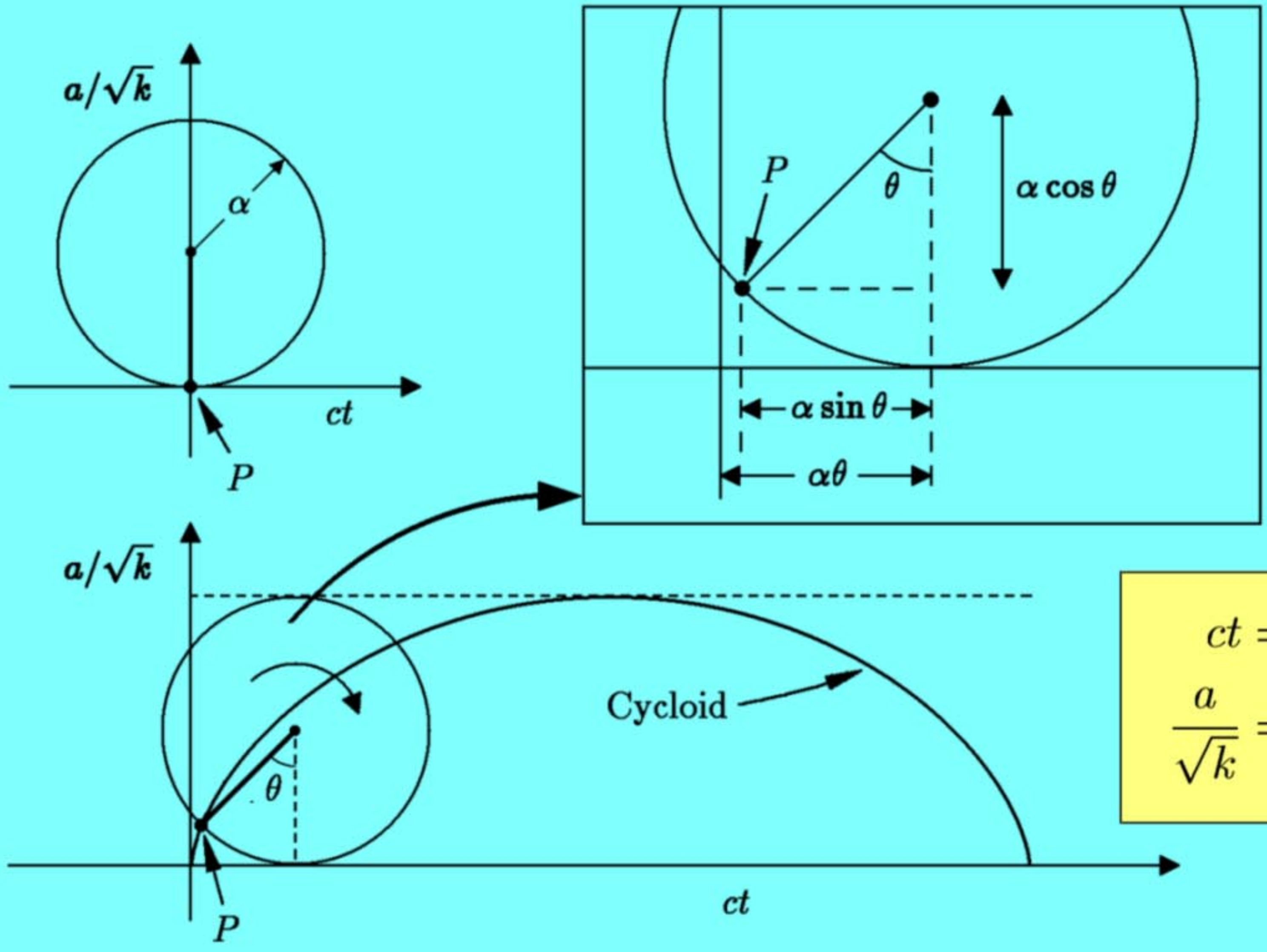
We finally decided that, depending on how one interpreted the question, any of the answers can arguably be true, so in the end we decided to give credit for any answer.

Parametric Solution for the Evolution of a Closed Matter-Dominated Universe

$$ct = \alpha(\theta - \sin\theta) \ ,$$

$$\frac{a}{\sqrt{k}} = \alpha(1 - \cos\theta) \ .$$

The angle θ is sometimes called the "development angle," because it describes the stage of development of the universe. The universe begins at $\theta = 0$, reaches its maximum expansion at $\theta = \pi$, and then is terminated by a big crunch at $\theta = 2\pi$.



$$ct = \alpha(\theta - \sin\theta) \; ,$$

$$\frac{a}{\sqrt{k}} = \alpha(1 - \cos\theta) \; .$$

Duration and Maximum Size

$$\frac{a}{\sqrt{k}} = \alpha(1 - \cos\theta) \quad \Longrightarrow \quad \boxed{\frac{a_{\rm max}}{\sqrt{k}} = 2\alpha\ ,}$$

where

$$\alpha = \frac{4\pi}{3}\frac{G\rho a^3}{k^{3/2}c^2}\ .$$

Similarly, $ct = \alpha(\theta - \sin\theta)$ implies that the total duration of the universe, from big bang to big crunch is

$$\boxed{t_{\rm total} = \frac{2\pi\alpha}{c} = \frac{\pi a_{\rm max}}{c\sqrt{k}}\ .}$$

Age of a Closed Matter-Dominated Universe

$$ct = \alpha(\theta - \sin\theta)$$

gives the age in terms of α and θ. But astronomers measure H and Ω. So we would like to express the age in terms of H and Ω.

Start with ρ:

$$\rho = \Omega\rho_c = \left(\frac{3H^2}{8\pi G}\right)\Omega\ .$$

The first-order Friedmann equation can then be rewritten as

$$H^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{a^2} \quad \Longrightarrow \quad H^2 = H^2\Omega - \frac{kc^2}{a^2}\ ,$$

so

$$\tilde{a} = \frac{a}{\sqrt{k}} = \frac{c}{|H|\sqrt{\Omega - 1}}\ .$$

$$\tilde{a} = \frac{a}{\sqrt{k}} = \frac{c}{|H|\sqrt{\Omega - 1}} \ .$$

In taking the square root, recall that $a > 0$, $k > 0$, while H changes sign — it is positive during the expansion phase, and negative during the collapse phase. So we need $|H|$, not just H, for the equation to be valid. Then

$$\alpha = \frac{4\pi}{3} \frac{G\rho\tilde{a}^3}{c^2} = \frac{c}{2|H|} \frac{\Omega}{(\Omega - 1)^{3/2}} \ .$$

To find age, we need to express α and θ in terms of H and Ω. To express θ, use expression for $\tilde{a}$ above, and 2nd parametric equation

$$\tilde{a} = \frac{a}{\sqrt{k}} = \alpha(1 - \cos\theta) \ .$$

Then

$$\frac{c}{|H|\sqrt{\Omega - 1}} = \frac{c}{2|H|} \frac{\Omega}{(\Omega - 1)^{3/2}} (1 - \cos\theta) \ ,$$

Then

$$\frac{c}{|H|\sqrt{\Omega - 1}} = \frac{c}{2|H|} \frac{\Omega}{(\Omega - 1)^{3/2}}(1 - \cos\theta) \ ,$$

which can be solved for either $\cos\theta$ or for Ω:

$$\boxed{\cos\theta = \frac{2 - \Omega}{\Omega} \ , \quad \Omega = \frac{2}{1 + \cos\theta} \ .}$$

Evolution of Ω: At $t = 0$, $\theta = 0$, so $\Omega = 1$. Any (matter-dominated) closed universe begins with $\Omega = 1$.

As θ increases from 0 to π, Ω grows from 1 to infinity. At $\theta = \pi$, a reaches its maximum size, and $H = 0$. So $\rho_c = 0$ and $\Omega = \infty$.

During the collapse phase, $\pi < \theta < 2\pi$, Ω falls from ∞ to 1.

What about $\sin\theta$?

$$\sin\theta = \pm\sqrt{1-\cos^2\theta} = \pm\frac{2\sqrt{\Omega-1}}{\Omega} \ .$$

$\sin\theta$ is positive during the expansion phase (while $0 < \theta < \pi$), and negative during the collapse phase (while $\pi < \theta < 2\pi$).

Evolution of a Closed Universe

$$ct = \alpha(\theta - \sin\theta) \ ,$$

$$\frac{a}{\sqrt{k}} = \alpha(1 - \cos\theta) \ .$$

Evolution of a Closed Universe

$$ct = \alpha(\theta - \sin\theta) \ ,$$

$$\frac{a}{\sqrt{k}} = \alpha(1 - \cos\theta) \ .$$

$$t = \frac{\Omega}{2|H|(\Omega - 1)^{3/2}} \left\{ \sin^{-1}\left(\pm \frac{2\sqrt{\Omega - 1}}{\Omega} \right) \mp \frac{2\sqrt{\Omega - 1}}{\Omega} \right\} \ .$$

$$t = \frac{\Omega}{2|H|(\Omega-1)^{3/2}} \left\{ \sin^{-1}\left(\pm \frac{2\sqrt{\Omega-1}}{\Omega} \right) \mp \frac{2\sqrt{\Omega-1}}{\Omega} \right\} .$$

Quadrant	θ	Phase	Ω	Sign Choice
1	0 to $\frac{\pi}{2}$	Expanding	1 to 2	Upper
2	$\frac{\pi}{2}$ to π	Expanding	2 to ∞	Upper
3	π to $\frac{3\pi}{2}$	Contracting	∞ to 2	Lower
4	$\frac{3\pi}{2}$ to 2π	Contracting	2 to 1	Lower

Evolution of Open Matter-Dominated Universes

$$ct = \alpha(\sinh\theta - \theta) \; ,$$

$$\frac{a}{\sqrt{\kappa}} = \alpha(\cosh\theta - 1) \; .$$

where $\kappa = -k$, and

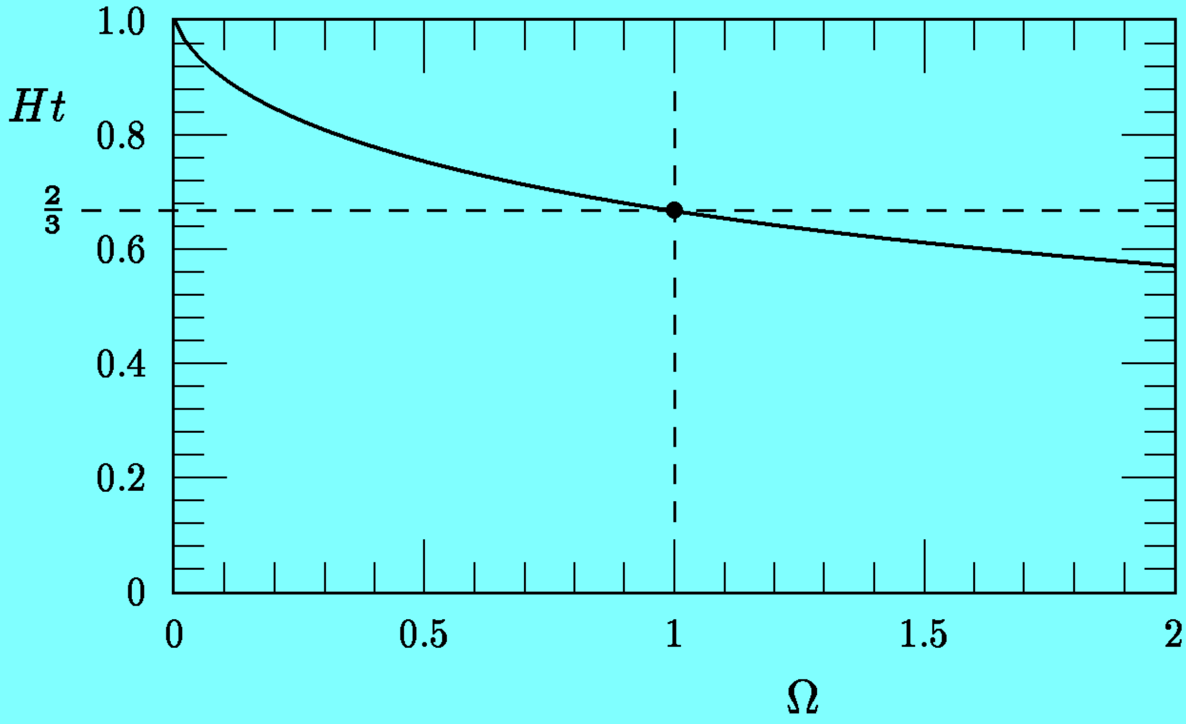
$$\tilde{a}(t) = \frac{a(t)}{\sqrt{\kappa}} \; , \qquad \alpha \equiv \frac{4\pi}{3}\frac{G\rho\tilde{a}^3}{c^2} \; .$$

θ evolves from 0 to ∞.

Age for Open, Flat, and Closed Matter-Dominated Universes

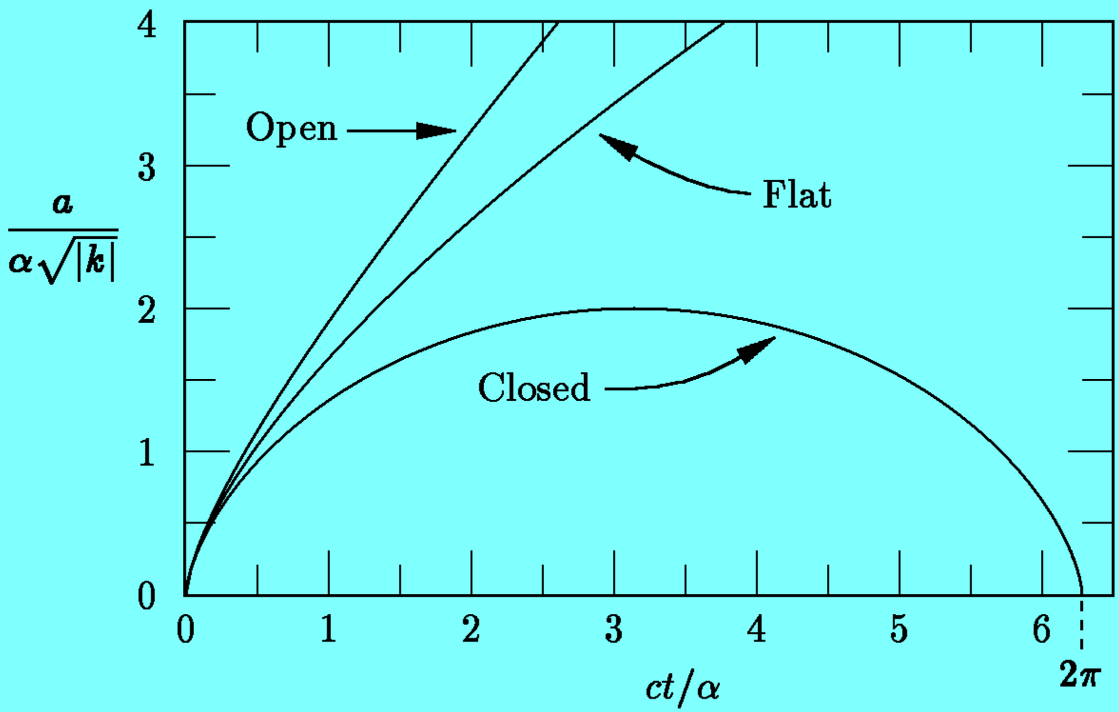
$$|H|\,t = \begin{cases} \dfrac{\Omega}{2(1-\Omega)^{3/2}}\left[\dfrac{2\sqrt{1-\Omega}}{\Omega} - \sinh^{-1}\left(\dfrac{2\sqrt{1-\Omega}}{\Omega}\right)\right] & \text{if } \Omega < 1 \\ 2/3 & \text{if } \Omega = 1 \\ \dfrac{\Omega}{2(\Omega-1)^{3/2}}\left[\sin^{-1}\left(\pm\dfrac{2\sqrt{\Omega-1}}{\Omega}\right) \mp \dfrac{2\sqrt{\Omega-1}}{\Omega}\right] & \text{if } \Omega > 1 \end{cases}$$

The Age of a Matter-Dominated Universe



The age of a matter-dominated universe, expressed as Ht (where t is the age and H is the Hubble expansion rate), as a function of Ω. The curve describes all three cases of an open ($\Omega < 1$), flat ($\Omega = 1$), and closed ($\Omega > 1$) universe.

Evolution of a Matter-Dominated Universe



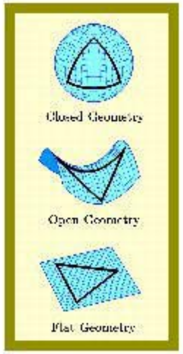
The evolution of a matter-dominated universe. Closed and open universes can be characterized by a single parameter α. With the scalings shown on the axis labels, the evolution of a matter-dominated universe is described in all cases by the curves shown in this graph.

8.286 Class 9, Part 2
October 5, 2020

INTRODUCTION TO NON-EUCLIDEAN SPACES

Ants on a Pringle

Mustafa Amin
18.10.2011

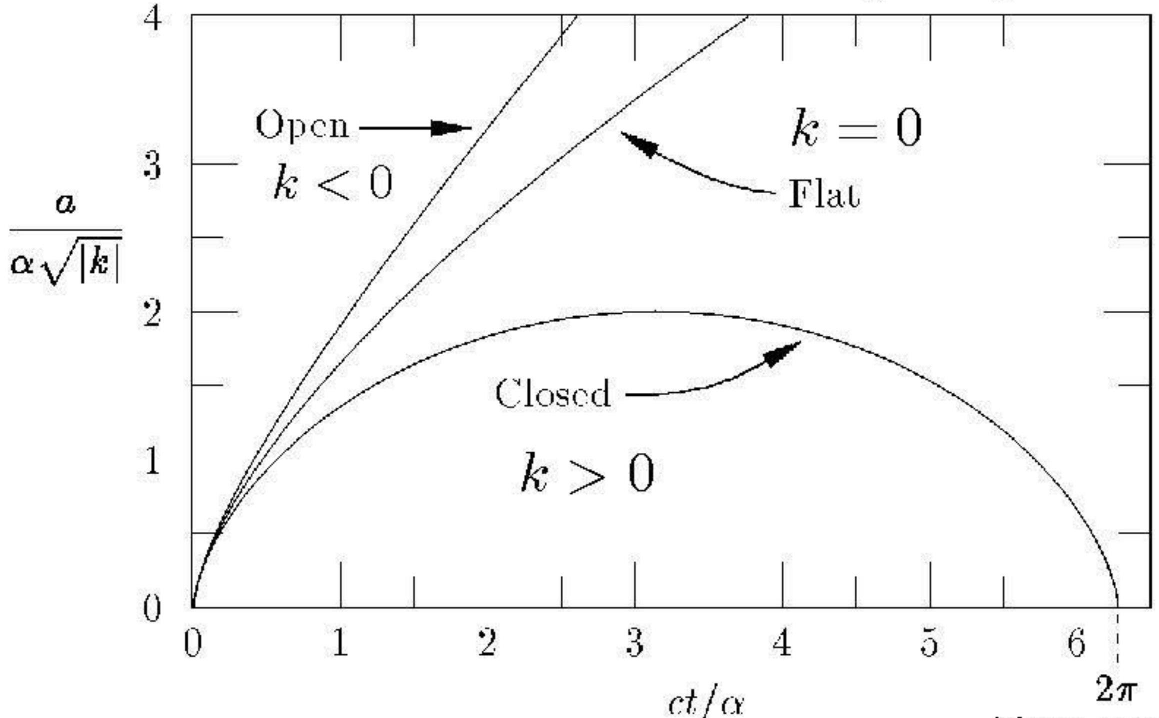


Slide created by Mustafa Amin

Recap

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}$$

Image: courtesy Alan Guth



Note: matter dominated universes only!

k ?

