8.286 Lecture 7
September 28, 2020

THE DYNAMICS OF NEWTONIAN COSMOLOGY, PART 3

Announcements

- ☆ Quiz 1 will take place this Wednesday (9/30/2020). Full details about the quiz are on the class website, and are on the *Review Problems for Quiz 1.* One problem on the quiz will be taken verbatim, or at least almost verbatim, from the problem sets or from the starred problems on the *Review Problems*.

- ☆ Quiz Logistics: You may start Quiz 1 anytime from 11:05 am Wed to 11:05 am Thurs. The default time is 11:05 am Wed. If you want to take it at a different time, you should email me before midnight on Tues night, telling me the time that you want to start.

- ☆ The quiz will be contained in a PDF file, which I am planning to distribute by email. You will each be expected to spend up to 85 minutes working on it, and then you will upload your answers to Canvas as a PDF file. I won't place any precise time limit on scanning or photographing and uploading, because the time needed for that can vary. If you have questions about the meaning of the questions, I will be available on Zoom during the September 30 class time, and we will arrange for either Bruno or me to be available by email as much as possible during the other quiz times.
- ☆ If you have any special circumstances that might make this procedure difficult, or if you need a postponement beyond the 24-hour window, please let me (guth@ctp.mit.edu) know.
- ☆ The recording of the review session for the quiz, by Bruno Scheihing, is on the website.

★ Special office hours this week:

Bruno: today (Mon 9/28): 6-7 pm.
Me: tomorrow (Tues 9/29): 5-6 pm.
No office hours Wed or Thurs.

★ Since people will be taking the quiz at different times, you will be on your honor, before you take the quiz, not to discuss it with anyone who has seen it.

Summary: Equations

Want: $r(r_i, t) \equiv$ radius at t of shell initially at r_i

Find: $r(r_i, t) = a(t) r_i$, where

$$\text{Friedmann Equations} \begin{cases} \ddot{a} = -\dfrac{4\pi}{3} G \rho(t) a \\ H^2 = \left(\dfrac{\dot{a}}{a}\right)^2 = \dfrac{8\pi}{3} G\rho - \dfrac{kc^2}{a^2} \quad \text{(Friedmann Eq.)} \end{cases}$$

and

$$\rho(t) \propto \frac{1}{a^3(t)} \, , \text{ or } \rho(t) = \left[\frac{a(t_1)}{a(t)}\right]^3 \rho(t_1) \text{ for any } t_1 .$$

☆ Note that t_i no longer plays any role. It does not appear on this slide!

Types of Solutions

$$\dot{a}^2 = \frac{8\pi G}{3}\frac{\rho(t_1)a^3(t_1)}{a(t)} - kc^2 \quad \text{(for any } t_1\text{)} \ .$$

For intuition, remember that $k = -2E/c^2$, where E is a measure of the energy of the system.

Types of Solutions:

1) $k < 0$ ($E > 0$): unbound system. $\dot{a}^2 > (-kc^2) > 0$, so the universe expands forever. **Open Universe.**

2) $k > 0$ ($E < 0$): bound system. $\dot{a}^2 \geq 0 \quad \Longrightarrow$

$$a_{\max} = \frac{8\pi G}{3}\frac{\rho(t_1)a^3(t_1)}{kc^2} \ .$$

Universe reaches maximum size and then contracts to a Big Crunch. **Closed Universe.**

3) $k = 0$ ($E = 0$): critical mass density.

$$H^2 = \frac{8\pi G}{3}\rho - \underbrace{\frac{kc^2}{a^2}}_{=0} \quad \Longrightarrow \quad \boxed{\rho \equiv \rho_c = \frac{3H^2}{8\pi G}\ .}$$

Flat Universe.

Summary: $\rho > \rho_c \Longleftrightarrow$ closed, $\rho < \rho_c \Longleftrightarrow$ open, $\rho = \rho_c \Longleftrightarrow$ flat.

Numerical value: For $H = 68$ km-s^{-1}-Mpc^{-1} (Planck 2015 plus other experiments),

$$\rho_c = 8.7 \times 10^{-27}\ \mathrm{kg/m^3} = 8.7 \times 10^{-30}\ \mathrm{g/cm^3}$$

$$\approx 5 \text{ proton masses per m}^3.$$

Definition: $\Omega \equiv \dfrac{\rho}{\rho_c}$.

