8.286 Lecture 6
September 23, 2020

THE DYNAMICS OF NEWTONIAN COSMOLOGY, PART 2

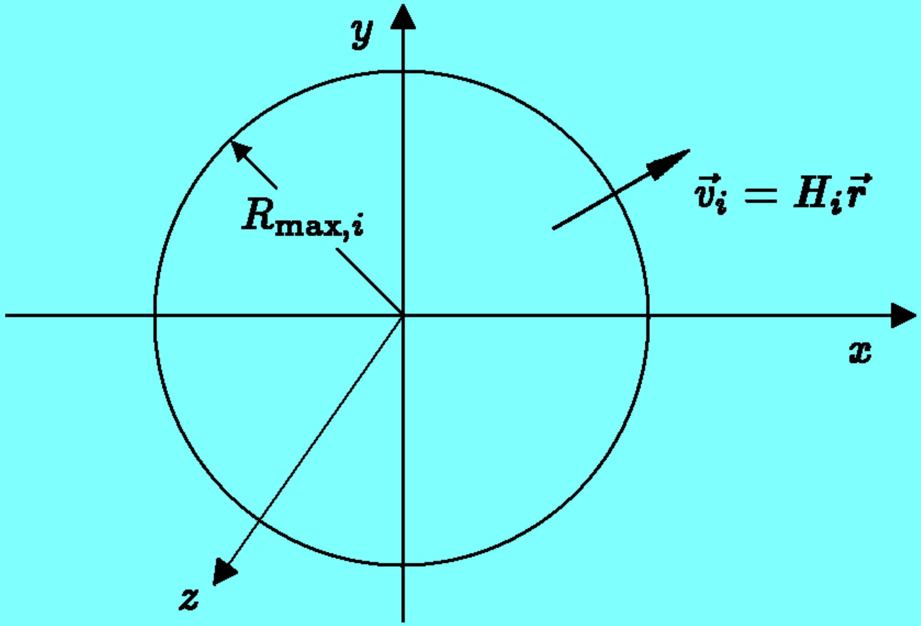
Announcements

- Problem Set 3 is due this Friday at 5 pm EDT.
- Quiz 1 will take place a week from Wednesday, on 9/30/2020. Full details about the quiz are on the class website, and are on the *Review Problems for Quiz 1.* One problem on the quiz will be taken verbatim, or at least almost verbatim, from the problem sets or from the starred problems on the *Review Problems*.
- Review session for the quiz, by Bruno Scheihing: Sunday, 9/27/2020, at 1:00 pm EDT. Same Zoom ID as our classes. Will be recorded.

Mathematical Model of a Uniformly Expanding Universe

- Desired properties: homogeneity, isotropy, and Hubble's law.
- The model should be finite, to avoid the conditional convergence problems discussed last time. At the end we will take the limit as the size approaches infinity.
- Newtonian dynamics: we choose the initial conditions, and then Newton's laws of motion will determine how it will evolve.
- To impose isotropy, we model the initial state as a solid sphere, of some radius $R_{\text{max},i}$.
- To impose homogeneity, we take the initial mass density to be constant, ρ_i. The matter is treated as a gas, that can thin as the universe expands. Think of a gas of very low speed particles, so the pressure is negligible.
- We take the initial velocities according to Hubble's law, with some initial expansion rate H_i

Mathematical Model of a Uniformly Expanding Universe



$$t_i \equiv \text{time of initial picture}$$

$$R_{\max,i} \equiv \text{initial maximum radius}$$

$$\rho_i \equiv \text{initial mass density}$$

$$\vec{v}_i = H_i \vec{r}\ .$$

Description of Evolution

- As the model universe evolves, the spherical symmetry will be preserved: each gas particle will continue on a radial trajectory, since there are no forces that might pull it tangentially.