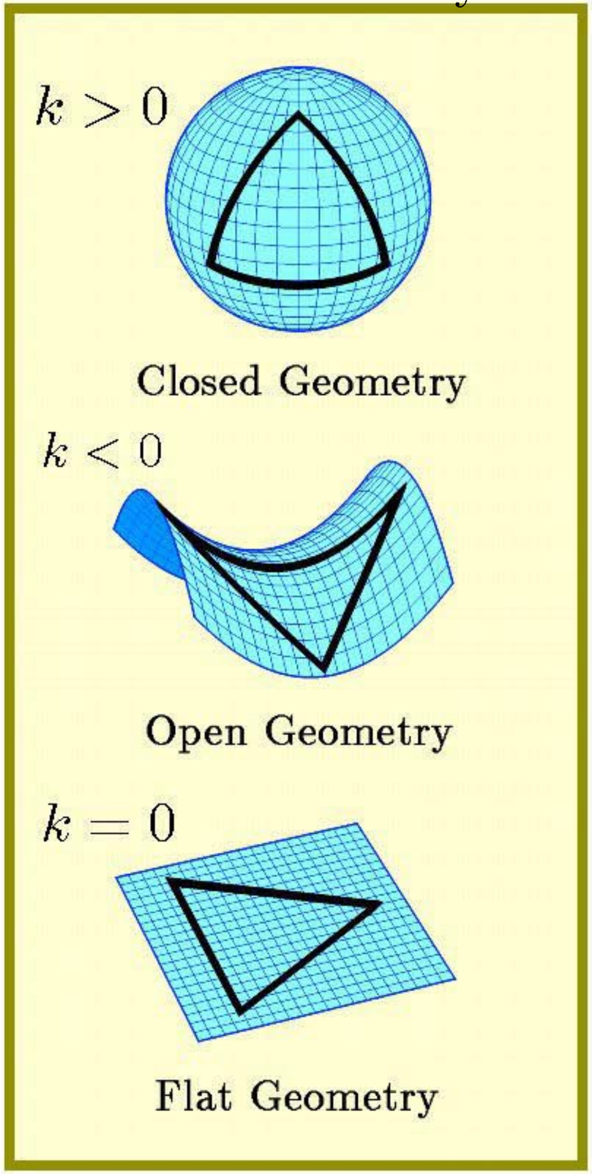
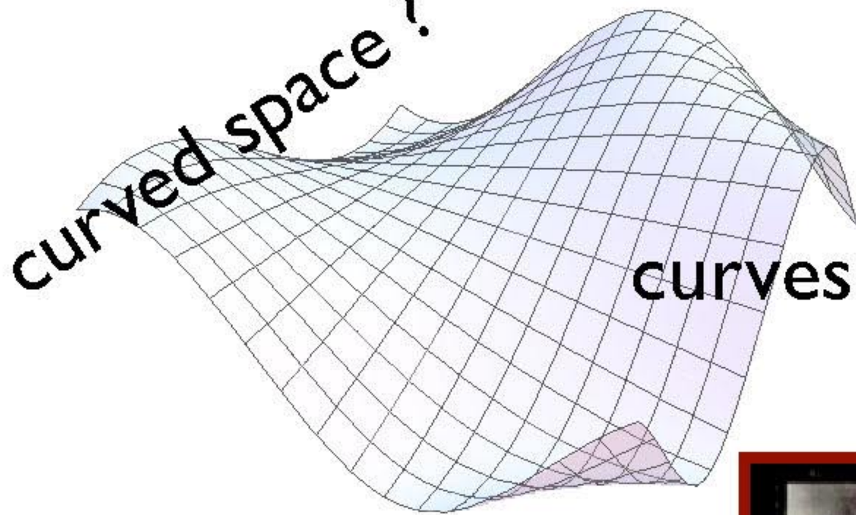


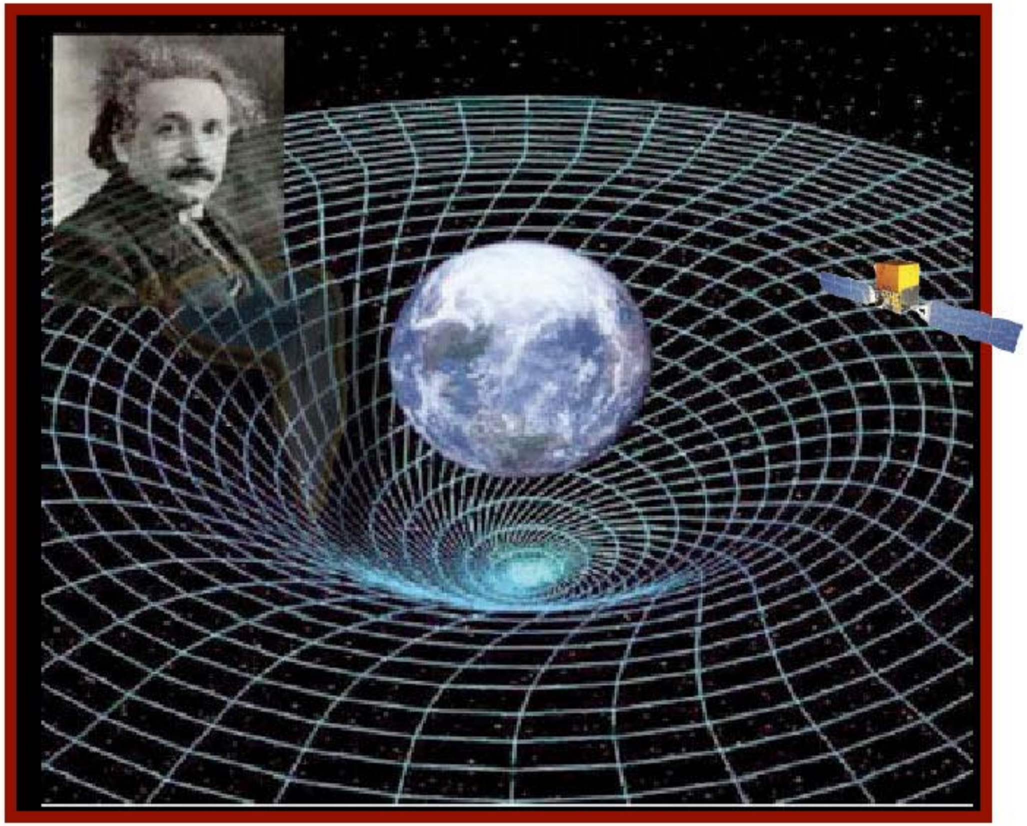
Image: courtesy Alan Guth



curved space ?

curves with respect to what ?

curved spacetime ?



non-Euclidean geometry

Note on image: I have added the "Dali" clock and Fermi satellite images to the original image created for the Gravity Probe B collaboration

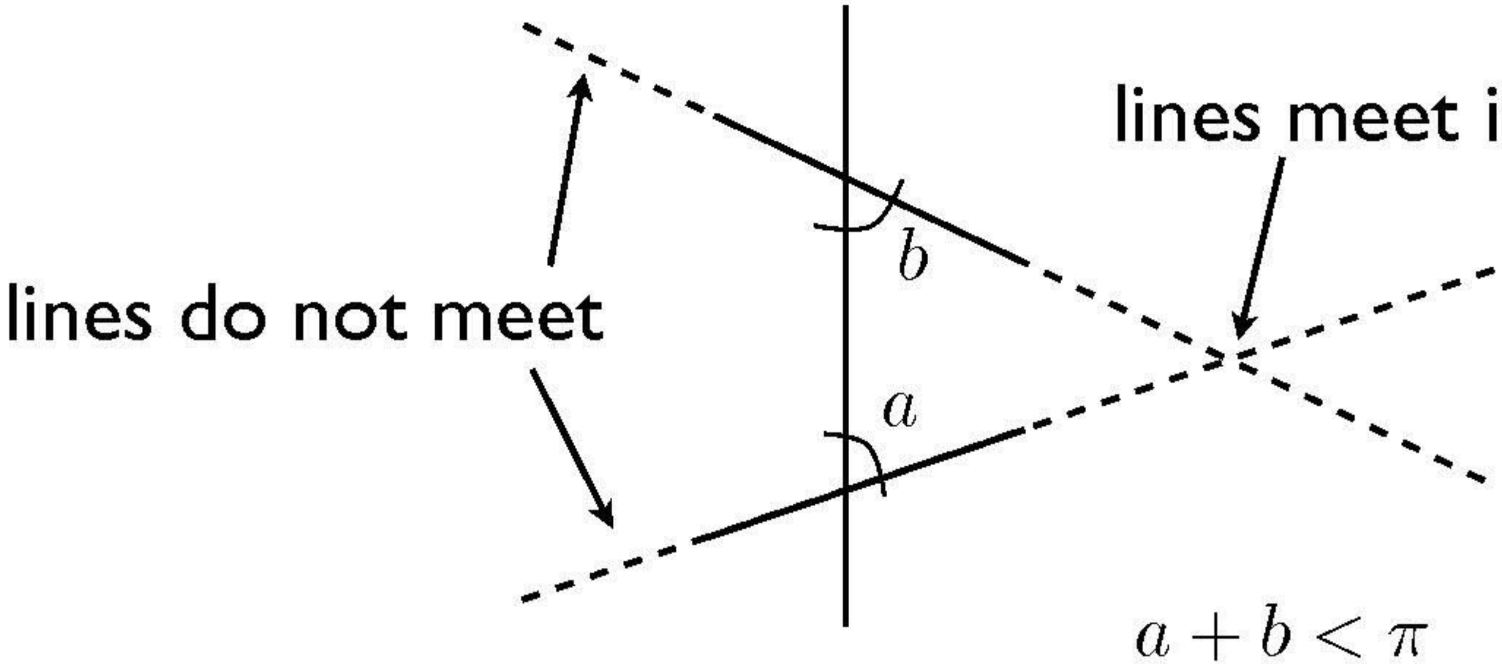
Euclid's *Postulates*

1. A straight line segment can be drawn joining any two points.
2. Any straight line segment can be extended indefinitely in a straight line.
3. Given a straight line segment , a circle can be drawn having the segment as radius and one endpoint as center.
4. All right angles are congruent.
5. If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines if produced indefinitely meet on that side on which the angles are less than two right angles

Corrected 10/10/13

Slide created by Mustafa Amin

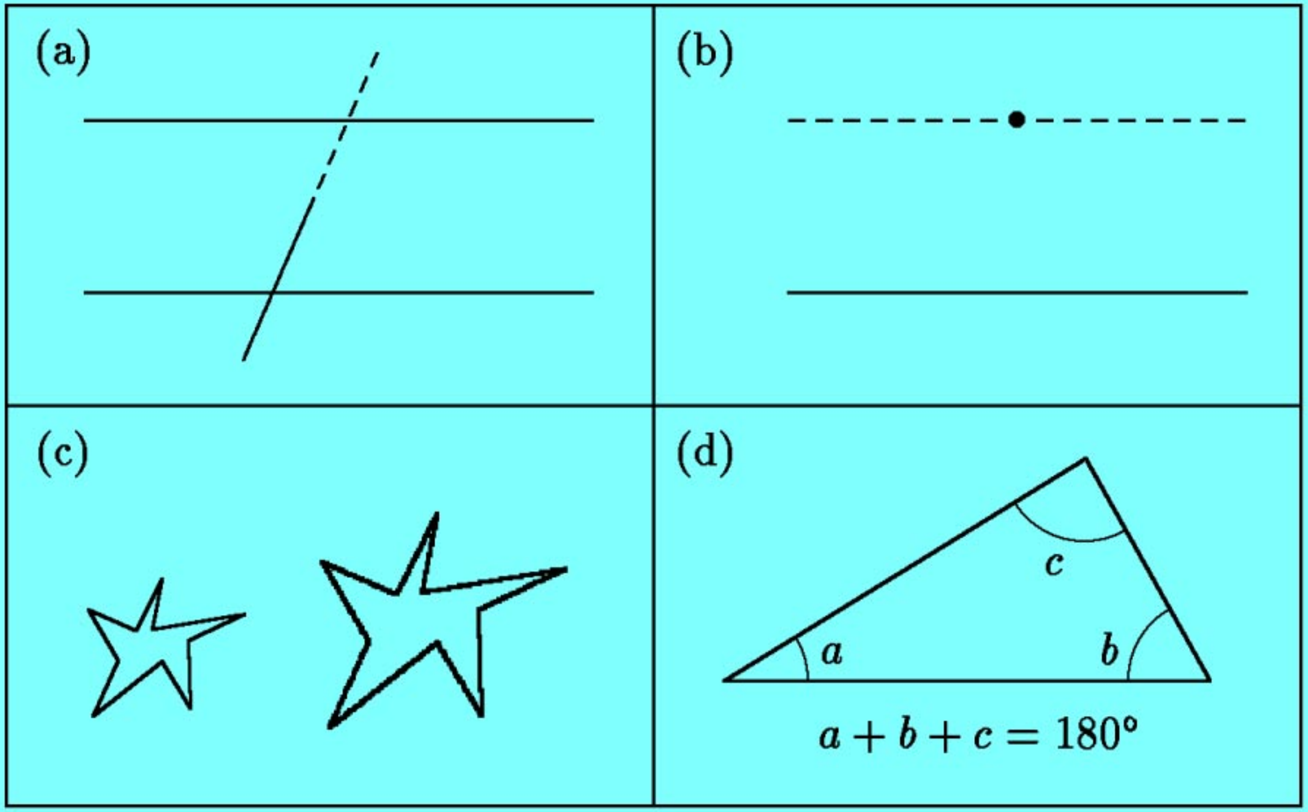
5th Postulate



5. If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines if produced indefinitely meet on that side on which the angles are less than two right angles

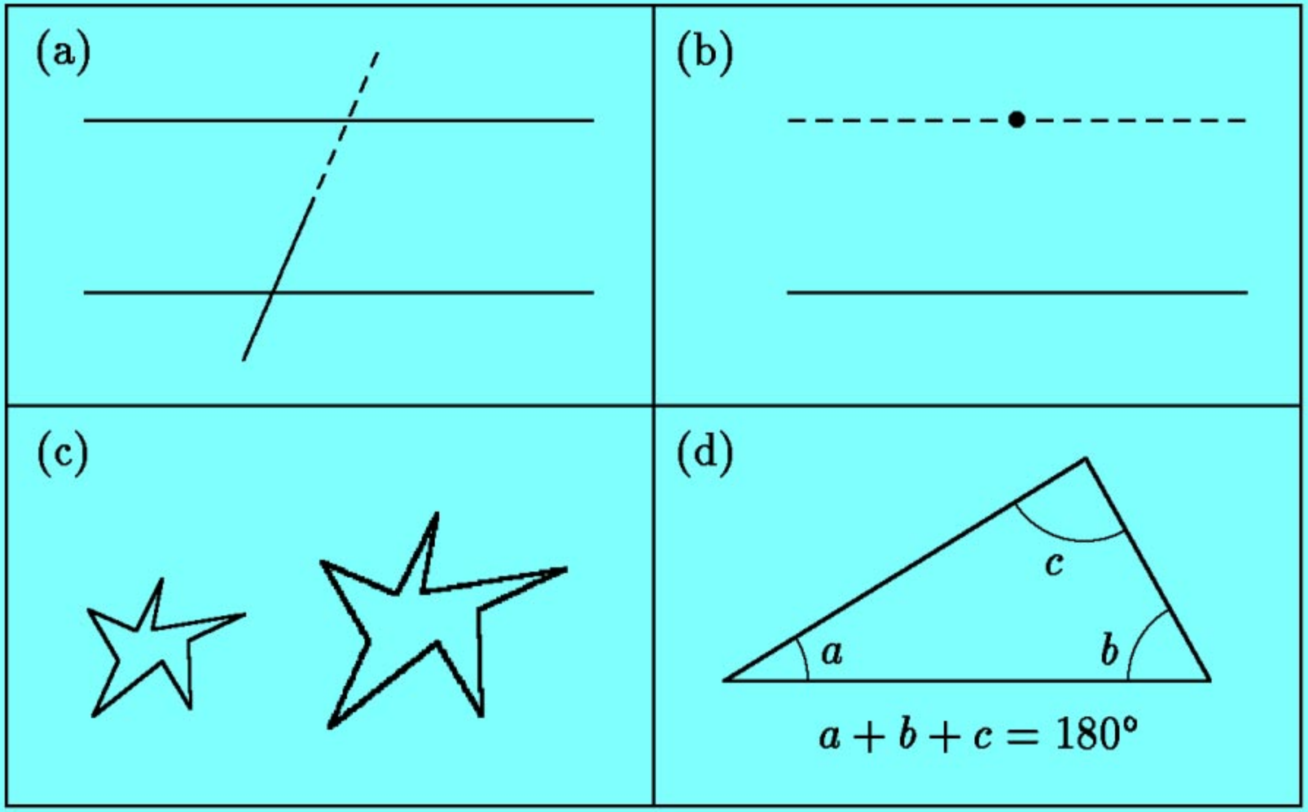
Corrected 10/10/13

Equivalent Statements of the 5th Postulate



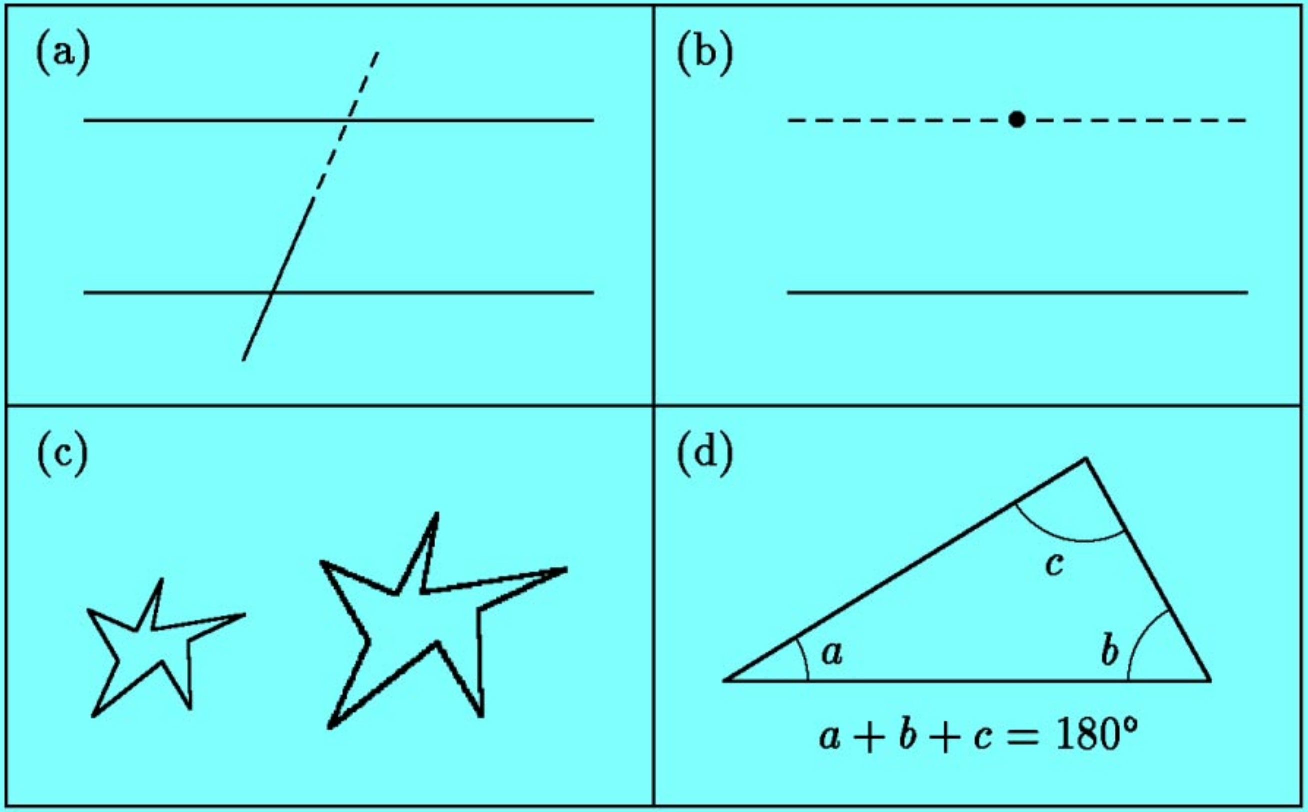
(a) "If a straight line intersects one of two parallels (i.e, lines which do not intersect however far they are extended), it will intersect the other also."

Equivalent Statements of the 5th Postulate



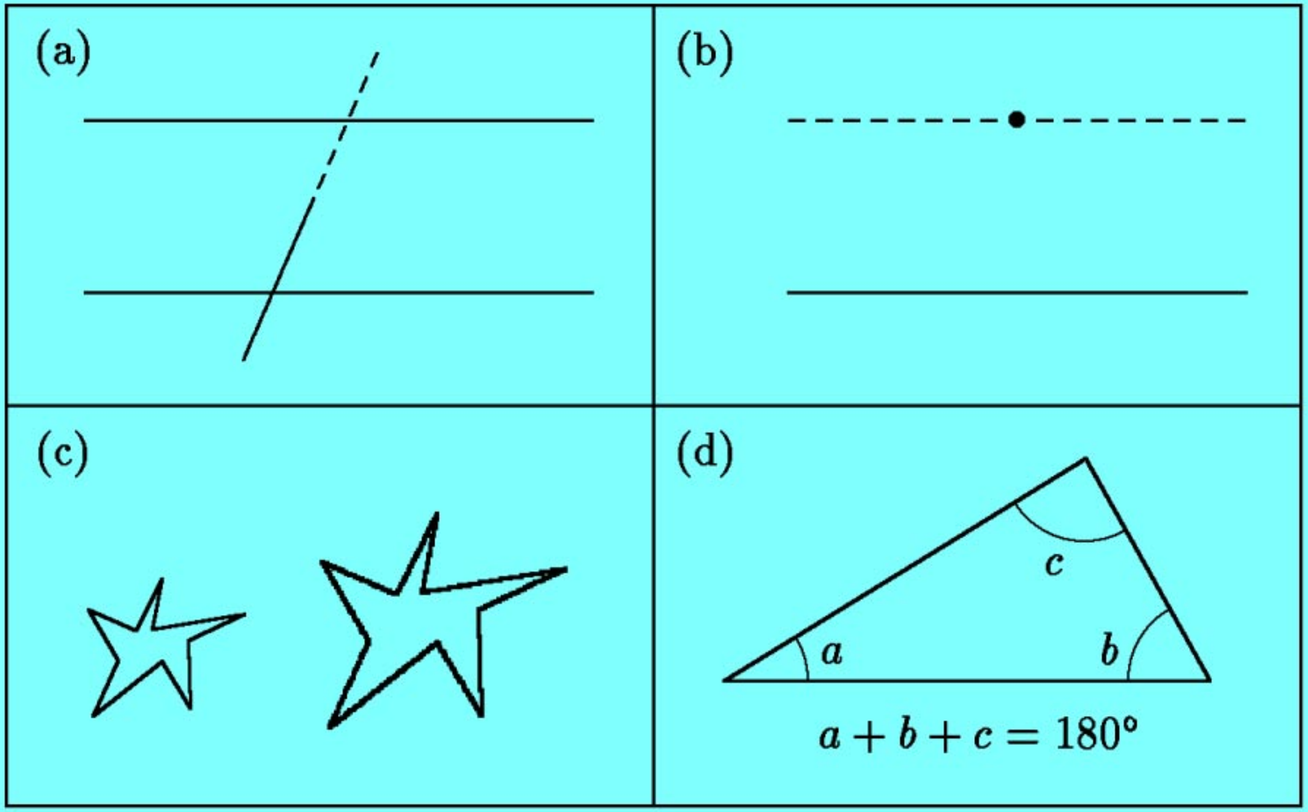
(b) "There is one and only one line that passes through any given point and is parallel to a given line."

Equivalent Statements of the 5th Postulate



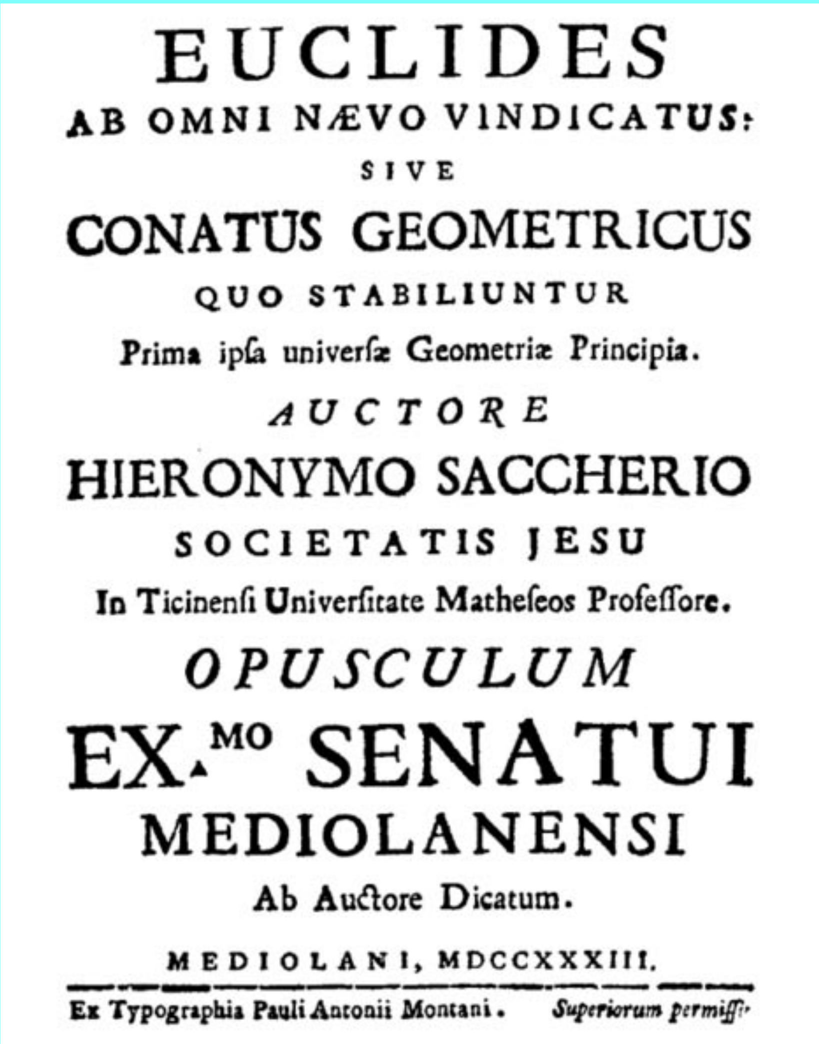
(c) "Given any figure there exists a figure, similar to it, of any size." (Two polygons are similar if their corresponding angles are equal, and their corresponding sides are proportional.)

Equivalent Statements of the 5th Postulate



(d) "There is a triangle in which the sum of the three angles is equal to two right angles (i.e., $180°$)."

Giovanni Geralamo Saccheri (1667–1733)


EUCLIDES
AB OMNI NÆVO VINDICATUS:
SIVE
CONATUS GEOMETRICUS
QUO STABILIUNTUR
Prima ipſa univerſæ Geometriæ Principia.
AUCTORE
HIERONYMO SACCHERIO
SOCIETATIS JESU
In Ticinenſi Univerſitate Matheſeos Profeſſore.
OPUSCULUM
EX.MO SENATUI
MEDIOLANENSI
Ab Auctore Dicatum.
MEDIOLANI, MDCCXXXIII.
Ex Typographia Pauli Antonii Montani. Superiorum permiſſ.

In 1733, Saccheri, a Jesuit priest, published *Euclides ab omni naevo vindicatus (Euclid Freed of Every Flaw).*

The book was a study of what geometry would be like if the 5th postulate were false.

He hoped to find an inconsistency, but failed.

Carl Friedrich Gauss (1777–1855)

German mathematician and physicist.

Born as the son of a poor working-class parents. His mother was illiterate and never even recorded the date of his birth.

His students included Richard Dedekind, Bernhard Riemann, Peter Gustav Lejeune Dirichlet, Gustav Kirchhoff, and August Ferinand Möbius.

János Bolyai (1802–1860)

Hungarian mathematician and army officer.

Son of Farkas Bolyai, a teacher of mathematics, physics, and chemistry at the Calvinist College in Marosvásárhely, Hungary.

Attended Marosvásárhely College and later studied military engineering at the Academy of Engineering at Vienna, because that is what his family could afford.

Served 11 years in the army engineering corps; during this time he developed his non-Euclidean geometry, which was published as an appendix to a book written by his father.

Retired from the army at age 31 due to poor health, and died in relative poverty at age 57, from pneumonia.

Nikolai Ivanovich Lobachevsky (1792–1856)

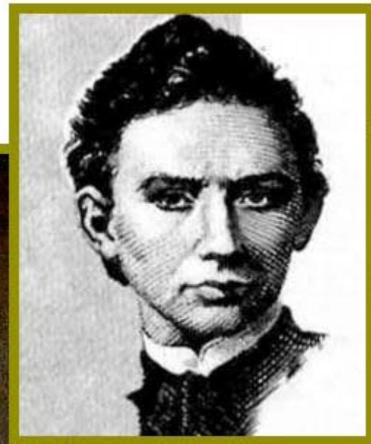
Russian mathematician and college teacher.

Born in Russia from Polish parents; father was a clerk in a land-surveying office, but died when Nikolai was only seven.

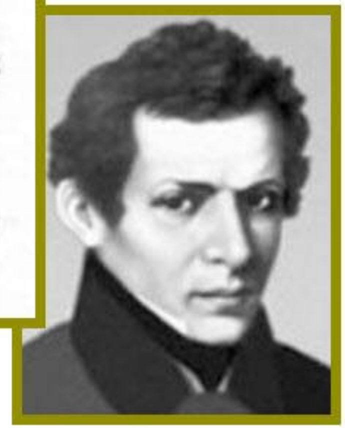
Moved to Kazan, attending Kazan Gymnasium and later was given a scholarship to Kazan University. He remained at Kazan University on the faculty.

Work on non-Euclidean geometry published in the *Kazan Messenger* in 1829, but was rejected for publication by the St. Petersburg Academy of Sciences.

He was "asked to retire" at age 54, and died 10 years later in poor health and in poverty. His work was never appreciated during his lifetime.



Bolyai

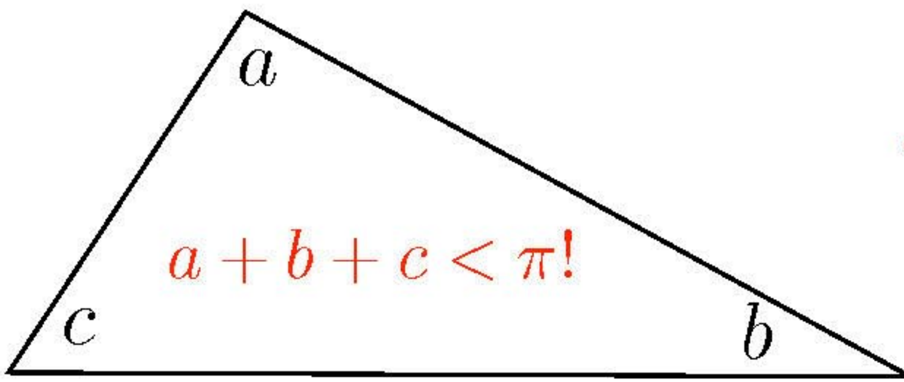


Lobachevski

Gauss

- infinite
- constant negative curvature

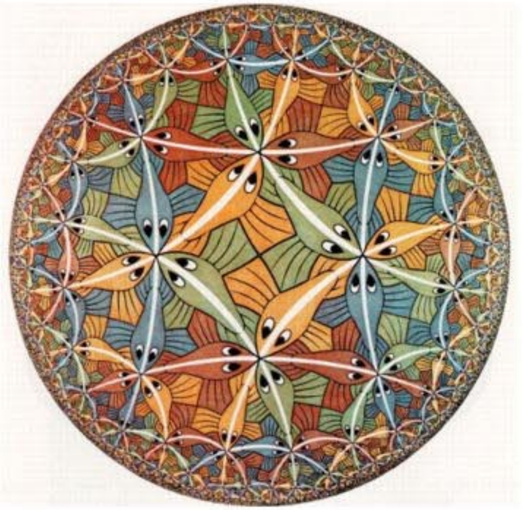
~~5th Postulate~~



Slide created by Mustafa Amin

GBL geometry with Klein

1. constant negative curvature
2. infinite
3. ~~5th postulate~~



(x_1, y_1)

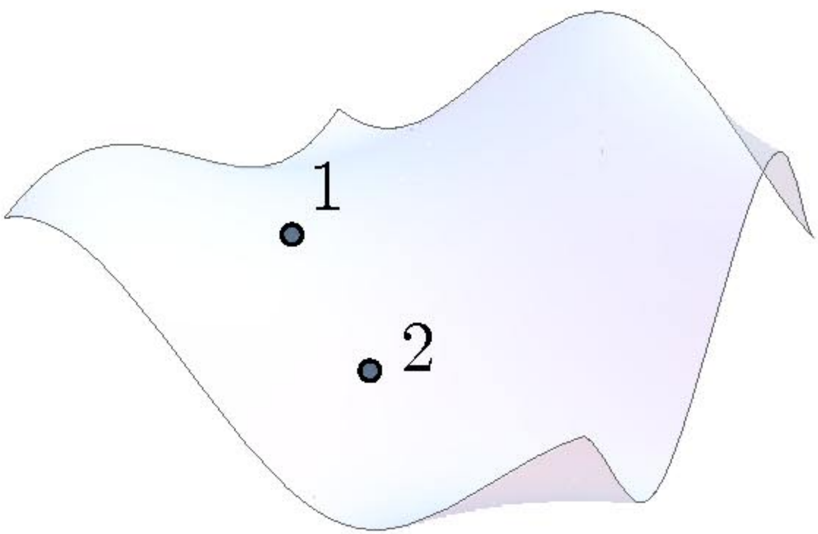
$$x^2 + y^2 < 1$$

$$d(1,2) = a \cosh^{-1}\left[\frac{1 - x_1 x_2 - y_1 y_2}{\sqrt{1 - x_1^2 - y_1^2}\sqrt{1 - x_2^2 - y_2^2}}\right]$$

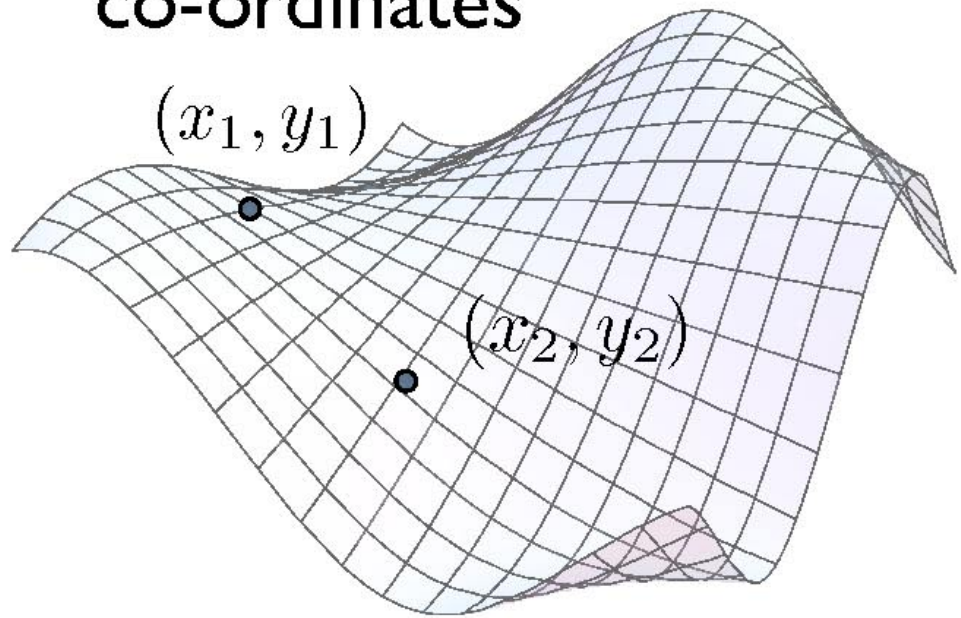
(x_2, y_2)

Note: no global embedding in 3D Euclidean space possible

Geometry (after Klein)



co-ordinates



distance function

$$d[(x_1, y_1), (x_2, y_2)]$$

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Intrinsic Geometry

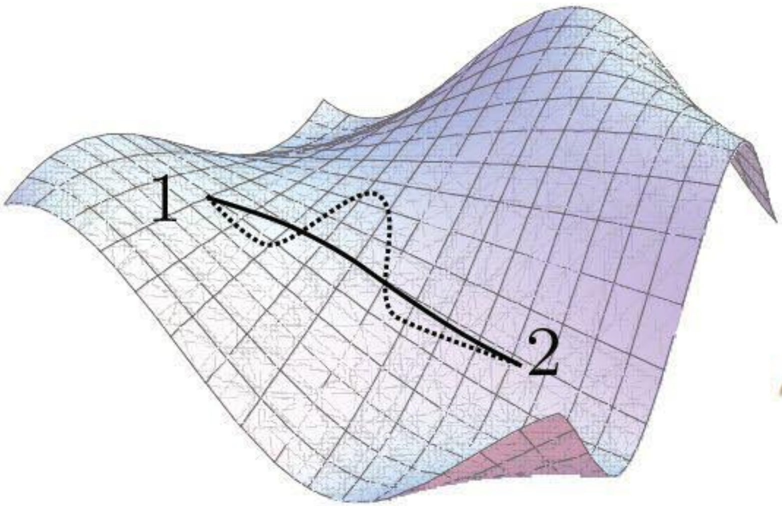

Image of ant from: http://www.termiteterry.com/

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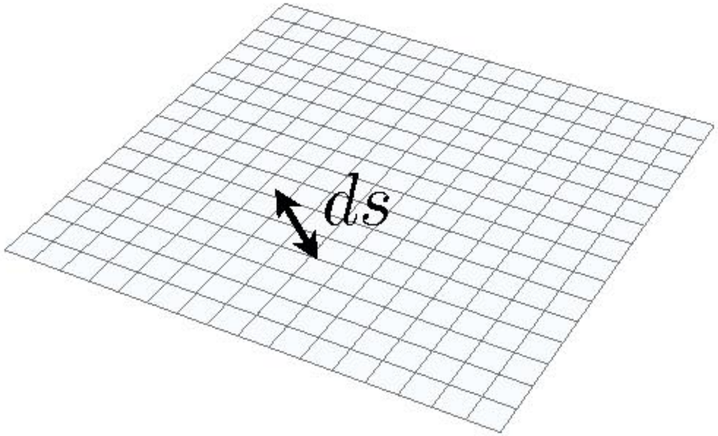
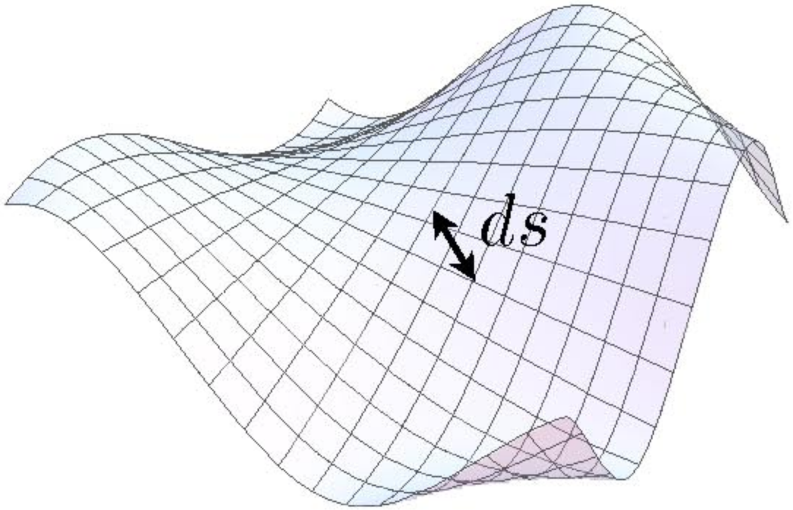
tiny distances



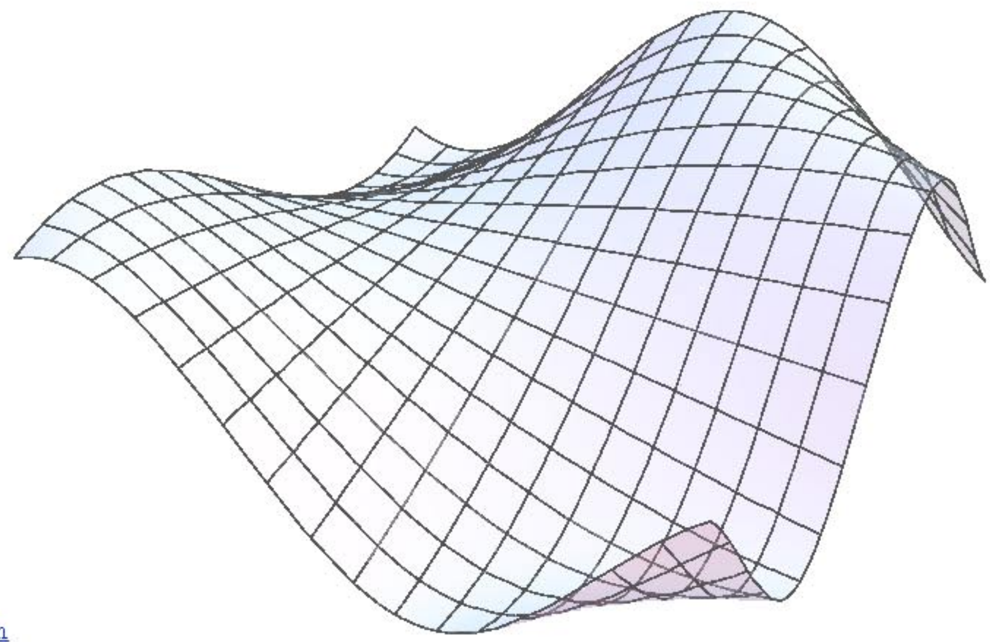


$$ds^2 = dx^2 + dy^2$$

$$ds^2 = g_{xx}(x,y)dx^2 + 2g_{xy}(x,y)dxdy + g_{yy}(x,y)dy^2$$

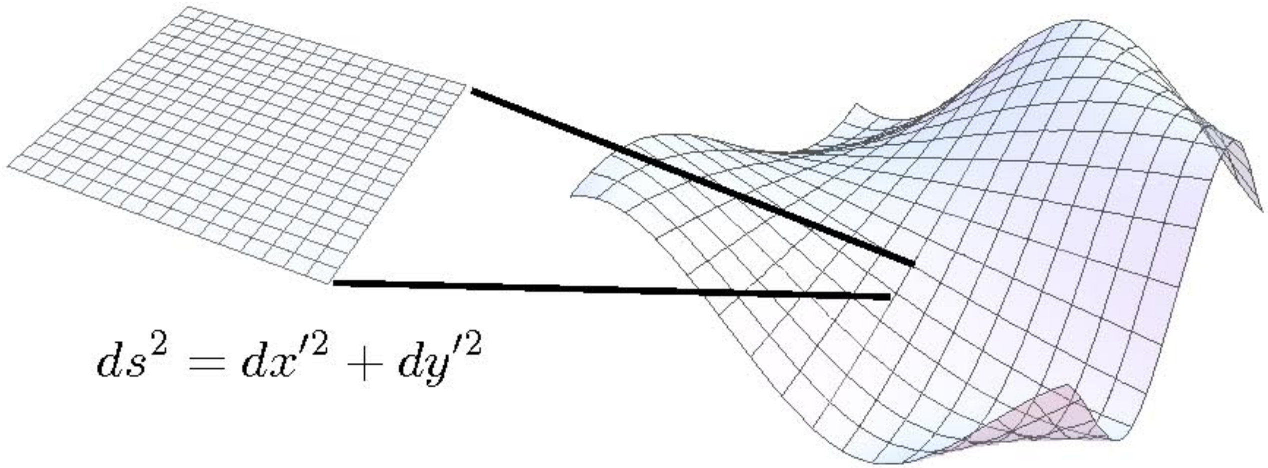
quadratic form

Image:www.easternct.edu/career/webresources.htm



$$ds^2 = g_{xx}(x,y)dx^2 + 2g_{xy}(x,y)dxdy + g_{yy}(x,y)dy^2$$

locally Euclidean



$$ds^2 = dx'^2 + dy'^2$$

$$ds^2 = g_{xx}(x,y)dx^2 + 2g_{xy}(x,y)dxdy + g_{yy}(x,y)dy^2$$

$$g_{xx}g_{yy} - g_{xy}^2 > 0$$

8.286 Class 10
October 7, 2020

INTRODUCTION TO NON-EUCLIDEAN SPACES PART 2

(Modified 10/23/20, to mark the end of the slides reached in class.)

Announcements

- ☆ "Remote learning check-in" survey is up and running:

  https://forms.gle/4GjAhH5YBvpoema18

  If you have not already filled it in, please do so by midnight tonight (after the vice-presidential debate).

- ☆ The survey is only to help Bruno and me make improvements to the course. We VALUE your feedback and suggestions.

https://www.nobelprize.org/prizes/physics/

The 2020 Physics Laureates

The Royal Swedish Academy of Sciences has decided to award the Nobel Prize in Physics 2020 with one half to Roger Penrose "for the discovery that black hole formation is a robust prediction of the general theory of relativity" and and the other half jointly to Reinhard Genzel and Andrea Ghez "for the discovery of a supermassive compact object at the centre of our galaxy".

Ill. Niklas Elmehed. © Nobel Media.

YouTube Search

https://www.youtube.com/watch?v=5bmJalWKTj8&feature=emb_logo

Roger Penrose: "I had this strange feeling of elation"

15,429 views • Oct 6, 2020 1.4K 4 SHARE SAVE

Nobel Prize
236K subscribers
SUBSCRIBE

In this phone interview with Adam Smith, recorded just after the announcement of the 2020 Nobel Prize in Physics, Roger Penrose recounts the story of how a particular crossroads held the key to his seminal 1965 paper on the theoretical basis of black holes. In the second half of the conversation he goes on to explain some of his latest work, describing how black holes are the basis for the second law of thermodynamics, and suggesting that signatures of black holes from a previous universe might be faintly apparent in the cosmic microwave background radiation.

Monthly Notices of the ROYAL ASTRONOMICAL SOCIETY
MNRAS **495**, 3403–3408 (2020)
Advance Access publication 2020 May 18
doi:10.1093/mnras/staa1343

Apparent evidence for Hawking points in the CMB Sky★

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ABSTRACT

This paper presents strong observational evidence of numerous previously unobserved anomalous circular spots, of significantly raised temperature, in the cosmic microwave background sky. The spots have angular radii between 0.03 and 0.04 rad (i.e. angular diameters between about 3° and 4°). There is a clear cut-off at that size, indicating that each anomalous spot would have originated from a highly energetic point-like source, located at the end of inflation – or else point-like at the conformally expanded Big Bang, if it is considered that there was no inflationary phase. The significant presence of these anomalous spots, was initially noticed in the *Planck* 70 GHz satellite data by comparison with 1000 standard simulations, and then confirmed by extending the comparison to 10 000 simulations. Such anomalous points were then found at precisely the same locations in the *WMAP (Wilkinson Microwave Anisotropy Probe)* data, their significance was confirmed by comparison with 1000 *WMAP* simulations. *Planck* and *WMAP* have very different noise properties and it seems exceedingly unlikely that the observed presence of anomalous points in the same directions on both maps may come entirely from the noise. Subsequently, further confirmation was found in the *Planck* data by comparison with 1000 FFP8.1 MC simulations (with $l \leq 1500$). The existence of such anomalous regions, resulting from point-like sources at the conformally stretched-out big bang, is a predicted consequence of conformal cyclic cosmology, these sources being the Hawking points of the theory, resulting from the Hawking radiation from supermassive black holes in a cosmic aeon prior to our own.

Key words: cosmic background radiation.

PACS: 04.20.Ha – 04.70.Dy – 98.80.Bp – 98.80.Ft.