- Spherical symmetry $\Longrightarrow$ all particles that start at the same initial radius will behave the same way. So, a particle that begins at radius r_i will be found at a later time t at some radius

$$\boxed{r = r(r_i, t)\ .}$$

- Our goal is to figure out what determines $r(r_i, t)$

- The only relevant force is gravity. Gravity and electromagnetism are the only (known) long-range forces. The universe appears to be electrically neutral, so long-range electric forces are not present.

Reminder: the Gravitational Field of a Shell of Matter

- ☆ For points outside the shell, the gravitational force is the same as if the total mass of the shell were concentrated at the center.
- ☆ For points inside the shell, the gravitational field is **zero.**
- ☆ Newton figured this out by integration. For us, Gauss's law makes it obvious.

Shell Crossings?

Can shells cross? I.e., can two shells that start at different r_i ever cross each other?

The answer is no, but we don't know that when we start.

But we do know that Hubble's law implies that any two shells are initially moving apart. Therefore there must be at least some interval before any shell crossings can happen.

We will write equations that are valid assuming no shell crossings.

These equations will be valid until any possible shell crossing.

If there was a shell crossing, these equations would have to show two shells becoming arbitrarily close.

We will find, however, that the equations imply uniform expansion, so no shell crossings ever happen in this system.

Equations of Motion

- Newtonian gravity of a shell:

  Inside: $\vec{g} = 0$.

  Outside: Same as point mass at center, with same M.

- $r(r_i, t) \equiv$ radius at t of shell initially at r_i.
- Let $M(r_i) \equiv$ mass inside r_i-shell $= \frac{4\pi}{3} r_i^3 \rho_i$ at all times.
- Pressure? When a gas with pressure $p > 0$ expands, it pushes on its surroundings and loses energy. Relativistically, energy = mass (times c^2). By assuming that $M(r_i)$ is constant, we are assuming that $p \simeq 0$.

Equations:

☆ For particles at radius r,

$$\boxed{\vec{g} = -\frac{GM(r_i)}{r^2}\,\hat{r}\ ,}$$

where

$$\boxed{M(r_i) = \frac{4\pi}{3} r_i^3 \rho_i\ .}$$

Since $\vec{g}$ is the acceleration,

$$\boxed{\ddot{r} = -\frac{GM(r_i)}{r^2} = -\frac{4\pi}{3}\frac{G r_i^3 \rho_i}{r^2}\ , \quad \text{where } r \equiv r(r_i, t),}$$

where an overdot indicates a derivative with respect to t.

$$\boxed{\ddot{r} = -\frac{GM(r_i)}{r^2} = -\frac{4\pi}{3}\frac{G r_i^3 \rho_i}{r^2}\ ,\ \text{where } r \equiv r(r_i, t),}$$

☆ For a second order equation like this, the solution is uniquely determined if the initial value of r and $\dot{r}$ are specified:

$$\boxed{r(r_i, t_i) = r_i\ ,}$$

and, by the Hubble law initial condition $\vec{v}_i = H_i \vec{r}_i$,

$$\boxed{\dot{r}(r_i, t_i) = H r_i\ .}$$

Miraculous Scaling Relations

$$\ddot{r} = -\frac{4\pi}{3}\frac{G r_i^3 \rho_i}{r^2} \;, \quad r(r_i, t_i) = r_i \;, \quad \dot{r}(r_i, t_i) = H r_i \;.$$

☆ Suppose we define

$$u(r_i, t) \equiv \frac{r(r_i, t)}{r_i} \;.$$

Then

$$\ddot{u} = \frac{\ddot{r}}{r_i} = -\frac{4\pi}{3}\frac{G\rho_i}{u^2} \;.$$

There is no r_i-dependence. This "miracle" depended on gravity being a $1/r^2$ force.