Astrophysics > Cosmology and Nongalactic Astrophysics

[Submitted on 6 Aug 2018 (v1), last revised 2 Mar 2020 (this version, v4)]

Apparent evidence for Hawking points in the CMB Sky

Daniel An, Krzysztof A. Meissner, Pawel Nurowski, Roger Penrose

This paper presents strong observational evidence of numerous previously unobserved anomalous circular spots, of significantly raised temperature, in the CMB sky. The spots have angular radii between 0.03 and 0.04 radians (i.e. angular diameters between about 3 and 4 degrees). There is a clear cut-off at that size, indicating that each anomalous spot would have originated from a highly energetic point-like source, located at the end of inflation -- or else point-like at the conformally expanded Big Bang, if it is considered that there was no inflationary phase. The significant presence of these anomalous spots, was initially noticed in the Planck 70 GHz satellite data by comparison with 1000 standard simulations, and then confirmed by extending the comparison to 10000 simulations. Such anomalous points were then found at precisely the same locations in the WMAP data, their significance confirmed by comparison with 1000 WMAP simulations. Planck and WMAP have very different noise properties and it seems exceedingly unlikely that the observed presence of anomalous points in the same directions on both maps may come entirely from the noise. Subsequently, further confirmation was found in the Planck data by comparison with 1000 FFP8.1 MC simulations (with $l \leq 1500$). The existence of such anomalous regions, resulting from point-like sources at the conformally stretched-out big bang, is a predicted consequence of conformal cyclic cosmology (CCC), these sources being the Hawking points of the theory, resulting from the Hawking radiation from supermassive black holes in a cosmic aeon prior to our own.

Subjects: **Cosmology and Nongalactic Astrophysics (astro-ph.CO)**
Cite as: arXiv:1808.01740 **[astro-ph.CO]**
(or arXiv:1808.01740v4 **[astro-ph.CO]** for this version)

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[v1] Mon, 6 Aug 2018 06:16:38 UTC (9 KB)
[v2] Sat, 17 Nov 2018 21:05:16 UTC (10 KB)
[v3] Mon, 17 Dec 2018 06:44:53 UTC (11 KB)
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arxiv.org/abs/1909.09672

Cornell University

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Astrophysics > Cosmology and Nongalactic Astrophysics

[Submitted on 20 Sep 2019 (v1), last revised 21 Jan 2020 (this version, v2)]

Re-evaluating evidence for Hawking points in the CMB

Dylan L. Jow, Douglas Scott

We investigate recent claims for a detection of "Hawking points" (positions on the sky with unusually large temperature gradients between rings) in the cosmic microwave background (CMB) temperature maps at the 99.98% confidence level. We find that, after marginalization over the size of the rings, an excess is detected in Planck satellite maps at only an 87% confidence level (i.e., little more than 1σ). Therefore, we conclude that there is no statistically significant evidence for the presence of Hawking points in the CMB.

Subjects: **Cosmology and Nongalactic Astrophysics (astro-ph.CO)**
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Cite as: arXiv:1909.09672 **[astro-ph.CO]**
(or arXiv:1909.09672v2 **[astro-ph.CO]** for this version)

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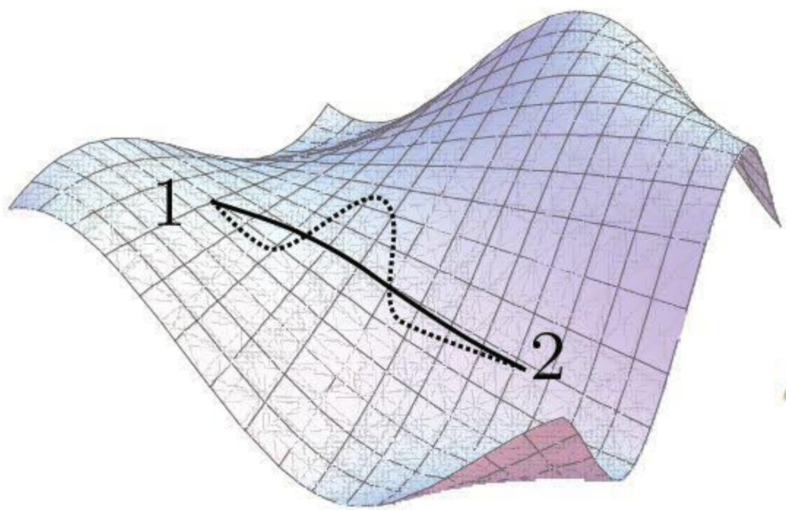
Intrinsic Geometry



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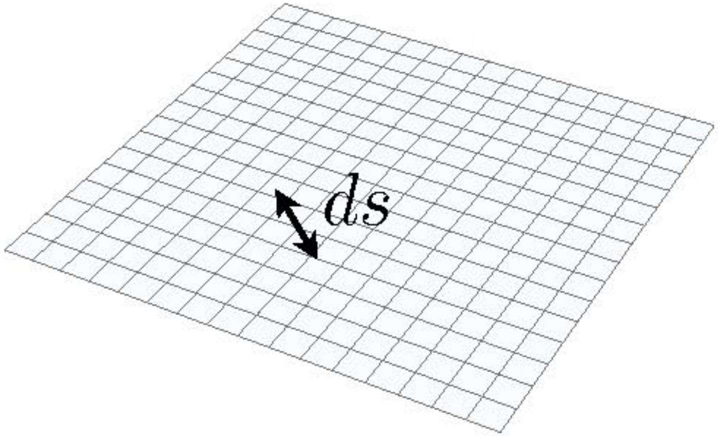
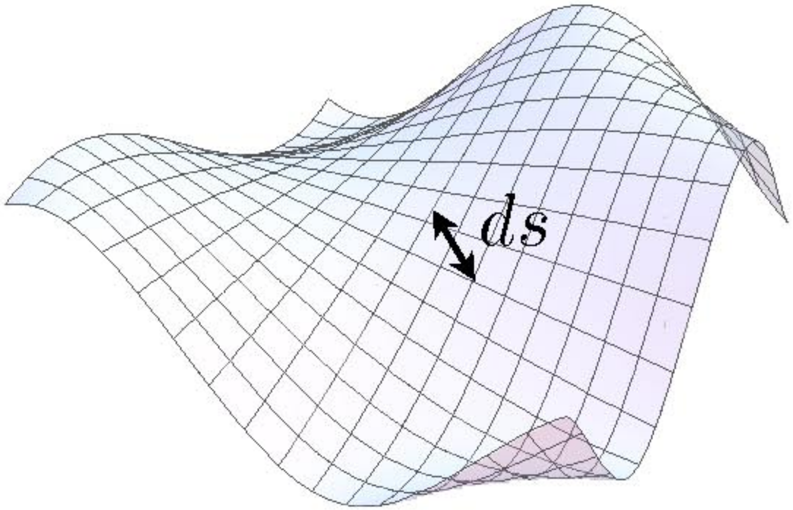
tiny distances



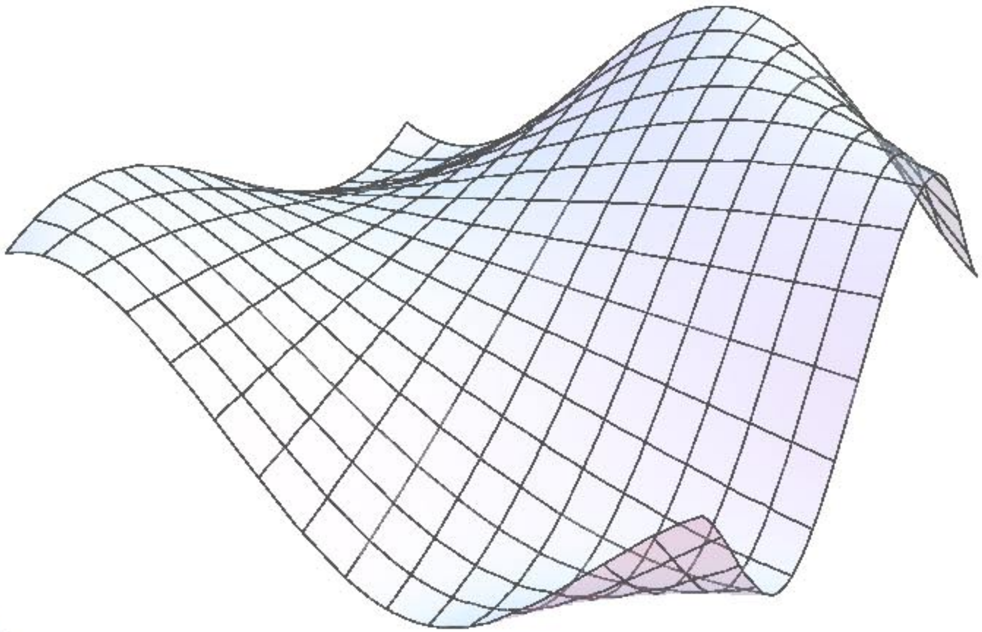


$ds^2 = dx^2 + dy^2$

$ds^2 = g_{xx}(x,y)dx^2 + 2g_{xy}(x,y)dxdy + g_{yy}(x,y)dy^2$

quadratic form

Image:www.easternct.edu/career/webresources.htm



$$ds^2 = g_{xx}(x,y)dx^2 + 2g_{xy}(x,y)dxdy + g_{yy}(x,y)dy^2$$

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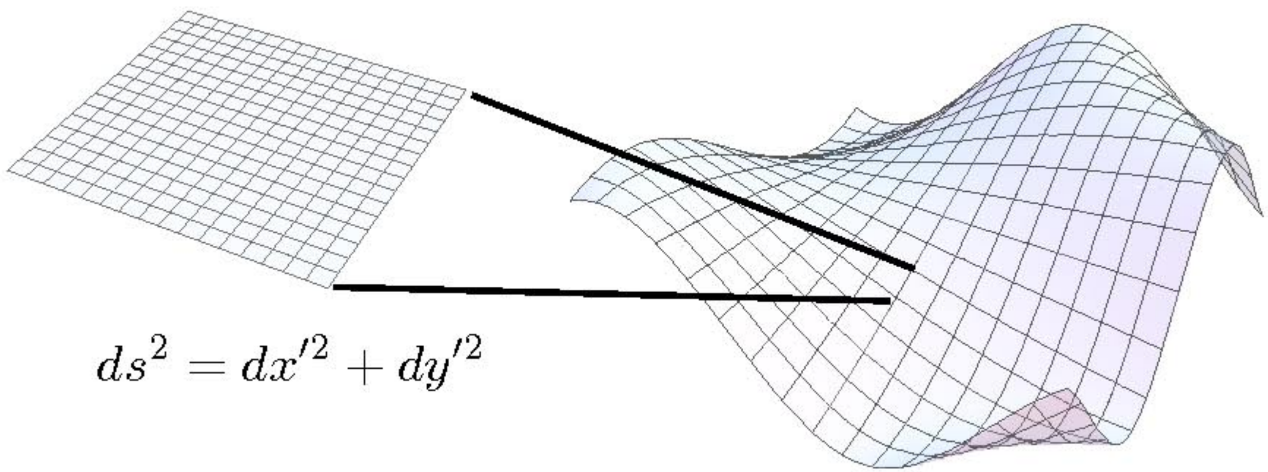
locally Euclidean

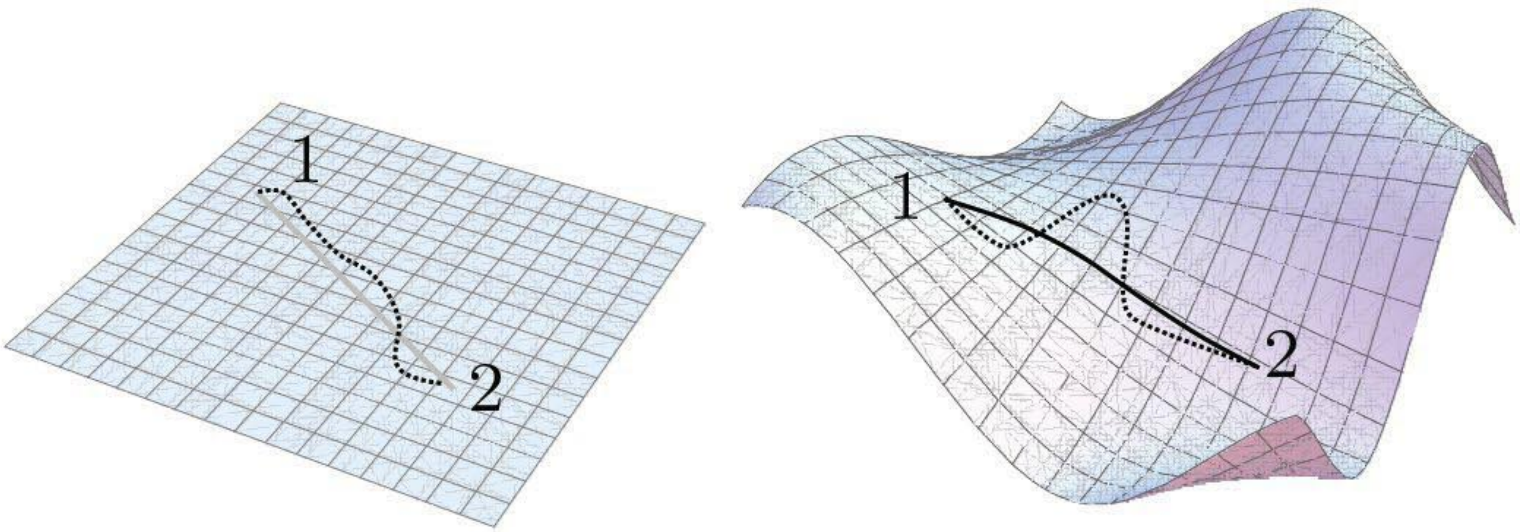


$$ds^2 = dx'^2 + dy'^2$$

$$ds^2 = g_{xx}(x,y)dx^2 + 2g_{xy}(x,y)dxdy + g_{yy}(x,y)dy^2$$

$$g_{xx}g_{yy} - g_{xy}^2 > 0$$

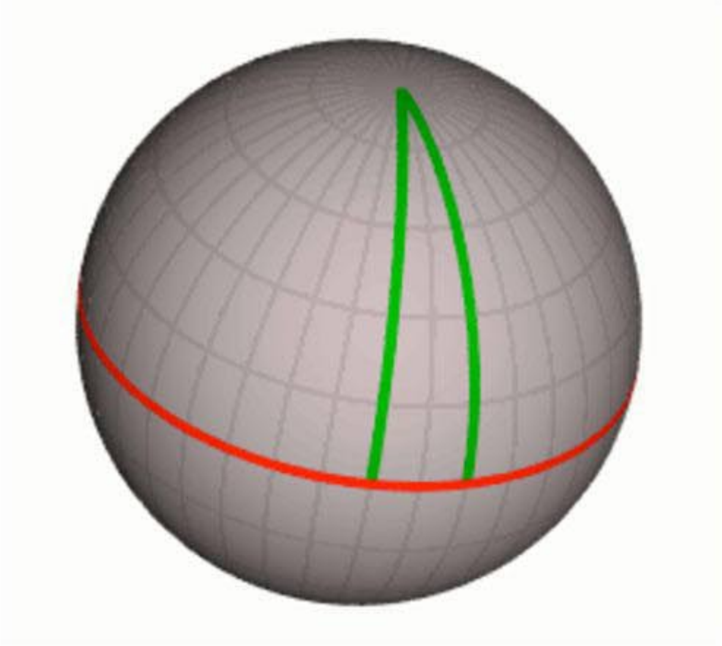
geodesics

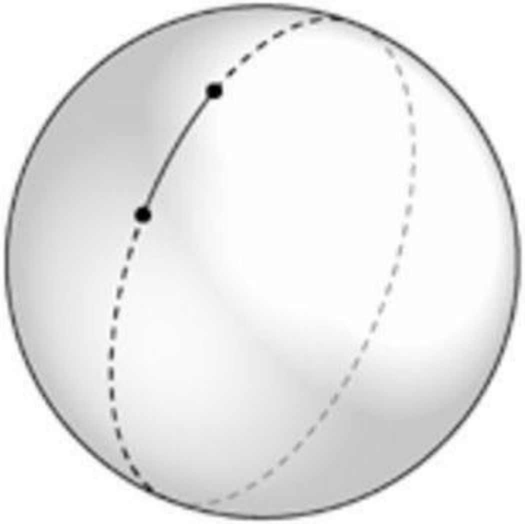


a geodesic is a curve along which the distance between two given points is extremised.

note: important!

sphere: geodesics





longitudes: yes
latitudes: no

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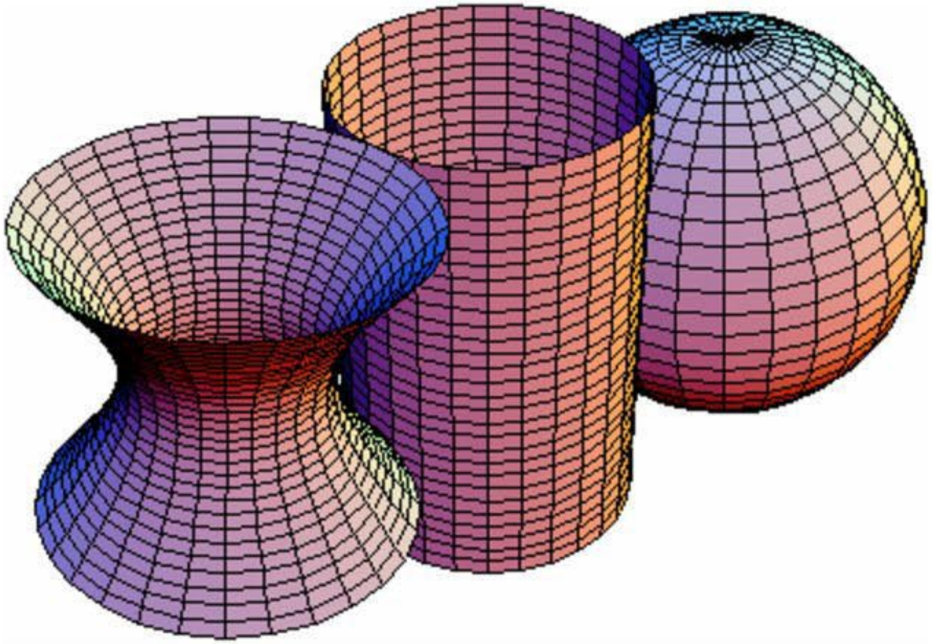
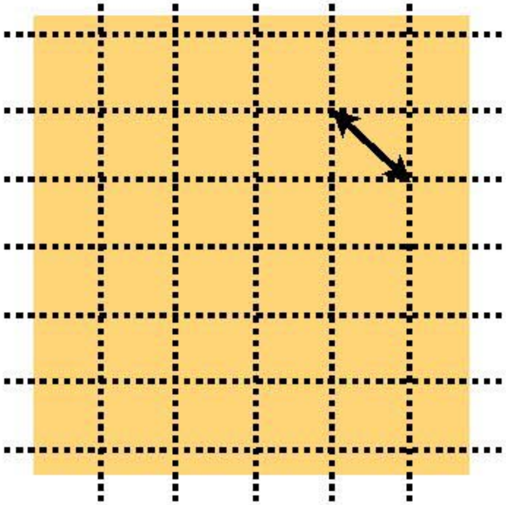
curved ?



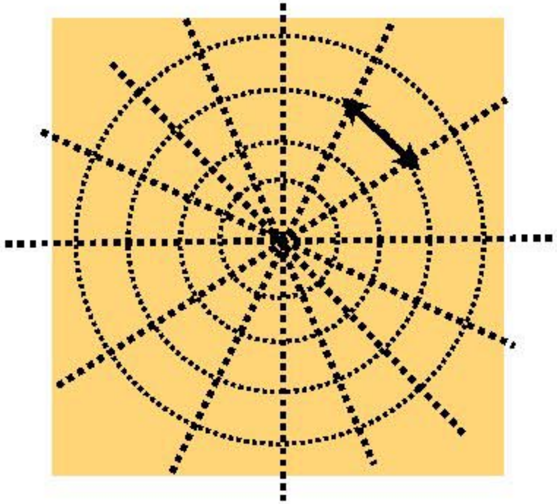
Above image:http://en.wikipedia.org/wiki/Gaussian_curvature

$$ds^2 = dx^2 + dy^2 \qquad ds^2 = g_{xx}(x,y)dx^2 + 2g_{xy}(x,y)dxdy + g_{yy}(x,y)dy^2$$

metric and co-ordinates



$$ds^2 = dx^2 + dy^2$$



$$ds^2 = dr^2 + r^2 d\theta^2$$

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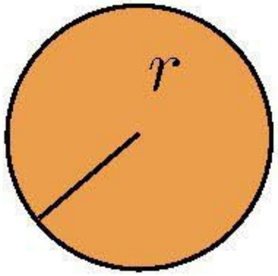
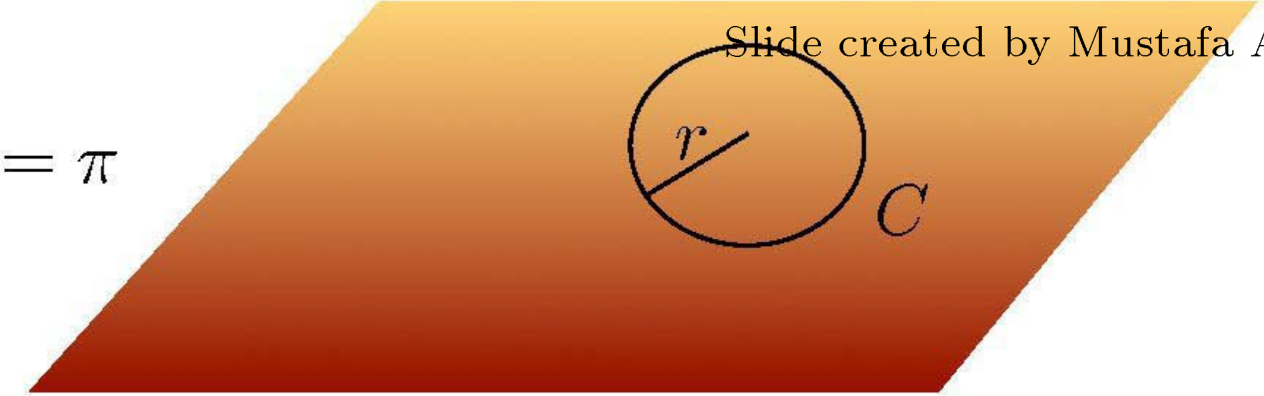
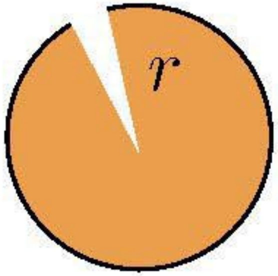
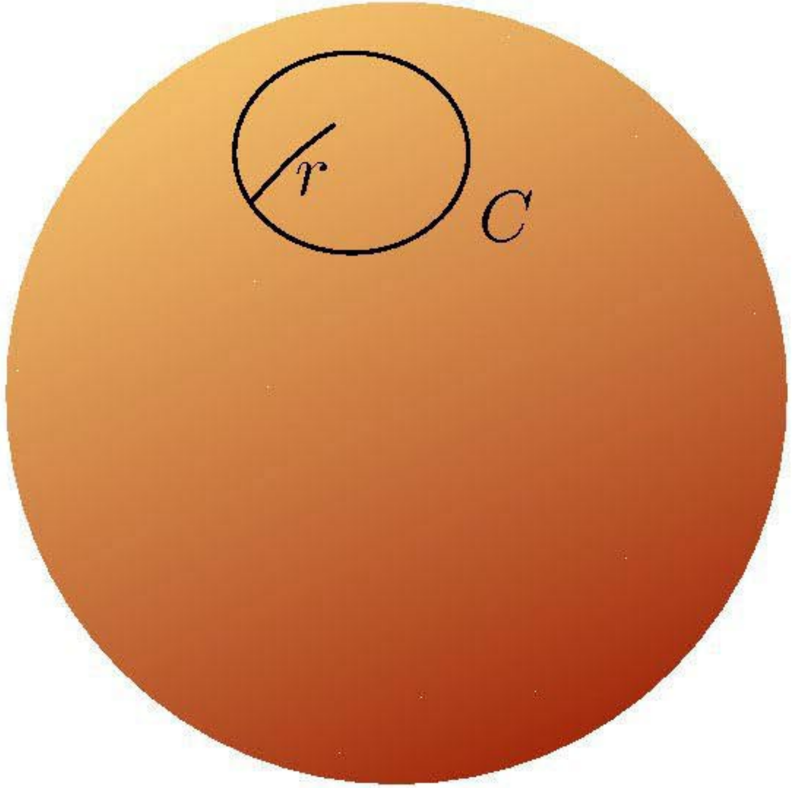


$$\frac{C}{2r} = \pi$$





$$\frac{C}{2r} < \pi$$



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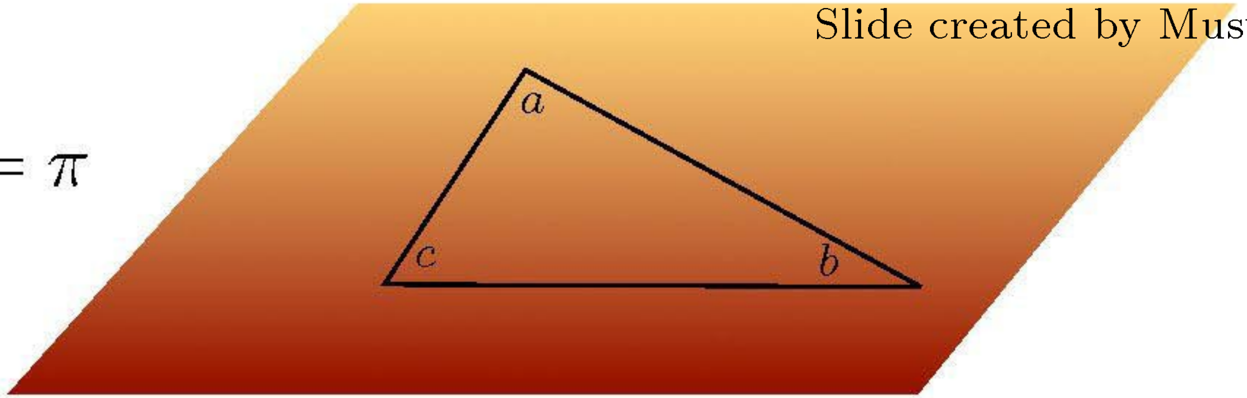
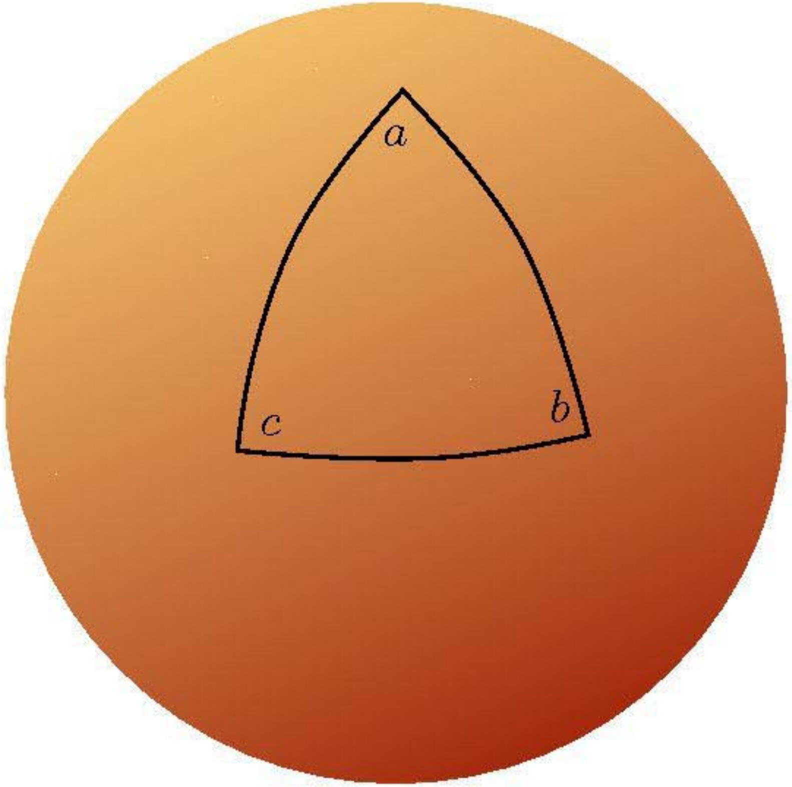
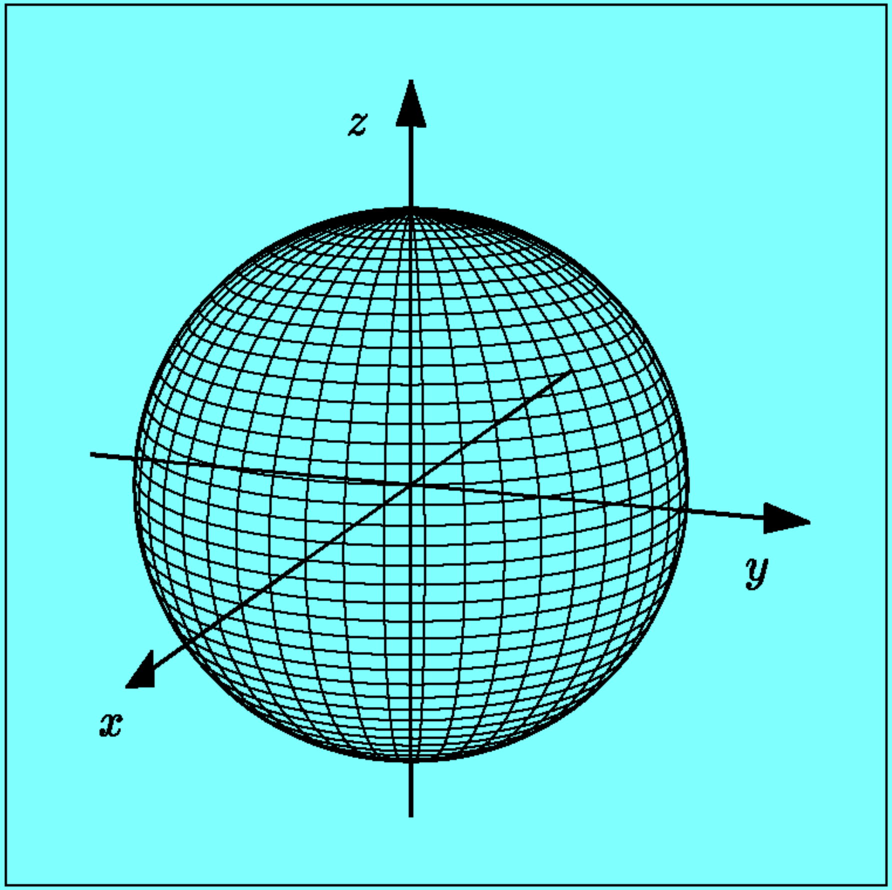
$$a + b + c = \pi$$



$$a + b + c > \pi$$



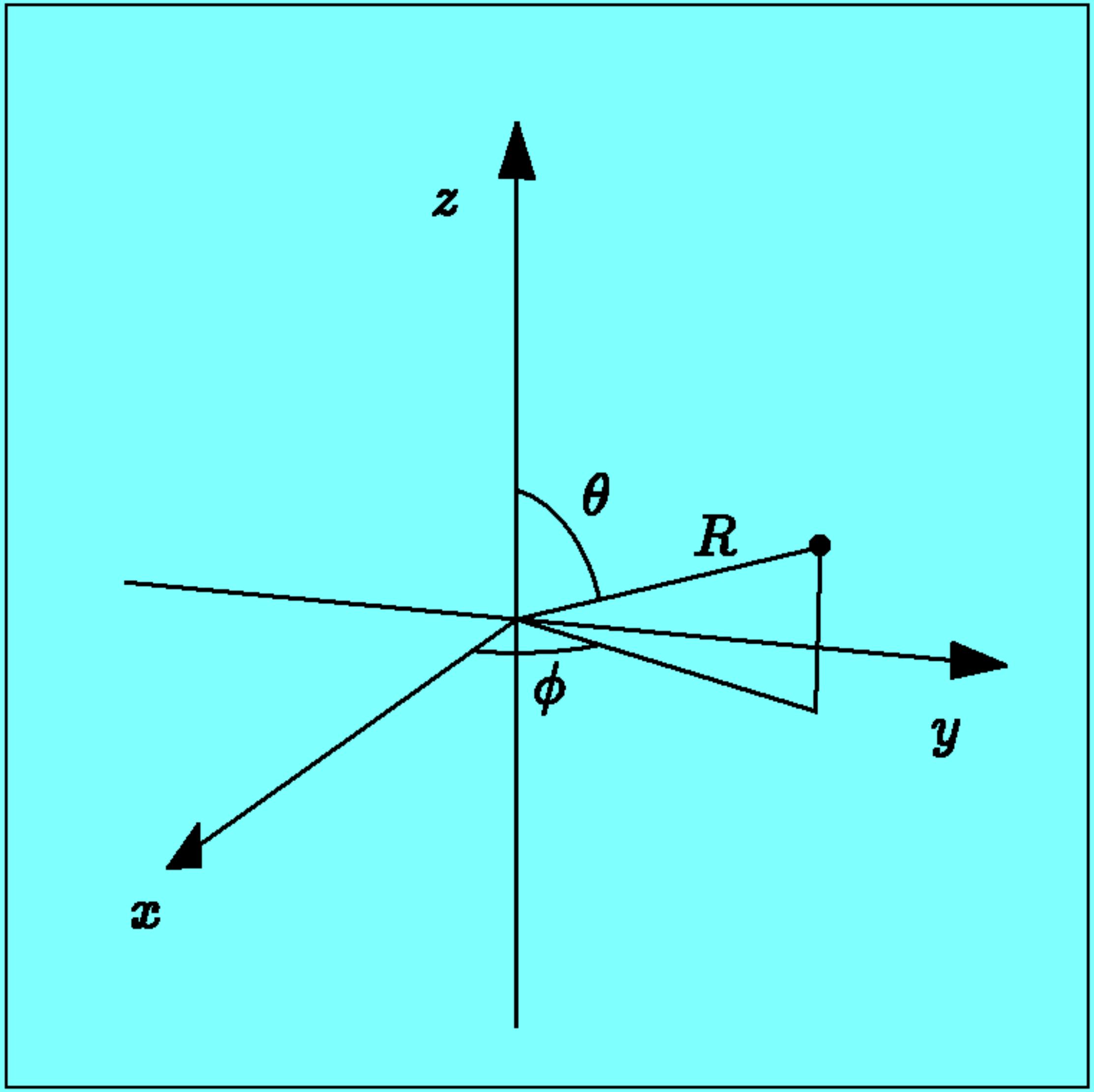
Non-Euclidean Geometry: The Surface of a Sphere



$$x^2 + y^2 + z^2 = R^2 \ .$$

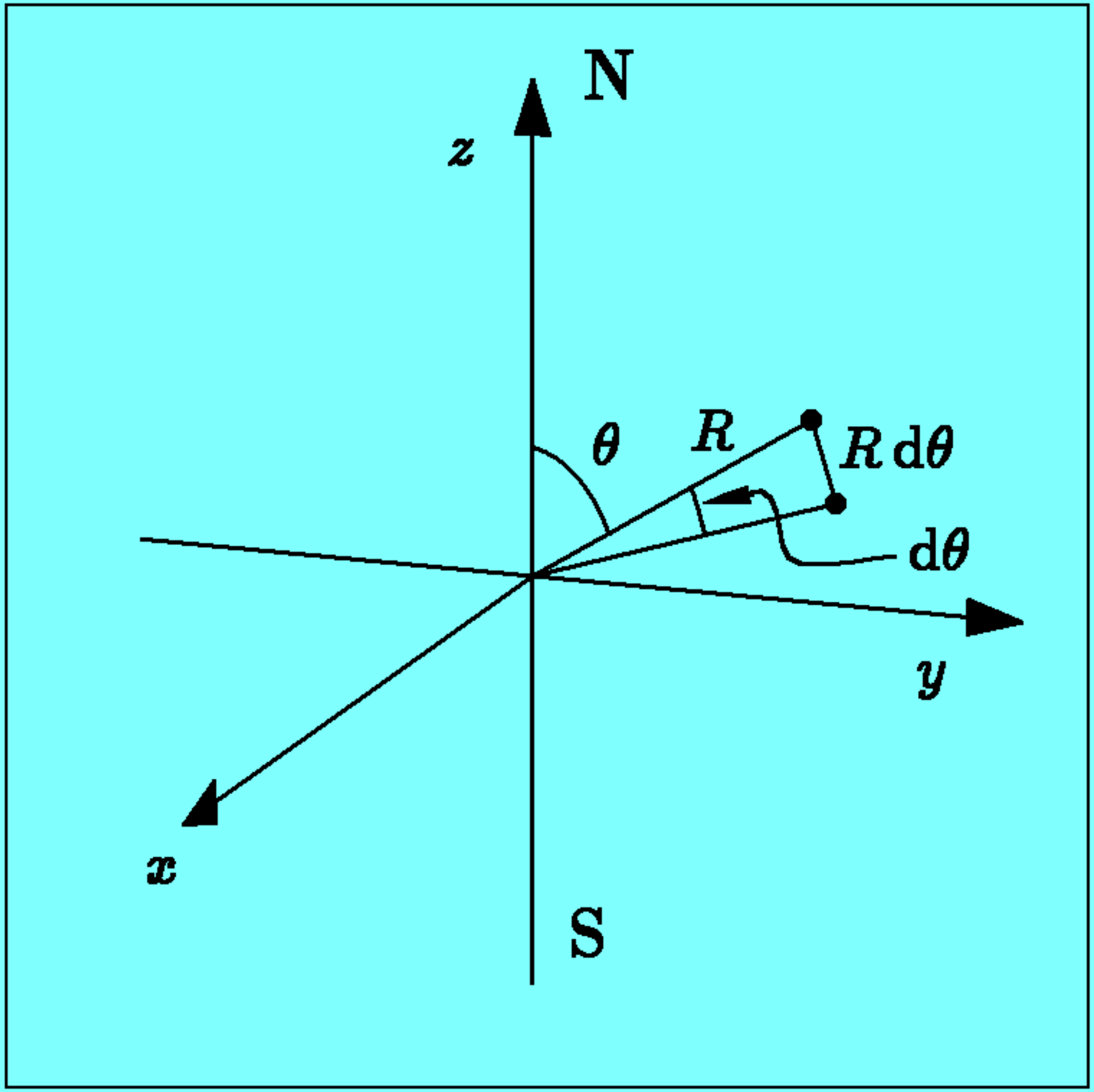
Polar Coordinates:

$$x = R \sin\theta \cos\phi$$
$$y = R \sin\theta \sin\phi$$
$$z = R \cos\theta \ ,$$



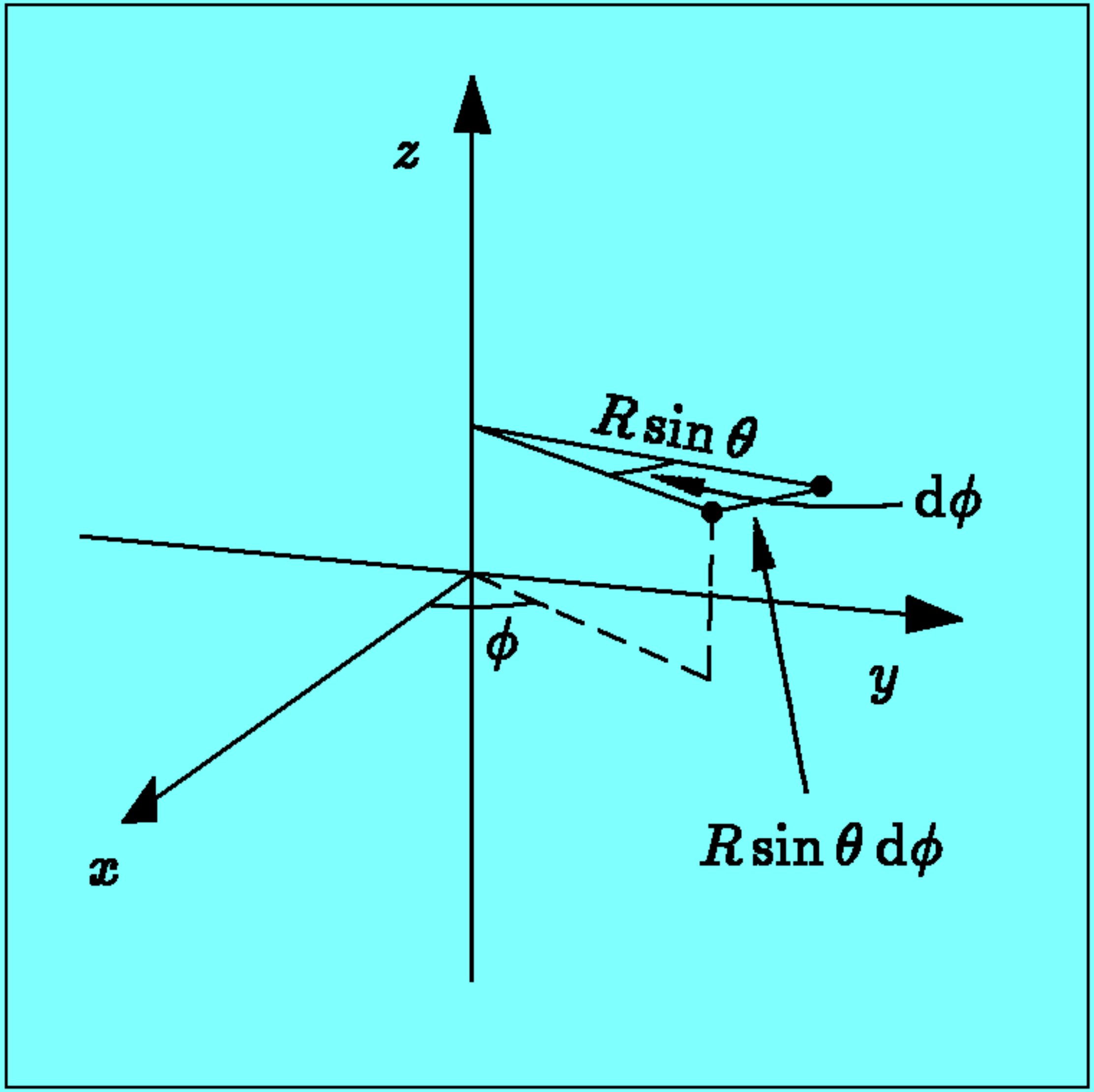
Varying θ:

$$ds = R\,d\theta$$



Varying ϕ:

$$ds = R\sin\theta\, d\phi$$



Varying θ and ϕ

Varying θ: $\quad ds = R\,d\theta$

Varying ϕ: $\quad ds = R\sin\theta\,d\phi$

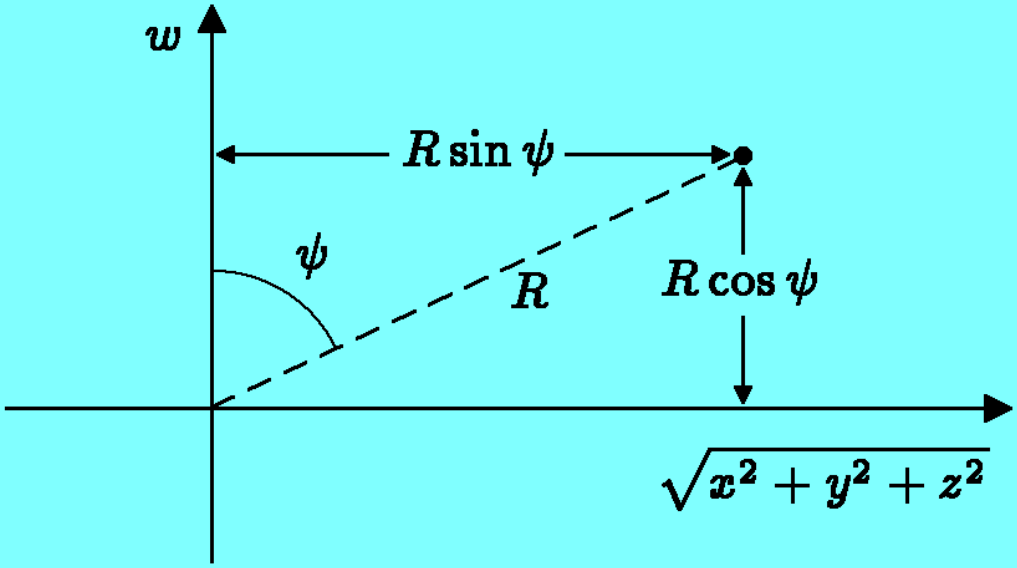
$$ds^2 = R^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)$$

A Closed Three-Dimensional Space

$$x^2 + y^2 + z^2 + w^2 = R^2$$

$$\begin{aligned} x &= R \sin\psi \sin\theta \cos\phi \\ y &= R \sin\psi \sin\theta \sin\phi \\ z &= R \sin\psi \cos\theta \\ w &= R \cos\psi \ , \end{aligned}$$

$$ds = R\, d\psi$$



Alan Guth
Massachusetts Institute of Technology

Metric for the Closed 3D Space

Varying ψ: $ds = R\,d\psi$

Varying θ or ϕ: $ds^2 = R^2 \sin^2\psi (d\theta^2 + \sin^2\theta\, d\phi^2)$

If the variations are orthogonal to each other, then

$$ds^2 = R^2\left[d\psi^2 + \sin^2\psi\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$$

Proof of Orthogonality of Variations

Let $d\vec{r}_\psi =$ displacement of point when ψ is changed to $\psi + d\psi$.