$$\ddot{r} = -\frac{4\pi}{3}\frac{G r_i^3 \rho_i}{r^2}\ , \quad r(r_i, t_i) = r_i\ , \quad \dot{r}(r_i, t_i) = H r_i\ .$$

$$u(r_i, t) \equiv \frac{r(r_i, t)}{r_i} \quad \Longrightarrow \quad \ddot{u} = -\frac{4\pi}{3}\frac{G\rho_i}{u^2}\ .$$

What about the initial conditions for $u(r_i, t)$?

$$u(r_i, t_i) = \frac{r(r_i, t_i)}{r_i} = 1\ , \quad \dot{u}(r_i, t_i) = \frac{\dot{r}(r_i, t_i)}{r_i} = H\ .$$

Since the differential equation and the intial conditions determine $u(r_i, t)$, it does not depend on r_i. We can rename it

$$u(r_i, t) \equiv a(t)\ , \qquad \text{so} \qquad r(r_i, t) = a(t)\, r_i\ .$$

This describes uniform expansion by a scale factor $a(t)$.

Time Dependence of $\rho(t)$

We know how the mass density depends on time, because we assumed that $M(r_i)$ — the total mass contained inside a shell of particles whose initial radius was r_i — does not change with time. The radius of the shell at time t is $a(t)r_i$. The mass density is just the mass divided by the volume,

$$\boxed{\rho(t) = \frac{M(r_i)}{\frac{4\pi}{3}a^3(t)r_i^3} = \frac{\frac{4\pi}{3}r_i^3\rho_i}{\frac{4\pi}{3}a^3(t)r_i^3} = \frac{\rho_i}{a^3(t)} \ .}$$

So

$$\boxed{\ddot{u} = -\frac{4\pi}{3}\frac{G\rho_i}{u^2} \quad \Longrightarrow \quad \ddot{a} = -\frac{4\pi}{3}\frac{G\rho_i}{a^2} \ .}$$

$$\Longrightarrow \quad \boxed{\ddot{a} = -\frac{4\pi}{3}G\rho(t)\,a(t) \ .} \quad \text{Friedmann equation.}$$

Nothing Depends on $R_{\mathrm{max},i}$

- An observer living in this model universe would see uniform expansion all around herself, and would only be aware of the boundary at R_{max} if she was close enough to the boundary to see it.

- Thus, we can take the limit $R_{\mathrm{max},i} \to \infty$ without doing anything, since nothing of interest depends on $R_{\mathrm{max},i}$.

A Conservation Law

☆ The equation for $\ddot{a}$ has the same form as an equation for the motion of a particle with a time-independent potential energy function. So, there is a conservation law:

$$\ddot{a} = -\frac{4\pi}{3}\frac{G\rho_i}{a^2} \quad \Longrightarrow \quad \dot{a}\left\{\ddot{a} + \frac{4\pi}{3}\frac{G\rho_i}{a^2}\right\} = 0 \quad \Longrightarrow \quad \frac{dE}{dt} = 0\ ,$$

where

$$E = \frac{1}{2}\dot{a}^2 - \frac{4\pi}{3}\frac{G\rho_i}{a}\ .$$

Summary: Equations

Want: $r(r_i, t) \equiv$ radius at t of shell initially at r_i

Find: $r(r_i, t) = a(t) r_i$, where

$$\text{Friedmann Equations} \begin{cases} \ddot{a} = -\dfrac{4\pi}{3} G\rho(t) a \\[2ex] H^2 = \left(\dfrac{\dot{a}}{a}\right)^2 = \dfrac{8\pi}{3} G\rho - \dfrac{kc^2}{a^2} \quad \text{(Friedmann Eq.)} \end{cases}$$

and

$$\rho(t) \propto \frac{1}{a^3(t)} \ , \ \text{or } \rho(t) = \left[\frac{a(t_1)}{a(t)}\right]^3 \rho(t_1) \text{ for any } t_1 .$$

☆ Note that t_i no longer plays any role. It does not appear on this slide!