Let $d\vec{r}_\theta =$ displacement of point when θ is changed to $\theta + d\theta$.

- $d\vec{r}_\theta$ has no w-component $\Longrightarrow$ $d\vec{r}_\psi \cdot d\vec{r}_\theta = d\vec{r}_\psi^{\,(3)} \cdot d\vec{r}_\theta^{\,(3)}$, where (3) denotes the projection into the x-y-z subspace.
- $d\vec{r}_\psi^{\,(3)}$ is radial; $d\vec{r}_\theta^{\,(3)}$ is tangential

$$\Longrightarrow \quad d\vec{r}_\psi^{\,(3)} \cdot d\vec{r}_\theta^{\,(3)} = 0$$

Implications of General Relativity

- ☆ $ds^2 = R^2\left[d\psi^2 + \sin^2\psi\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$, where R is radius of curvature.
- ☆ According to GR, matter causes space to curve.
- ☆ R cannot be arbitrary. Instead, $R^2(t) = \dfrac{a^2(t)}{k}$.
- ☆ Finally,

$$\mathrm{d}s^2 = a^2(t)\left\{\frac{\mathrm{d}r^2}{1-kr^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right\} \ ,$$

where $r = \dfrac{\sin\psi}{\sqrt{k}}$. Called the Robertson-Walker metric.

8.286 Class 11
October 13, 2020

INTRODUCTION TO NON-EUCLIDEAN SPACES PART 3

(Modified 10/23/20 to add note about the meaning of $g_{ij}\left(x^k\right)$ at the bottom of slide 16.)

Announcements

Questions about quiz grading (or problem set grading):

Please ask either Bruno or me. We try to grade accurately, but sometimes we make mistakes. We are **always** happy to discuss this with you, and are happy to make changes when grading errors are found.

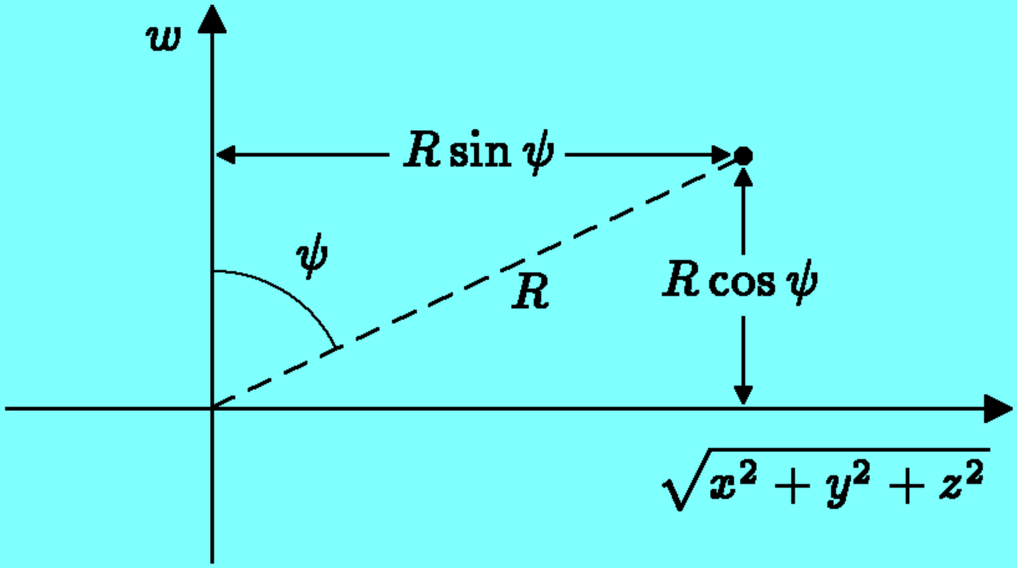
Review from previous class:

A Closed Three-Dimensional Space

$$x^2 + y^2 + z^2 + w^2 = R^2$$

$$\begin{aligned} x &= R \sin\psi \sin\theta \cos\phi \\ y &= R \sin\psi \sin\theta \sin\phi \\ z &= R \sin\psi \cos\theta \\ w &= R \cos\psi \ , \end{aligned}$$

$$\mathrm{d}s = R\, d\psi$$



Alan Guth
Massachusetts Institute of Technology

Review from previous class:

Metric for the Closed 3D Space

Varying ψ: $\quad \mathrm{d}s = R\, d\psi$

Varying θ or ϕ: $\quad \mathrm{d}s^2 = R^2 \sin^2\psi (\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2)$

If the variations are orthogonal to each other, then

$$\mathrm{d}s^2 = R^2 \left[\mathrm{d}\psi^2 + \sin^2\psi \left(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2\right)\right]$$

Review from previous class, but enlarged:

Implications of General Relativity

- ☆ $ds^2 = R^2\left[d\psi^2 + \sin^2\psi\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$, where R is radius of curvature.
- ☆ According to GR, matter causes space to curve. So R, the curvature radius, should be determined by the matter.
- ☆ From the metric, or from the picture of a sphere of radius R in a 4D Euclidean embedding space, it is clear that R determines the size of the space. But $a(t)$, the scale factor, also determines the size of the space. So they must be proportional.
- ☆ But R is in meters, $a(t)$ in meters/notch. So dimensional consistency $\Longrightarrow$ $R \propto a(t)/\sqrt{k}$, since $[k] = \text{notch}^{-2}$.
- ☆ In fact,

$$R^2(t) = \frac{a^2(t)}{k} \ .$$

(I do not know any way to explain why the proportionality constant is 1, except by using the full equations of GR.)

- $ds^2 = R^2 \left[d\psi^2 + \sin^2\psi \left(d\theta^2 + \sin^2\theta \, d\phi^2\right)\right]$, where R is radius of curvature.
- In fact,

$$R^2(t) = \frac{a^2(t)}{k} \ .$$

- So,

$$ds^2 = \frac{a^2(t)}{k} \left[d\psi^2 + \sin^2\psi \left(d\theta^2 + \sin^2\theta \, d\phi^2\right)\right] \ .$$

- It is common to introduce a new radial variable $r \equiv \sin\psi/\sqrt{k}$, so $dr = \cos\psi \, d\psi/\sqrt{k} = \sqrt{1-kr^2} \, d\psi/\sqrt{k}$. In terms of r,

$$ds^2 = a^2(t) \left\{ \frac{dr^2}{1-kr^2} + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right) \right\} \ .$$

This is the spatial part of the Robertson-Walker metric.

Open Universes

- ☆ For $k > 0$ (closed universe),

$$ds^2 = a^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right\}$$

describes a homogeneous isotropic universe.

- ☆ For $k < 0$ (open universe),

$$ds^2 = a^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right\}$$

still describes a homogeneous isotropic universe.

- ☆ Properties are very different. The closed universe reaches its equator at $r = 1/\sqrt{k}$, which is a finite distance from the origin,

$$a(t)\int_0^{1/\sqrt{k}} \frac{dr}{\sqrt{1-kr^2}} = \frac{\pi a(t)}{2\sqrt{k}}\ .$$

The total volume is finite. For the open universe, r has no limit, and the volume is infinite.

From Space to Spacetime

In special relativity,

$$s_{AB}^2 \equiv (x_A - x_B)^2 + (y_A - y_B)^2 + (z_A - z_B)^2 - c^2 (t_A - t_B)^2 \ .$$

s_{AB}^2 is Lorentz-invariant — it has the same value for all inertial reference frames.

From Space to Spacetime

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s_{AB}^2 is Lorentz-invariant — it has the same value for all inertial reference frames.

Meaning of s_{AB}^2:

If positive, it is the distance^2 between the two events in the inertial frame in which they are simultaneous. (Spacelike.)

If negative, then $s_{AB}^2 = -c^2\,\Delta\tau^2$, where $\Delta\tau$ is the time interval between the two events in the inertial frame in which they occur at the same place. (Timelike.)

If zero, it implies that a light pulse could travel from the earlier to the later event. (Lightlike.)

From Space to Spacetime

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$$s_{AB}^2 \equiv (x_A - x_B)^2 + (y_A - y_B)^2 + (z_A - z_B)^2 - c^2 (t_A - t_B)^2 \ .$$

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If zero, it implies that a light pulse could travel from the earlier to the later event. (Lightlike.)

☆ If you are interested, Lecture Notes 5 has an appendix which derives the Lorentz transformation from time dilation, Lorentz contraction, and the relativity of simultaneity, and shows that s_{AB}^2 is invariant.

Infinitesimal Separations and the Metric

- Following Gauss, we focus on the distance between infinitesimally separated points. So

$$s_{AB}^2 \equiv (x_A - x_B)^2 + (y_A - y_B)^2 + (z_A - z_B)^2 - c^2\,(t_A - t_B)^2$$

is replaced by

$$\boxed{\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 - c^2\,\mathrm{d}t^2\ ,}$$

which is called the *Minkowski metric.*

- The interpretation is the same as before: $\mathrm{d}s^2 > 0 \implies$ distance^2 in frame where events are simultaneous; $\mathrm{d}s^2 < 0 \implies \mathrm{d}s^2 = -c^2\,\mathrm{d}\tau^2$, where $\mathrm{d}\tau$ = time difference in frame where events are at same place; $\mathrm{d}s^2 = 0 \implies$ light can travel from one event to the other.

- This will be our springboard to metric used in general relativity.

Coordinates in Curves Spaces

- In Newtonian physics or special relativity, coordinates have a direct physical meaning: they directly measure distances or time intervals.
- In curves spaces, there is generally no way to construct coordinates that are directly connected to distances.
- For example, on the surface of the Earth we measure East-West position by longitude, but the distance for a longitude distance of 1 degree depends on the latitude.
- Bottom line: in general relativity (or in any curved space), coordinates are just arbitrary markers, with any set of coordinates in principle as good as any other.
- Distances are determined from the coordinates, using the metric.
- If one changes from one coordinate system to another, one changes the metric so that distances remain unchanged.

General Relativity: the Equivalence Principle and Free-Falling Observers

- Consider a person holding a rock inside an elevator, initially at rest.

General Relativity: the Equivalence Principle and Free-Falling Observers

☆ Consider a person holding a rock inside an elevator, initially at rest. The person feels the force of gravity pulling down on the rock, and the force of gravity pressing his feet against the floor.

General Relativity: the Equivalence Principle and Free-Falling Observers

- ☆ Consider a person holding a rock inside an elevator, initially at rest. The person feels the force of gravity pulling down on the rock, and the force of gravity pressing his feet against the floor.
- ☆ Now imagine that the elevator cable is cut, so the elevator falls — we assume that there is no friction or air resistance.

General Relativity: the Equivalence Principle and Free-Falling Observers

- Consider a person holding a rock inside an elevator, initially at rest. The person feels the force of gravity pulling down on the rock, and the force of gravity pressing his feet against the floor.

- Now imagine that the elevator cable is cut, so the elevator falls — we assume that there is no friction or air resistance. The elevator, person, and rock all accelerate together. The person no longer feels his feet pressed to the floor; if he lets go of the rock, it floats. The effects of gravity have disappeared.

General Relativity: the Equivalence Principle and Free-Falling Observers

- Consider a person holding a rock inside an elevator, initially at rest. The person feels the force of gravity pulling down on the rock, and the force of gravity pressing his feet against the floor.

- Now imagine that the elevator cable is cut, so the elevator falls — we assume that there is no friction or air resistance. The elevator, person, and rock all accelerate together. The person no longer feels his feet pressed to the floor; if he lets go of the rock, it floats. The effects of gravity have disappeared.

- The *Equivalence Principle* says that the disappearance of gravity is precise: as long as the elevator is small enough so that the gravitational field is uniform, then there is absolutely no way that the person in the free-falling elevator can detect the gravitational field of the Earth.

- The *Equivalence Principle* says that the disappearance of gravity is precise: as long as the elevator is small enough so that the gravitational field is uniform, then there is absolutely no way that the person in the free-falling elevator can detect the gravitational field of the Earth.

- The person in the elevator is called a *free-falling observer*, and the local coordinate system that he would construct in his immediate vicinity is called a free-falling coordinate system. The metric for the free-falling coordinates, in the immediate vicinity of the person, is described by the Minkowski metric. It is called *locally Minkowskian*.

- We mentioned earlier that any quadratic metric for space (i.e., a positive definite metric) is locally Euclidean. If the metric is negative for one direction, then it is always locally Minkowskian.

Adding Time to the Robertson-Walker Metric

$$ds^2 = -c^2\,dt^2 + a^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\} \ .$$

Why does dt^2 term look like it does:

- The coefficient of dt^2 term must be independent of position, due to homogeneity.
- Terms such as $dt\,dr$ or $dt\,d\phi$ cannot appear, due to isotropy. That is, a term $dt\,dr$ would behave differently for $dr > 0$ and $dr < 0$, creating an asymmetry between the $+r$ and -r directions.
- The coefficient must be negative, to match the sign in Minkowski space for a locally free-falling coordinate system.

Adding Time to the Robertson-Walker Metric

$$ds^2 = -c^2\,dt^2 + a^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\} .$$

Meaning:

- If $ds^2 > 0$, it is the square of the spatial separation measured by a local free-falling observer for whom the two events happen at the same time.
- If $ds^2 < 0$, it is $-c^2$ times the square of the time separation measured by a local free-falling observer for whom the two events happen at the same location.
- If $ds^2 = 0$, then the two events can be joined by a light pulse.

Summary: Metrics of Interest

Minkowski Metric: (Special relativity)

$$
\begin{aligned}
ds^2 &= -c^2 dt^2 + dx^2 + dy^2 + dz^2 \\
&= -c^2 dt^2 + dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2) \; .
\end{aligned}
$$

Robertson–Walker Metric:

$$
ds^2 = -c^2 \, dt^2 + a^2(t) \left\{ \frac{dr^2}{1 - kr^2} + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right) \right\} \; .
$$

Meaning: If $ds^2 > 0$, ds is distance in freely falling frame in which events are simultaneous. If $ds^2 < 0$, $ds^2 = -c^2 d\tau^2$, where $d\tau$ is time interval in freely falling frame in which events occur at same point. If $ds^2 = 0$, events are lightlike separated.

Geodesics in General Relativity

A geodesic is a path connecting two points in spacetime, with the property that the length of the curve is stationary with respect to small changes in the path. It can be a maximum, minimum, or saddle point.

In a curved spacetime, a geodesic is the closest thing to a straight line that exists.

In general relativity, if no forces act on a particle other than gravity, the particle travels on a geodesic.

Geodesics in Two Spatial Dimensions

Metric:

$$ds^2 = g_{xx}dx^2 + g_{xy}dx\,dy + g_{yx}dy\,dx + g_{yy}dy^2 \ .$$

Let $x^1 \equiv x$, $x^2 \equiv y$, so x^i is either, as $i = 1$ or 2.

$$ds^2 = \sum_{i=1}^{2} \sum_{j=1}^{2} g_{ij}(x^k)\,dx^i\,dx^j$$

$$= g_{ij}(x^k)\,dx^i\,dx^j \ .$$

Einstein summation convention: repeated indices within one term are summed over coordinate indices (1 and 2), unless otherwise specified.

The sum is always over one upper index and one lower, but we will not discuss why some indices are written as upper and some as lower.

$g_{ij}\left(x^k\right)$ indicates that g_{ij} is a function of all the components of x^k, i.e., x^1 and x^2.

The Length of Path

Consider a path from A to B.

Path description: $x^i(\lambda)$, where λ is parameter running from 0 to λ_f.

$$x^i(0) = x^i_A, \qquad x^i(\lambda_f) = x^i_B \ .$$

Between λ and $\lambda + \mathrm{d}\lambda$,

$$\mathrm{d}x^i = \frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\mathrm{d}\lambda \ ,$$

so

$$\mathrm{d}s^2 = g_{ij}\left(x^k(\lambda)\right) \frac{\mathrm{d}x^i}{\mathrm{d}\lambda} \frac{\mathrm{d}x^j}{\mathrm{d}\lambda} \mathrm{d}\lambda^2 \ ,$$

and then

$$\mathrm{d}s = \sqrt{g_{ij}\left(x^k(\lambda)\right) \frac{\mathrm{d}x^i}{\mathrm{d}\lambda} \frac{\mathrm{d}x^j}{\mathrm{d}\lambda}} \mathrm{d}\lambda \ ,$$

and

$$S[x^i(\lambda)] = \int_0^{\lambda_f} \sqrt{g_{ij}\left(x^k(\lambda)\right) \frac{\mathrm{d}x^i}{\mathrm{d}\lambda} \frac{\mathrm{d}x^j}{\mathrm{d}\lambda}} \mathrm{d}\lambda \ .$$

8.286 Class 12
October 14, 2020

INTRODUCTION TO NON-EUCLIDEAN SPACES PART 4

(Modified 10/22/20 to correct some indices on pp. 9, 10, 11, and 16.)

Announcements

Reminder: Problem Set 5 is due this Friday at 5:00 pm.

Review from last class.

Summary: Metrics of Interest

Minkowski Metric: (Special relativity)

$$\begin{aligned} ds^2 &= -c^2 dt^2 + dx^2 + dy^2 + dz^2 \\ &= -c^2 dt^2 + dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2) \ . \end{aligned}$$

Robertson-Walker Metric:

$$ds^2 = -c^2 \, dt^2 + a^2(t) \left\{ \frac{dr^2}{1 - kr^2} + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right) \right\} \ .$$

Meaning: If $ds^2 > 0$, ds is distance in freely falling frame in which events are simultaneous. If $ds^2 < 0$, $ds^2 = -c^2 d\tau^2$, where $d\tau$ is time interval in freely falling frame in which events occur at same point. If $ds^2 = 0$, events are lightlike separated.

General Relativity: the Equivalence Principle and Free-Falling Observers

- Consider a person holding a rock inside an elevator, initially at rest. The person feels the force of gravity pulling down on the rock, and the force of gravity pressing his feet against the floor.

- Now imagine that the elevator cable is cut, so the elevator falls — we assume that there is no friction or air resistance. The elevator, person, and rock all accelerate together. The person no longer feels his feet pressed to the floor; if he lets go of the rock, it floats. The effects of gravity have disappeared.

- The *Equivalence Principle* says that the disappearance of gravity is precise: as long as the elevator is small enough so that the gravitational field is uniform, then there is absolutely no way that the person in the free-falling elevator can detect the gravitational field of the Earth.

Review from last class.

- The *Equivalence Principle* says that the disappearance of gravity is precise: as long as the elevator is small enough so that the gravitational field is uniform, then there is absolutely no way that the person in the free-falling elevator can detect the gravitational field of the Earth.

- The person in the elevator is called a *free-falling observer*, and the local coordinate system that he would construct in his immediate vicinity is called a *free-falling coordinate system.* The metric for the free-falling coordinates, in the immediate vicinity of the person, is described by the Minkowski metric. It is called *locally Minkowskian.*

- We mentioned earlier that any quadratic metric for space (i.e., a positive definite metric) is locally Euclidean. If the metric is negative for one direction, then it is always locally Minkowskian.

- ☆ The *Equivalence Principle* says that the disappearance of gravity is precise: as long as the elevator is small enough so that the gravitational field is uniform, then there is absolutely no way that the person in the free-falling elevator can detect the gravitational field of the Earth.

- ☆ The person in the elevator is called a *free-falling observer*, and the local coordinate system that he would construct in his immediate vicinity is called a *free-falling coordinate system.* The metric for the free-falling coordinates, in the immediate vicinity of the person, is described by the Minkowski metric. It is called *locally Minkowskian.*

- ☆ We mentioned earlier that any quadratic metric for space (i.e., a positive definite metric) is locally Euclidean. If the metric is negative for one direction, then it is always locally Minkowskian.

- ☆ **(Added today):** Not as simple as it sounds! If you calculate the bending of a light beam by gravity this way, you will get only half the GR answer. The correct free-falling coordinate system is not just an accelerating version of a Euclidean coordinate system, but also takes into account the bending of space caused by gravity (in GR).

Geodesics in General Relativity

A geodesic is a path connecting two points in spacetime, with the property that the length of the curve is stationary with respect to small changes in the path. It can be a maximum, minimum, or saddle point.

In a curved spacetime, a geodesic is the closest thing to a straight line that exists.

In general relativity, if no forces act on a particle other than gravity, the particle travels on a geodesic.

Review from last class.

Geodesics in Two Spatial Dimensions

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Let $x^1 \equiv x$, $x^2 \equiv y$, so x^i is either, as $i = 1$ or 2.

$$ds^2 = \sum_{i=1}^{2} \sum_{j=1}^{2} g_{ij}(x^k)\, dx^i\, dx^j$$

$$= g_{ij}(x^k)\, dx^i\, dx^j \ .$$

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Review from last class.

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Path description: $x^i(\lambda)$, where λ is parameter running from 0 to λ_f.

$$x^i(0) = x^i_A, \qquad x^i(\lambda_f) = x^i_B \ .$$

Between λ and $\lambda + \mathrm{d}\lambda$,

$$\mathrm{d}x^i = \frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\mathrm{d}\lambda \ ,$$

so

$$\mathrm{d}s^2 = \boldsymbol{g_{ij}(x^k)\,\mathrm{d}x^i\,\mathrm{d}x^j} = g_{ij}\left(x^k(\lambda)\right)\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\,\mathrm{d}\lambda^2 \ ,$$

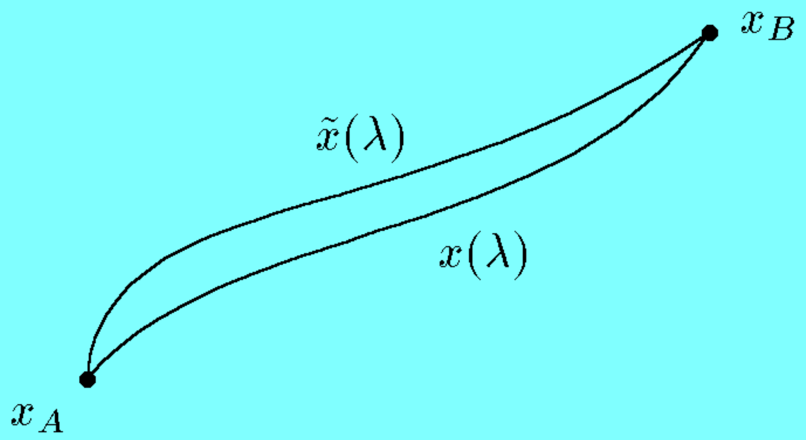
and then

$$\mathrm{d}s = \sqrt{g_{ij}\left(x^k(\lambda)\right)\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}}\,\mathrm{d}\lambda \ ,$$

and

$$S[x^i(\lambda)] = \int_0^{\lambda_f}\sqrt{g_{ij}\left(x^k(\lambda)\right)\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}}\,\mathrm{d}\lambda \ .$$

Varying the Path



$$\tilde{x}^i(\lambda) = x^i(\lambda) + \alpha w^i(\lambda) \ ,$$

where

$$w^i(0) = 0 \ , \qquad w^i(\lambda_f) = 0 \ .$$

Geodesic condition:

$$\left. \frac{\mathrm{d}\, S\left[\tilde{x}^i(\lambda)\right]}{\mathrm{d}\alpha} \right|_{\alpha=0} = 0 \qquad \text{for all } w^i(\lambda) \quad .$$

$$\tilde{x}^i(\lambda) = x^i(\lambda) + \alpha w^i(\lambda) \ .$$

$$S\left[\tilde{x}^i(\lambda)\right] = \int_0^{\lambda_f} \sqrt{g_{ij}\big(x^k(\lambda)\big)\,\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}}\,\mathrm{d}\lambda \ .$$

Define

$$A(\lambda,\alpha) = g_{ij}\left(\tilde{x}^k(\lambda)\right)\frac{\mathrm{d}\tilde{x}^i}{\mathrm{d}\lambda}\frac{\mathrm{d}\tilde{x}^j}{\mathrm{d}\lambda} \ ,$$

so we can write

$$S\left[\tilde{x}^i(\lambda)\right] = \int_0^{\lambda_f} \sqrt{A(\lambda,\alpha)}\,\mathrm{d}\lambda \ .$$

Using chain rule, $\dfrac{\mathrm{d}f\big(x(\alpha),y(\alpha)\big)}{\mathrm{d}\alpha} = \dfrac{\partial f(x,y)}{\partial x}\dfrac{\mathrm{d}x(\alpha)}{\mathrm{d}\alpha} + \dfrac{\partial f(x,y)}{\partial y}\dfrac{\mathrm{d}y(\alpha)}{\mathrm{d}\alpha}$,

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}g_{ij}\left(\tilde{x}^k(\lambda)\right)\bigg|_{\alpha=0} = \left[\frac{\partial g_{ij}}{\partial \tilde{x}^k}\frac{\partial \tilde{x}^k}{\partial \alpha}\right]_{\alpha=0} = \frac{\partial g_{ij}}{\partial x^k}\big(x^\ell(\lambda)\big)\,\frac{\partial \tilde{x}^k}{\partial \alpha}\bigg|_{\alpha=0} = \frac{\partial g_{ij}}{\partial x^k}\big(x^\ell(\lambda)\big)w^k \ ,$$

$$\tilde{x}^i(\lambda) = x^i(\lambda) + \alpha w^i(\lambda) \ .$$

$$A(\lambda, \alpha) = g_{ij}\left(\tilde{x}^k(\lambda)\right) \frac{\mathrm{d}\tilde{x}^i}{\mathrm{d}\lambda} \frac{\mathrm{d}\tilde{x}^j}{\mathrm{d}\lambda} \ .$$

Using chain rule, $\dfrac{\mathrm{d} f\left(x(\alpha), y(\alpha)\right)}{\mathrm{d}\alpha} = \dfrac{\partial f(x,y)}{\partial x} \dfrac{\mathrm{d}x(\alpha)}{\mathrm{d}\alpha} + \dfrac{\partial f(x,y)}{\partial y} \dfrac{\mathrm{d}y(\alpha)}{\mathrm{d}\alpha}$,

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} g_{ij}\left(\tilde{x}^k(\lambda)\right)\bigg|_{\alpha=0} = \left[\frac{\partial g_{ij}}{\partial \tilde{x}^k} \frac{\partial \tilde{x}^k}{\partial \alpha}\right]_{\alpha=0} = \frac{\partial g_{ij}}{\partial x^k}\left(x^\ell(\lambda)\right) \left.\frac{\partial \tilde{x}^k}{\partial \alpha}\right|_{\alpha=0} = \frac{\partial g_{ij}}{\partial x^k}\left(x^\ell(\lambda)\right) w^k \ .$$

Furthermore,

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\left(\frac{\mathrm{d}\tilde{x}^i}{\mathrm{d}\lambda}\right) = \frac{\mathrm{d}}{\mathrm{d}\alpha}\left[\frac{\mathrm{d}x^i(\lambda)}{\mathrm{d}\lambda} + \alpha \frac{\mathrm{d}w^i(\lambda)}{\mathrm{d}\lambda}\right] = \frac{\mathrm{d}w^i(\lambda)}{\mathrm{d}\lambda} \ .$$

$$S\left[\tilde{x}^i(\lambda)\right] = \int_0^{\lambda_f} \sqrt{A(\lambda,\alpha)}\,\mathrm{d}\lambda\ ,$$

where

$$A(\lambda,\alpha) = g_{ij}\left(\tilde{x}^k(\lambda)\right)\frac{\mathrm{d}\tilde{x}^i}{\mathrm{d}\lambda}\frac{\mathrm{d}\tilde{x}^j}{\mathrm{d}\lambda}\ ,$$

with

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} g_{ij}\left(\tilde{x}^k(\lambda)\right)\Big|_{\alpha=0} = \frac{\partial g_{ij}}{\partial x^k}\left(x^\ell(\lambda)\right)w^k\ , \qquad \frac{\mathrm{d}}{\mathrm{d}\alpha}\left(\frac{\mathrm{d}\tilde{x}^i}{\mathrm{d}\lambda}\right) = \frac{\mathrm{d}w^i(\lambda)}{\mathrm{d}\lambda}\ .$$

Then

$$\frac{\mathrm{d}S\left[\tilde{x}^i(\lambda)\right]}{\mathrm{d}\alpha}\Bigg|_{\alpha=0} = \frac{1}{2}\int_0^{\lambda_f}\frac{1}{\sqrt{A(\lambda,0)}}\left\{\frac{\partial g_{ij}}{\partial x^k}w^k\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda} + \right.$$

$$\left. + g_{ij}\frac{\mathrm{d}w^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda} + g_{ij}\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}w^j}{\mathrm{d}\lambda}\right\}\mathrm{d}\lambda\ ,$$

where the metric g_{ij} is to be evaluated at $x^\ell(\lambda)$.

$$\left.\frac{\mathrm{d}S\left[\tilde{x}^i(\lambda)\right]}{\mathrm{d}\alpha}\right|_{\alpha=0} = \frac{1}{2}\int_0^{\lambda_f} \frac{1}{\sqrt{A(\lambda,0)}} \left\{ \frac{\partial g_{ij}}{\partial x^k} w^k \frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda} + \right.$$

$$\left. + g_{ij}\frac{\mathrm{d}w^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda} + g_{ij}\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}w^j}{\mathrm{d}\lambda} \right\} \mathrm{d}\lambda \ .$$

Manipulating "dummy" indices: in third term, replace $i \to j$ and $j \to i$, and recall that $g_{ij} = g_{ji}$. Then 2nd & 3rd term are equal:

$$\left.\frac{\mathrm{d}S\left[\tilde{x}^i(\lambda)\right]}{\mathrm{d}\alpha}\right|_{\alpha=0} = \frac{1}{2}\int_0^{\lambda_f} \frac{1}{\sqrt{A(\lambda,0)}} \left\{ \frac{\partial g_{ij}}{\partial x^k} w^k \frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda} + 2 g_{ij}\frac{\mathrm{d}w^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda} \right\} \mathrm{d}\lambda \ .$$

Repeating,

$$\left.\frac{\mathrm{d}S\left[\tilde{x}^i(\lambda)\right]}{\mathrm{d}\alpha}\right|_{\alpha=0} = \frac{1}{2}\int_0^{\lambda_f} \frac{1}{\sqrt{A(\lambda,0)}}\left\{\frac{\partial g_{ij}}{\partial x^k} w^k \frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda} + 2g_{ij}\frac{\mathrm{d}w^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\right\}\mathrm{d}\lambda\ .$$

Integration by Parts: Integral depends on both w^k and $\mathrm{d}w^i/\mathrm{d}\lambda$. Can eliminate $\mathrm{d}w^i/\mathrm{d}\lambda$ by integrating by parts:

$$\int_0^{\lambda_f}\left[\frac{1}{\sqrt{A}}g_{ij}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\right]\frac{\mathrm{d}w^i}{\mathrm{d}\lambda}\,\mathrm{d}\lambda = \int_0^{\lambda_f}\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\frac{1}{\sqrt{A}}g_{ij}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}w^i\right]\mathrm{d}\lambda$$
$$- \int_0^{\lambda_f}\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\frac{1}{\sqrt{A}}g_{ij}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\right]w^i\,\mathrm{d}\lambda\ .$$

But

$$\int_0^{\lambda_f}\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\frac{1}{\sqrt{A}}g_{ij}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}w^i\right]\mathrm{d}\lambda = \left.\left[\frac{1}{\sqrt{A}}g_{ij}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}w^i\right]\right|_{\lambda=0}^{\lambda=\lambda_f} = 0\ ,$$

since $w^i(\lambda)$ vanishes at $\lambda=0$ and $\lambda=\lambda_f$.

$$\left.\frac{\mathrm{d}S\left[\tilde{x}^i(\lambda)\right]}{\mathrm{d}\alpha}\right|_{\alpha=0} = \frac{1}{2}\int_0^{\lambda_f} \frac{1}{\sqrt{A(\lambda,0)}}\left\{\frac{\partial g_{ij}}{\partial x^k} w^k \frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda} + 2 g_{ij}\frac{\mathrm{d}w^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\right\}\mathrm{d}\lambda\ .$$

$$\left.\frac{\mathrm{d}S}{\mathrm{d}\alpha}\right|_{\alpha=0} = \frac{1}{2}\int_0^{\lambda_f} \left\{\frac{1}{\sqrt{A}}\frac{\partial g_{ij}}{\partial x^k}\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda} w^k - 2\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\frac{1}{\sqrt{A}} g_{ij}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\right] w^i\right\}\mathrm{d}\lambda\ .$$

$$\left.\frac{\mathrm{d}S}{\mathrm{d}\alpha}\right|_{\alpha=0} = \frac{1}{2}\int_0^{\lambda_f} \left\{ \frac{1}{\sqrt{A}} \frac{\partial g_{ij}}{\partial x^k} \frac{\mathrm{d}x^i}{\mathrm{d}\lambda} \frac{\mathrm{d}x^j}{\mathrm{d}\lambda} w^k - 2 \frac{\mathrm{d}}{\mathrm{d}\lambda} \left[\frac{1}{\sqrt{A}} g_{ij} \frac{\mathrm{d}x^j}{\mathrm{d}\lambda} \right] w^i \right\} \mathrm{d}\lambda \; .$$

Complication: one term is proportional to w^k, and the other is proportional to w^i. But with more index juggling, we can fix that. In 1st term replace $i \to j, j \to k, k \to i$:

$$\left.\frac{\mathrm{d}S}{\mathrm{d}\alpha}\right|_{\alpha=0} = \int_0^{\lambda_f} \left\{ \frac{1}{2\sqrt{A}} \frac{\partial g_{jk}}{\partial x^i} \frac{\mathrm{d}x^j}{\mathrm{d}\lambda} \frac{\mathrm{d}x^k}{\mathrm{d}\lambda} - \frac{\mathrm{d}}{\mathrm{d}\lambda} \left[\frac{1}{\sqrt{A}} g_{ij} \frac{\mathrm{d}x^j}{\mathrm{d}\lambda} \right] \right\} w^i(\lambda)\, \mathrm{d}\lambda \; .$$

To vanish ***for all*** $\boldsymbol{w^i(\lambda)}$ which vanish at $\lambda = 0$ and $\lambda = \lambda_f$, the quantity in curly brackets must vanish. If not, then suppose that $\{\} \neq 0$ at some $\lambda = \lambda_0$. By continuity, $\{\} \neq 0$ in some neighborhood of λ_0. Choose $w^i(\lambda)$ to be positive in this neighborhood, and zero everywhere else, and one has a contradiction.

So

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \left[\frac{1}{\sqrt{A}} g_{ij} \frac{\mathrm{d}x^j}{\mathrm{d}\lambda} \right] = \frac{1}{2\sqrt{A}} \frac{\partial g_{jk}}{\partial x^i} \frac{\mathrm{d}x^j}{\mathrm{d}\lambda} \frac{\mathrm{d}x^k}{\mathrm{d}\lambda} \quad .$$

Repeating,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\frac{1}{\sqrt{A}} g_{ij} \frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\right] = \frac{1}{2\sqrt{A}} \frac{\partial g_{jk}}{\partial x^i} \frac{\mathrm{d}x^j}{\mathrm{d}\lambda} \frac{\mathrm{d}x^k}{\mathrm{d}\lambda} \quad .$$

This is ***complicated,*** since A is complicated.

Simplify by choice of parameterization: This result is valid for any parameterization. We don't need that! We can choose λ to be the path length. Since

$$\mathrm{d}s = \sqrt{g_{ij}\left(x^\ell(\lambda)\right) \frac{\mathrm{d}x^i}{\mathrm{d}\lambda} \frac{\mathrm{d}x^j}{\mathrm{d}\lambda}}\, \mathrm{d}\lambda = \sqrt{A}\, \mathrm{d}\lambda \ ,$$

we see that $\mathrm{d}\lambda = \mathrm{d}s$ implies

$$A = 1 \qquad (\text{for } \lambda = \text{path length}).$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[g_{ij} \frac{\mathrm{d}x^j}{\mathrm{d}s}\right] = \frac{1}{2} \frac{\partial g_{jk}}{\partial x^i} \frac{\mathrm{d}x^j}{\mathrm{d}s} \frac{\mathrm{d}x^k}{\mathrm{d}s} \quad .$$

Alternative Form of Geodesic Equation

Most books write the geodesic equation differently, as

$$\frac{d^2x^i}{ds^2} = -\Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} \ ,$$

where

$$\Gamma^i_{jk} = \frac{1}{2}g^{i\ell}\left(\partial_j g_{\ell k} + \partial_k g_{\ell j} - \partial_\ell g_{jk}\right)$$

and $g^{i\ell}$ is the matrix inverse of g_{ij}. The quantity Γ^i_{jk} is called the affine connection.

If you are interested, see the lecture notes. If you are not interested, you can skip this.

BLACK HOLES (Fun!)

The Schwarzschild Metric:

For any spherically symmetric distribution of mass, outside the mass the metric is given by the Schwarzschild metric,

$$ds^2 = -c^2 d\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2$$
$$+ r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \; ,$$

where M is the total mass, G is Newton's gravitational constant, c is the speed of light, and θ and ϕ have the usual polar-angle ranges.

Schwarzschild Horizon

$$ds^2 = -c^2 d\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \ .$$

The metric is singular at

$$r = R_S \equiv \frac{2GM}{c^2} \ ,$$

where the coefficient of $c^2 dt^2$ vanishes, and the coefficient of dr^2 is infinite.

Surprisingly, this singularity is not real — it is a coordinate artifact. There are other coordinate systems where the metric is smooth at R_S.

But R_S is a ***horizon:*** If you fall past the horizon, there is no return, even if you are photon.

Schwarzschild Radius of the Sun

$$R_{S,\odot} = \frac{2GM}{c^2}$$

$$= \frac{2 \times 6.673 \times 10^{-11}\ \text{m}^3\text{-kg}^{-1}\text{-s}^{-2} \times 1.989 \times 10^{30}\ \text{kg}}{(2.998 \times 10^{8}\ \text{m-s}^{-1})^2}$$

$$= 2.95\ \text{km}\ .$$

- ☆ If the Sun were compressed to this radius, it would become a black hole. Since the Sun is much larger than R_S, and the Schwarzschild metric is only valid outside the matter, there is no Schwarzschild horizon in the Sun.
- ☆ At the center of our galaxy is a supermassive black hole, with $M = 4.1 \times 10^6\ M_\odot$. This gives $R_S = 1.2 \times 10^{10}$ meters $\approx 1/4$ of radius of orbit of Mercury ≈ 17 times radius of Sun.

Radial Geodesics in the Schwarzschild Metric

$$\mathrm{d}s^2 = -c^2\mathrm{d}\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right)c^2\mathrm{d}t^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1}\mathrm{d}r^2$$
$$+ r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2 \;.$$

Consider a particle released from rest at $r = r_0$.

r is a "radial coordinate," but not the radius, since it is not the distance from some center. If r is varied by $\mathrm{d}r$, the distance traveled is not $\mathrm{d}r$, but $\mathrm{d}r/\sqrt{1 - 2GM/rc^2}$. r can be called the "circumferential radius," since the term $r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)$ in the metric implies that the circumference of a circle about the origin is $2\pi r$.

By symmetry, the particle will fall straight down, with no change in θ or ϕ. Spherical symmetry implies that all directions in θ and ϕ are equivalent, so any motion in θ–ϕ space would violate this symmetry.

Particle Trajectories in Spacetime

Particle trajectories are timelike, so we use proper time τ to parameterize them, where $\mathrm{d}s^2 \equiv -c^2\mathrm{d}\tau^2$. This implies that $A = -c^2$, instead of $A = 1$, but as long as A is constant, it drops out of the geodesic equation.

By tradition, the spacetime indices in general relativity are denoted by Greek letters such as μ, ν, λ, σ, and are summed from 0 to 3, where $x^0 \equiv t$.

The geodesic equation

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[g_{ij}\frac{\mathrm{d}x^j}{\mathrm{d}s}\right] = \frac{1}{2}\frac{\partial g_{jk}}{\partial x^i}\frac{\mathrm{d}x^j}{\mathrm{d}s}\frac{\mathrm{d}x^k}{\mathrm{d}s}$$

is then rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[g_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right] = \frac{1}{2}\frac{\partial g_{\lambda\sigma}}{\partial x^\mu}\frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau} .$$

8.286 Class 13
October 19, 2020

INTRODUCTION TO NON-EUCLIDEAN SPACES PART 5

BLACK-BODY RADIATION AND THE EARLY HISTORY OF THE UNIVERSE

Announcements

Reminder: Problem Set 6 is due this Friday at 5:00 pm.

Quiz 2 will be next Wednesday, October 28. Procedures will be the same as for Quiz 1. Precise coverage will be announced soon. Lecture Notes 6 will be included only through the *Dynamics of a Flat Radiation-Dominated Universe*, ending at the top of p. 12.

BLACK HOLES (Fun!)

The Schwarzschild Metric:

For any spherically symmetric distribution of mass, outside the mass the metric is given by the Schwarzschild metric,

$$ds^2 = -c^2 d\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2) \,,$$

where M is the total mass, G is Newton's gravitational constant, c is the speed of light, and θ and ϕ have the usual polar-angle ranges.

Review from last class

Schwarzschild Horizon

$$ds^2 = -c^2 d\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2) \ .$$

The metric is singular at

$$r = R_S \equiv \frac{2GM}{c^2} \ ,$$

where the coefficient of $c^2 dt^2$ vanishes, and the coefficient of dr^2 is infinite.

Surprisingly, this singularity is not real — it is a coordinate artifact. There are other coordinate systems where the metric is smooth at R_S.

But R_S is a ***horizon:*** If you fall past the horizon, there is no return, even if you are photon.

Schwarzschild Radius of the Sun

$$R_{S,\odot} = \frac{2GM}{c^2}$$

$$= \frac{2 \times 6.673 \times 10^{-11}\ \text{m}^3\text{-kg}^{-1}\text{-s}^{-2} \times 1.989 \times 10^{30}\ \text{kg}}{(2.998 \times 10^8\ \text{m-s}^{-1})^2}$$

$$= 2.95\ \text{km} .$$

- ☆ If the Sun were compressed to this radius, it would become a black hole. Since the Sun is much larger than R_S, and the Schwarzschild metric is only valid outside the matter, there is no Schwarzschild horizon in the Sun.
- ☆ At the center of our galaxy is a supermassive black hole, with $M = 4.1 \times 10^6\ M_\odot$. This gives $R_S = 1.2 \times 10^{10}$ meters $\approx 1/4$ of radius of orbit of Mercury ≈ 17 times radius of Sun.

Review from last class

Radial Geodesics in the Schwarzschild Metric

$$ds^2 = -c^2 d\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \ .$$

Consider a particle released from rest at $r = r_0$.

r is a "radial coordinate," but not the radius, since it is not the distance from some center. If r is varied by dr, the distance traveled is not dr, but $dr/\sqrt{1 - 2GM/rc^2}$. r can be called the "circumferential radius," since the term $r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$ in the metric implies that the circumference of a circle about the origin is $2\pi r$.

By symmetry, the particle will fall straight down, with no change in θ or ϕ. Spherical symmetry implies that all directions in θ and ϕ are equivalent, so any motion in θ–ϕ space would violate this symmetry.

Review from last class

Particle Trajectories in Spacetime

Particle trajectories are timelike, so we use proper time τ to parameterize them, where $\mathrm{d}s^2 \equiv -c^2 \mathrm{d}\tau^2$. This implies that $A = -c^2$, instead of $A = 1$, but as long as A is constant, it drops out of the geodesic equation.

By tradition, the spacetime indices in general relativity are denoted by Greek letters such as μ, ν, λ, σ, and are summed from 0 to 3, where $x^0 \equiv t$.