The Return of the 'Notch'

- ☆ Definition: $r(r_i, t) = a(t) r_i$.
- ☆ In the previous derivation, r_i was the initial radius of some particle, measured in meters. But when we finished, r_i was being used only as a coordinate to label shells, where the shell corresponding to $r_i = 1$ had a radius of one meter only at time t_i.
- ☆ But t_i no longer appears, and will not be mentioned again! So, the connection between the numerical value of r_i and the length of a meter has disappeared from the formalism.
- ☆ Bottom line: r_i is the radial coordinate in a comoving coordinate system, measured in units that have no particular meaning. I will refer to the units of r_i as "notches," but you should be aware that the term is not standard.

Conventions for the Notch

Us: For us, the notch is an arbitrary unit that we use to mark off intervals on the comoving coordinate system. We are free to use a different definition every time we use the notch.

Ryden: $a(t_0) = 1$ (where $t_0 = \text{now}$). (In our language, Ryden's convention is $a(t_0) = 1\ \text{m/notch}$.)

Many Other Books: if $k \neq 0$, then $k = \pm 1$.

In our language, this means $k = \pm 1/\text{notch}^2$. To see the units of k, recall that the Friedmann equation is

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{a^2} \; .$$

We will use $[x]$ to mean the units of x, and we will use T and L to denote the units of time and length, respectively. The units of the left-hand side are $1/T^2$, with the units of a canceling. So

$$[k] = \frac{1}{T^2}\left[\frac{a}{c}\right]^2 = \frac{1}{T^2}\left[\frac{L/\text{notch}}{L/T}\right]^2 = \frac{1}{\text{notch}^2} \; .$$

Types of Solutions

$$\dot{a}^2 = \frac{8\pi G}{3} \frac{\rho(t_1) a^3(t_1)}{a(t)} - kc^2 \quad (\text{for any } t_1) \; .$$

For intuition, remember that $k \propto -E$, where E is a measure of the energy of the system.

Types of Solutions:

1) $k < 0$ $(E > 0)$: unbound system. $\dot{a}^2 > (-kc^2) > 0$, so the universe expands forever. **Open Universe.**

2) $k > 0$ $(E < 0)$: bound system. $\dot{a}^2 \geq 0 \quad \Longrightarrow$

$$a_{\max} = \frac{8\pi G}{3} \frac{\rho(t_1) a^3(t_1)}{kc^2} \; .$$

Universe reaches maximum size and then contracts to a Big Crunch. **Closed Universe.**

3) $k = 0$ $(E = 0)$: critical mass density.

$$H^2 = \frac{8\pi G}{3}\rho - \underbrace{\frac{kc^2}{a^2}}_{=0} \quad \Longrightarrow \quad \boxed{\rho \equiv \rho_c = \frac{3H^2}{8\pi G}\ .}$$

Flat Universe.

Summary: $\rho > \rho_c \Longleftrightarrow$ closed, $\rho < \rho_c \Longleftrightarrow$ open, $\rho = \rho_c \Longleftrightarrow$ flat.

Numerical value: For $H = 68$ km-s^{-1}-Mpc^{-1} (Planck 2015 plus other experiments),

$$\rho_c = 8.7 \times 10^{-27}\ \mathrm{kg/m}^3 = 8.7 \times 10^{-30}\ \mathrm{g/cm}^3$$

$$\approx 5 \text{ proton masses per m}^3.$$

Definition: $\Omega \equiv \dfrac{\rho}{\rho_c}$.

Evolution of a Flat Universe

If $k = 0$, then

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho = \frac{\text{const}}{a^3} \quad \Longrightarrow \quad \frac{da}{dt} = \frac{\text{const}}{a^{1/2}}$$

$$\Longrightarrow \quad a^{1/2}\,da = \text{const}\,dt \quad \Longrightarrow \quad \frac{2}{3}a^{3/2} = (\text{const})t + c' \ .$$

Choose the zero of time to make $c' = 0$, and then

$$\boxed{a(t) \propto t^{2/3} \ .}$$