The geodesic equation

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[g_{ij}\frac{\mathrm{d}x^j}{\mathrm{d}s}\right] = \frac{1}{2}\frac{\partial g_{jk}}{\partial x^i}\frac{\mathrm{d}x^j}{\mathrm{d}s}\frac{\mathrm{d}x^k}{\mathrm{d}s}$$

is then rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[g_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right] = \frac{1}{2}\frac{\partial g_{\lambda\sigma}}{\partial x^\mu}\frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau} .$$

Radial Trajectory Equations

Only $\mathrm{d}r/\mathrm{d}\tau$ and $\mathrm{d}t/\mathrm{d}\tau$ are nonzero. But they are related by the metric:

$$c^2\mathrm{d}\tau^2 = \left(1 - \frac{2GM}{rc^2}\right) c^2\,\mathrm{d}t^2 - \left(1 - \frac{2GM}{rc^2}\right)^{-1} \mathrm{d}r^2$$

implies that

$$c^2 = \left(1 - \frac{2GM}{rc^2}\right) c^2 \left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 - \left(1 - \frac{2GM}{rc^2}\right)^{-1} \left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 \; .$$

Then, looking at the $\mu = r$ geodesic equation,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[g_{\mu\nu}\,\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right] = \frac{1}{2}\frac{\partial g_{\lambda\sigma}}{\partial x^\mu}\,\frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau}$$

implies that

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[g_{rr}\frac{\mathrm{d}r}{\mathrm{d}\tau}\right] = \frac{1}{2}\partial_r g_{rr}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + \frac{1}{2}\partial_r g_{tt}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 \; ,$$

where

$$g_{rr} = \left(1 - \frac{2GM}{rc^2}\right)^{-1} \; , \quad g_{tt} = -c^2\left(1 - \frac{2GM}{rc^2}\right) \; .$$

Repeating,

$$c^2 = \left(1 - \frac{2GM}{rc^2}\right) c^2 \left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 - \left(1 - \frac{2GM}{rc^2}\right)^{-1} \left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 .$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[g_{rr}\frac{\mathrm{d}r}{\mathrm{d}\tau}\right] = \frac{1}{2}\partial_r g_{rr}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + \frac{1}{2}\partial_r g_{tt}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 ,$$

where

$$g_{rr} = \left(1 - \frac{2GM}{rc^2}\right)^{-1} , \quad g_{tt} = -c^2\left(1 - \frac{2GM}{rc^2}\right) .$$

Expand

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[g_{rr}\frac{\mathrm{d}r}{\mathrm{d}\tau}\right]$$

with the product rule, replace $(\mathrm{d}t/\mathrm{d}\tau)^2$ using the equation above, and simplify. Result:

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} = -\frac{GM}{r^2} ,$$

which looks just like Newton, but it is not really the same. Here τ is the proper time as measured by the infalling object, and r is not the radial distance.

Solving the Equation

$$\frac{d^2 r}{d\tau^2} = -\frac{GM}{r^2} \quad .$$

Like Newton's equation, multiply by $dr/d\tau$, and it can then be written as

$$\frac{d}{d\tau}\left\{\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 - \frac{GM}{r}\right\} = 0 \ .$$

Quantity in curly brackets is conserved. Initial value (on release from rest at r_0) is $-GM/r_0$, so it always has this value. Then

$$\frac{dr}{d\tau} = -\sqrt{2GM\left(\frac{1}{r} - \frac{1}{r_0}\right)} = -\sqrt{\frac{2GM(r_0 - r)}{r r_0}} \ .$$

Repeating,

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = -\sqrt{2GM\left(\frac{1}{r} - \frac{1}{r_0}\right)} = -\sqrt{\frac{2GM(r_0 - r)}{r r_0}} \ .$$

Bring all r-dependent factors to one side, and bring $\mathrm{d}\tau$ to the other side, and integrate:

$$\tau(r_f) = -\int_{r_0}^{r_f} \mathrm{d}r \sqrt{\frac{r r_0}{2GM(r_0 - r)}}$$

$$= \sqrt{\frac{r_0}{2GM}} \left\{ r_0 \tan^{-1}\left(\sqrt{\frac{r_0 - r_f}{r_f}}\right) + \sqrt{r_f(r_0 - r_f)} \right\} \ ,$$

where $\tan^{-1} \equiv \arctan$.

Conclusion: object will reach $r = 0$ in a finite proper time τ.

$$\tau(r_f) = \sqrt{\frac{r_0}{2GM}} \left\{ r_0 \tan^{-1}\left(\sqrt{\frac{r_0 - r_f}{r_f}}\right) + \sqrt{r_f(r_0 - r_f)} \right\} .$$

Setting $r_f = 0$ to find the proper time when the object reaches $r = 0$,

$$\tau(0) = \sqrt{\frac{r_0}{2GM}} \left\{ r_0 \tan^{-1}(\infty) + 0 \right\}$$

$$= \boxed{\frac{\pi}{2}\sqrt{\frac{r_0^3}{2GM}}} .$$

Falling from the Schwarzschild Horizon to $r = 0$

Recall,

$$\tau(0) = \frac{\pi}{2}\sqrt{\frac{r_0^3}{2GM}}\ .$$

For $r_0 = R_S$,

$$\tau = \frac{\pi GM}{c^3}\ .$$

For $r_0 = R_S$,

$$\tau = \frac{\pi G M}{c^3} \ .$$

For the Sun, this gives

$$\tau = 1.55 \times 10^{-5} \text{ s.}$$

For the black hole in the center of our galaxy,

$$\tau = 6.34 \text{ s.}$$

Note that inside the black hole,

$$\mathrm{d}s^2 = -c^2\mathrm{d}\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right)c^2\mathrm{d}t^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\ ,$$

but

$$\left(1 - \frac{2GM}{rc^2}\right) < 0\ ,$$

which implies that t is spacelike, and r is timelike! The calculation that we just did is still correct. The singularity at $r = 0$ cannot be avoided for the same reason that we cannot prevent ourselves from reaching tomorrow!

But Coordinate Time t is Different!

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = -\sqrt{2GM\left(\frac{1}{r} - \frac{1}{r_0}\right)} = -\sqrt{\frac{2GM(r_0 - r)}{rr_0}} \ .$$

$$c^2 = \left(1 - \frac{2GM}{rc^2}\right) c^2 \left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 - \left(1 - \frac{2GM}{rc^2}\right)^{-1} \left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 \ .$$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t} = \frac{\mathrm{d}r/\mathrm{d}\tau}{\mathrm{d}t/\mathrm{d}\tau}$$

$$= \frac{\mathrm{d}r/\mathrm{d}\tau}{\sqrt{h^{-1}(r) + c^{-2}h^{-2}(r)\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2}} \ ,$$

where $h^{-1}(r) \equiv 1/h(r)$, not the inverse function, and

$$h(r) \equiv 1 - \frac{R_S}{r} = 1 - \frac{2GM}{rc^2} \ .$$

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = -\sqrt{2GM\left(\frac{1}{r} - \frac{1}{r_0}\right)} = -\sqrt{\frac{2GM(r_0 - r)}{r r_0}} \ .$$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}r/\mathrm{d}\tau}{\sqrt{h^{-1}(r) + c^{-2}h^{-2}(r)\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2}} \ ,$$

where

$$h(r) \equiv 1 - \frac{R_S}{r} = 1 - \frac{2GM}{rc^2} \ .$$

Look at behavior near horizon; $h^{-1}(r)$ blows up:

$$h^{-1}(r) = \frac{r}{r - R_S} \approx \frac{R_S}{r - R_S} \ .$$

Denominator of $\mathrm{d}r/\mathrm{d}t$ is dominated by 2nd term, which gives

$$\frac{\mathrm{d}r}{\mathrm{d}t} \approx -ch(r) = -c\left(\frac{r - R_S}{R_S}\right) \ .$$

Repeating,

$$\frac{\mathrm{d}r}{\mathrm{d}t} \approx -c\left(\frac{r - R_S}{R_S}\right) \ .$$

Rearranging,

$$\mathrm{d}t = -\frac{R_S}{c}\frac{\mathrm{d}r}{r - R_S} \ .$$

We can find the time needed to fall from some r_i near the horizon, to a smaller r_f which is nearer to the horizon:

$$t(r_f) \approx -\frac{R_S}{c}\int_{r_i}^{r_f} \frac{\mathrm{d}r'}{r' - R_S} \approx \boxed{\frac{R_S}{c}\ln\left(\frac{r_i - R_S}{r_f - R_S}\right) \ .}$$

Thus t diverges logarithmically as $r_f \to R_S$, so the object does not reach R_S for any finite value of t.

8.286 Class 13
October 19, 2020

BLACK-BODY RADIATION AND THE EARLY HISTORY OF THE UNIVERSE

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- Meaning: Mass and energy are **equivalent**. They are just two different ways of expressing exactly the same thing. The total energy of any system is equal to the total mass of the system — sometimes called the relativistic mass — times c^2, the square of the speed of light.
- One can imagine measuring the mass/energy of an object in either kilograms, joules, or kilowatt-hours, with

$$1 \text{ kg} = 8.9876 \times 10^{16} \text{ joule} = 2.497 \times 10^{10} \text{ kW-hr}.$$

$E = mc^2$ and the World Power Supply

- The total amount of power produced in the world, on average, is about 1.89×10^{10} kW, according to the International Energy Agency.
- This amounts to about 2.5 kW per person.
- If a 15 gallon tank of gasoline could be converted *entirely* into usable energy, it would power the world for $2\frac{1}{2}$ days.
- However, it is not so easy! Even with nuclear power, when a uranium-235 nucleus undergoes fission, only about 0.09% of its mass is converted to energy.

$E = mc^2$ and Particle Masses

☆ Nuclear and particle physicists tend to measure the mass of elementary particles in energy units, usually using either MeV (10^6 eV) or GeV (10^9 eV) as the unit of energy, where

$$1\,\text{eV} = 1 \text{ electron volt} = 1.6022 \times 10^{-19} \text{ J},$$

and then

$$1\,\text{GeV} = 1.7827 \times 10^{-27} \text{ kg}.$$

The mass of a proton is 0.938 GeV, and the mass of an electron is 0.511 MeV.

Energy and Momentum in Special Relativity

- We have talked about the kinematic consequences of special relativity (time dilation, Lorentz contraction, and the relativity of simultaneity), but now we need to bring in the dynamical consequences, involving energy and momentum.
- In special relativity, the definitions of energy and momentum are different from those in Newtonian mechanics.
- Why? Because special relativity is based on the principle that the laws of physics in any inertial reference frame are the same, and furthermore, in order for the speed of light be the same in any inertial reference frame, these frames cannot be related to each other as in Newtonian physics. They must instead be related by Lorentz transformations, which take into account the kinematic effects mentioned above.

- Two important laws of physics are the conservation of energy and momentum.
- If energy and momentum kept their Newtonian definitions, then, if they were conserved in one frame, they would not be conserved in other frames.
- The requirement that the conservation equations hold in all frames requires the standard special relativity definitions.

Energy, Momentum, and the Energy-Momentum Four-Vector

- The energy-momentum four-vector is defined by starting with the momentum three-vector $(p^1, p^2, p^3) \equiv (p^x, p^y, p^z)$, and appending a fourth component

$$p^0 = \frac{E}{c} \; ,$$

so the four-vector can be written as

$$\boxed{p^\mu = \left(\frac{E}{c}, \vec{p}\right) \; .}$$

As with the three-vector momentum, the energy-momentum four-vector can be defined for a system of particles as the sum of the vectors for the individual particles.

- The 4-vector p^μ transforms, when we change frames of reference, according to the Lorentz transformation, exactly like the 4-vector $x^\mu = (ct, \vec{x})$.
- Furthermore, the total energy-momentum 4-vector is conserved — in any inertial frame of reference.

8.286 Class 14

October 21, 2020

BLACK-BODY RADIATION AND THE EARLY HISTORY OF THE UNIVERSE, PART 2

(Corrected on 10/23/20: on p. 17, the value of h_0 was changed from 67 to 0.67.)

Announcements

Reminder: Problem Set 6 is due this Friday at 5:00 pm.

Quiz 2 will be next Wednesday, October 28. Procedures will be the same as for Quiz 1. Review Problems for Quiz 2 have been posted, and they contain a complete description of what will be covered on the quiz.

WARNING: don't let your wonderful success on Quiz 1 cause you to become complacent. The material has gotten harder. The last time I taught this course, in 2018, the class average was 85.0 on the first quiz, and fell to 69.7 on the second. Don't let that happen this year!

Review session by Bruno Scheihing: Monday, 10/26/20, at 7:30 pm.

Special office hours next week:

Bruno: Monday 10/26/20 at 4:00 pm

Me: Tuesday 10/27/20 at 6:00 pm

To be posted today: Compiled course documents (lecture notes, problem sets, solutions, quiz review problems, Quiz 1), on Lecture Notes page. Compiled lecture slides, on main web page.

$$E = mc^2$$

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- **Meaning: Mass and energy are equivalent.** They are just two different ways of expressing exactly the same thing. The total energy of any system is equal to the total mass of the system — sometimes called the relativistic mass — times c^2, the square of the speed of light.
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the mass of an electron is 0.511 MeV.

Alan Guth
Massachusetts Institute of Technology

Energy and Momentum in Special Relativity

If energy and momentum are to be conserved in all inertial reference frames, then the Newtonian definitions must be modified.

Review from last class

Energy, Momentum, and the Energy-Momentum Four-Vector

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$$p^0 = \frac{E}{c} ,$$

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As with the three-vector momentum, the energy-momentum four-vector can be defined for a system of particles as the sum of the vectors for the individual particles.

Review from last class

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- Furthermore, the total energy-momentum 4-vector is conserved — in any inertial frame of reference.

Relation of Energy and Momentum to Rest Mass and Velocity

The mass of a particle in its own rest frame is called its *rest mass*, which we denote by m_0. At velocity $\vec{v}$

$$\vec{p} = \gamma m_0 \vec{v} \; ,$$

$$E = \gamma m_0 c^2 = \sqrt{\left(m_0 c^2\right)^2 + |\vec{p}|^2 \, c^2} \; ,$$

where as usual γ is defined by

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \; .$$

Lorentz Invariance of p^2

$$\vec{p} = \gamma m_0 \vec{v} \ ,$$

$$E = \gamma m_0 c^2 = \sqrt{(m_0 c^2)^2 + |\vec{p}|^2 c^2} \ ,$$

Like the Lorentz-invariant interval that we discussed as $\mathrm{d}s^2 = |\mathrm{d}\vec{x}^2| - c^2 \mathrm{d}t^2$, the energy-momentum four-vector has a Lorentz-invariant square:

$$p^2 \equiv |\vec{p}|^2 - \left(p^0\right)^2 = |\vec{p}|^2 - \frac{E^2}{c^2} = -\left(m_0 c\right)^2 \ .$$

For a particle at rest,

$$E = m_0 c^2 \ .$$

Energy Exchange in a Simple Chemical Reaction

Consider the reaction

$$p + e^- \longrightarrow H + \gamma \ .$$

Assuming that the proton and electron begin at rest, and ignoring the very small kinetic energy of the hydrogen atom when it recoils from the emitted photon, conservation of energy implies that

$$m_H = m_p + m_e - E_\gamma / c^2 \ .$$

The energy given off when the proton and electron bind is called the **binding energy** of the hydrogen atom. It is 13.6 eV.

Relativistic Mass

- Since $E = mc^2$, we can define the *relativistic mass* of any particle or system as simply

$$m_{\rm rel} \equiv \frac{E}{c^2} \ .$$

- Some authors avoid using the concept of relativistic mass, reserving the word "mass" to mean rest mass m_0. Relativistic mass is certainly a redundant concept, since anything that can be described in terms of $m_{\rm rel}$ can also be described in terms of E.
- For cosmology the concept of relativistic mass will be helpful, since relativistic mass is the source of gravity. By calling E/c^2 a mass, we are indicating our recognition that it is the source of gravity.

The Source of Gravity in General Relativity

This is beyond the level of what we need, but for those who are interested, I mention that the Einstein field equations imply that the source of gravitational fields is the *energy-momentum tensor* $T^{\mu\nu}$, where μ and ν are 4-vector indices that take on values from 0 to 3.

$T^{00} = u =$ energy density,

$T^{0i} = T^{i0}$ is $\frac{1}{c}$ times the flow of energy in the i'th direction (i=1,2,3) and is also c times the density of the i'th component of momentum,

$T^{ij} = T^{ji}$ is the flow in the j'th direction of the i'th component of momentum. T^{ij} is often diagonal, with $T^{ij} = p\,\delta^{ij}$, where p is the pressure.

For a homogeneous, isotropic universe model, only u and p will serve as sources for gravity.

Mass of Radiation

☆ Electromagnetic radiation has energy. The energy density is given by

$$u = \frac{1}{2}\left[\epsilon_0 \left|\vec{E}\right|^2 + \frac{1}{\mu_0}\left|\vec{B}\right|^2\right] \ .$$

We won't need this equation, but we need to know that electromagnetic radiation **has** an energy density u.

☆ Energy density implies a (relativistic) mass density

$$\rho = u/c^2 \ .$$

(*Relativistic mass* is defined to be the energy divided by c^2.)

Energy and Momentum of Photons

Photons have zero rest mass.

In general,

$$p^2 = |\vec{p}\,|^2 - \frac{E^2}{c^2} = -\,(m_0 c)^2 \quad ,$$

but for photons, $m_0 = 0$, so

$$|\vec{p}\,|^2 - \frac{E^2}{c^2} = 0 \;, \quad \text{or} \quad E = c|\vec{p}\,| \;.$$

Radiation in an Expanding Universe

- From the end of inflation (maybe about 10^{-35} second, to be discussed later) until stars form, the number of photons is almost exactly conserved.
- Therefore,

$$n_\gamma \propto \frac{1}{a^3(t)} \ .$$

- Bu1t the frequency of each photon redshifts:

$$\nu \propto \frac{1}{a(t)} \ .$$

$$n_\gamma \propto \frac{1}{a^3(t)} , \qquad \nu \propto \frac{1}{a(t)} .$$

- But according to quantum mechanics, the energy of each photon is

$$E = h\nu ,$$

so the energy of each photon is proportional to $1/a(t)$.

- Finally,

$$n_\gamma \propto \frac{1}{a^3(t)} \quad , \quad E_\gamma \propto \frac{1}{a(t)} \qquad \Longrightarrow \qquad \rho_\gamma = \frac{u_\gamma}{c^2} \propto \frac{1}{a^4(t)} .$$

The Radiation Dominated Era

Radiation energy density today (including photons and neutrinos):

$$u_r = 7.01 \times 10^{-14}\ \text{J/m}^3\ , \qquad \rho_r = u_r/c^2 = 7.80 \times 10^{-34}\ \text{g/cm}^3\ .$$

Total mass density today, ρ_0, is equal to within uncertainties to the critical density,

$$\rho_c = \frac{3H_0^2}{8\pi G} = 1.88\, h_0^2 \times 10^{-29}\ \text{g/cm}^3\ ,$$

where

$$H_0 = 100\, h_0\, \text{km-s}^{-1}\text{-Mpc}^{-1}\ , \qquad h_0 \approx 0.67\ ,$$

which gives the present value of Ω_r as $\boxed{\Omega_r \approx 9.2 \times 10^{-5}}$.

Since $\rho_r \propto 1/a^4(t)$, while $\rho_m \propto 1/a^3(t)$.

ρ_m = mass density of nonrelativistic matter, baryonic matter plus dark matter. It follows that

$$\rho_r/\rho_m \propto 1/a(t) \ .$$

Today $\rho_m \approx 0.30\rho_c$, so $\rho_r/\rho_m \approx 9.2 \times 10^{-5}/0.30 \approx 3.1 \times 10^{-4}$. Thus

$$\frac{\rho_r(t)}{\rho_m(t)} = \frac{a(t_0)}{a(t)} \times 3.1 \times 10^{-4} \ .$$

$t_{\rm eq}$ is defined to be the time of matter-radiation equality. Thus

$$\frac{\rho_r(t_{\rm eq})}{\rho_m(t_{\rm eq})} \equiv 1 = \frac{a(t_0)}{a(t_{\rm eq})} \times 3.1 \times 10^{-4} \ .$$

Since $a(t_0)/a(t_{\rm eq}) = 1 + z_{\rm eq}$,

$$z_{\rm eq} = \frac{1}{3.1 \times 10^{-4}} - 1 \approx 3200 \ .$$

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Time of matter-radiation equality:

We are not ready to calculate this accurately, but for now we can estimate it by assuming that between $t_{\rm eq}$ and now, $a(t) \propto t^{2/3}$, as in a matter-dominated flat universe. Then

$$(t_{\rm eq}/t_0)^{2/3} = 3.1 \times 10^{-4} \ ,$$

so

$$t_{\rm eq} = 5.5 \times 10^{-6}\, t_0 = 5.5 \times 10^{-6} \times 13.8 \text{ Gyr} \approx 75{,}000 \text{ years.}$$

Ryden (p. 96) gives 50,000 years, which is more accurate.

Dynamics of the Radiation-Dominated Era

$$\rho \propto \frac{1}{a^3} \implies \dot{\rho} = -3\frac{\dot{a}}{a}\,\rho\ , \quad \rho(t) \propto \frac{1}{a^4(t)} \implies \dot{\rho} = -4\frac{\dot{a}}{a}\,\rho\ .$$

$\dot{\rho}$ and pressure p: (Problem 4, Problem Set 6)

$$dU = -p\,dV \qquad \implies \qquad \frac{d}{dt}\left(a^3\rho c^2\right) = -p\frac{d}{dt}(a^3)$$

$$\implies \qquad \dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right)\ .$$

Friedmann equations:

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{a^2}$$

$$\ddot a = -\frac{4\pi}{3}G\rho a \ ,$$

$$\dot\rho = -3\frac{\dot a}{a}\,\rho$$

$\left(\begin{array}{c}\text{matter-dominated}\\ \text{universe}\end{array}\right)$

Any two of the above equations implies the third. So they become inconsistent if

$$\dot\rho = -3\frac{\dot a}{a}\left(\rho + \frac{p}{c^2}\right) \ .$$

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{a^2}\ , \qquad \ddot a = -\frac{4\pi}{3}G\rho a\ .$$

Any two of the above equations implies the third. So they become inconsistent if

$$\dot\rho = -3\frac{\dot a}{a}\left(\rho + \frac{p}{c^2}\right)\ .$$

So, if we believe the equation for $\dot\rho$, we must modify one of the two Friedmann equations. First order equation represents conservation of energy: pressure does not belong! (Pressures can change suddenly, as when dynamite explodes, so it does not make sense to have pressure in a conservation equation.) So modify the 2nd order equation, deriving it from the first order equation and the $\dot\rho$ equation:

$$\ddot a = -\frac{4\pi}{3}G\left(\rho + \frac{3p}{c^2}\right)a\ .$$

Dynamics of a Flat Radiation-dominated Universe

$$H^2 = \frac{8\pi G}{3}\rho \ , \ \rho \propto 1/a^4 \quad \Longrightarrow \quad \left(\frac{\dot{a}}{a}\right)^2 = \frac{\text{const}}{a^4} \ .$$

Then

$$a\,\mathrm{d}a = \sqrt{\text{const}}\,\mathrm{d}t \quad \Longrightarrow \quad \frac{1}{2}a^2 = \sqrt{\text{const}}\,t + \text{const}' \ .$$

So, setting our clocks so that $\text{const}' = 0$,

$$a(t) \propto \sqrt{t} \qquad \text{(flat radiation-dominated)} \ .$$

$$H(t) = \frac{\dot{a}}{a} = \frac{1}{2t} \qquad \text{(flat radiation-dominated)} \ .$$

$$\ell_{p,\text{horizon}}(t) = a(t) \int_0^t \frac{c}{a(t')} dt'$$

$$= 2ct \qquad \text{(flat radiation-dominated)} \ .$$

$$H^2 = \frac{8\pi G}{3} \rho \quad \Longrightarrow \quad \rho = \frac{3}{32\pi G t^2} \ .$$