Evolution of a Flat Universe

If $k = 0$, then

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho = \frac{\text{const}}{a^3} \qquad \Longrightarrow \qquad \frac{da}{dt} = \frac{\text{const}}{a^{1/2}}$$

$$\Longrightarrow \qquad a^{1/2}\, da = \text{const}\, dt \qquad \Longrightarrow \qquad \frac{2}{3} a^{3/2} = (\text{const})t + c' \ .$$

Choose the zero of time to make $c' = 0$, and then

$$\boxed{a(t) \propto t^{2/3} \ .}$$

Age of a Flat Matter-Dominated Universe

$$a(t) \propto t^{2/3} \qquad \Longrightarrow \qquad H = \frac{\dot{a}}{a} = \frac{2}{3t} \qquad \Longrightarrow$$

$$\boxed{t = \frac{2}{3} H^{-1}}$$

For $H = 67.7 \pm 0.5$ km-s^{-1}-Mpc^{-1}, age $= 9.56$ – 9.70 billion years — but stars are older. Conclusion: our universe is nearly flat, but not matter-dominated.

Big Bang Singularity

- ☆ $a(0) = 0$, so the mass density ρ at $t = 0$ is infinite.
- ☆ This instant of infinite mass density is called a singularity.
- ☆ But, as we extrapolate backwards to early t, ρ becomes higher than any mass density that we know about.
- ☆ Hence, there is no reason to trust the model back to $t = 0$.

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- ☆ Conclusion: the singularity is a feature of the model, but not necessarily the real universe.
- ☆ Quantum gravity? The singularity is a feature of the *classical* theory, but might be avoided by a quantum gravity treatment — but we don't know.
- ☆ In eternal inflation models, to be discussed near the end of the term, the event 13.8 billion years ago was not a singularity, but rather the decay of the repulsive-gravity material that drove the inflation. There might still have been a singularity deeper in the past.

Horizon Distance

Definition: the horizon distance is the present distance of the furthest particles from which light has had time to reach us.

To find it, use comoving coordinates. The coordinate velocity of light is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{c}{a(t)} \; ,$$

so the maximum coordinate distance that light could have traveled by time t (starting at $t = 0$) is

$$\ell_{c,\mathrm{horizon}}(t) = \int_0^t \frac{c}{a(t')}\,\mathrm{d}t' \; .$$

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The horizon distance is the maximum *physical* distance that light could have traveled, so

$$\boxed{\ell_{\text{phys,horizon}}(t) = a(t)\int_0^t \frac{c}{a(t')}\mathrm{d}t' \ .}$$

For a flat, matter-dominated universe, $a(t) \propto t^{2/3}$, so

$$\boxed{\ell_{\text{phys,horizon}}(t) = 3ct = 2cH^{-1} \ ,}$$

since $t = \frac{2}{3}H^{-1}$.

Equations for a Matter-Dominated Universe

("Matter-dominated" = dominated by nonrelativistic matter.)

Friedmann equations:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{a^2} \ ,$$

$$\ddot{a} = -\frac{4\pi}{3}G\rho(t)a \ .$$

Matter conservation:

$$\rho(t) \propto \frac{1}{a^3(t)} \ , \ \text{or } \rho(t) = \left[\frac{a(t_1)}{a(t)}\right]^3 \rho(t_1) \text{ for any } t_1 .$$

Any two of the above equations can allow us to find the third.

Evolution of a Closed Universe

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{a^2} \; , \quad \rho(t)a^3(t) = \text{constant} \; , \;\; k > 0 \; .$$

Recall $[a(t)] = \text{meter/notch}$, $[k] = 1/\text{notch}^2$.

Define new variables:

$$\tilde{a}(t) \equiv \frac{a(t)}{\sqrt{k}} \; , \qquad \tilde{t} \equiv ct \qquad \text{(both with units of distance)}$$

Multiplying Friedmann eq by $a^2/(kc^2)$:

$$\frac{1}{kc^2}\left(\frac{da}{dt}\right)^2 = \frac{8\pi}{3}\frac{G\rho a^2}{kc^2} - 1 \; .$$

Recalling

$$\tilde{a}(t) \equiv \frac{a(t)}{\sqrt{k}} , \qquad \tilde{t} \equiv ct,$$

we find

$$\frac{1}{kc^2}\left(\frac{da}{dt}\right)^2 = \frac{8\pi}{3}\frac{G\rho a^2}{kc^2} - 1$$

$$= \frac{8\pi}{3}\frac{G\rho a^3}{k^{3/2}c^2}\frac{\sqrt{k}}{a} - 1 .$$

Rewrite as

$$\left(\frac{d\tilde{a}}{d\tilde{t}}\right)^2 = \frac{2\alpha}{\tilde{a}} - 1 ,$$

where

$$\alpha \equiv \frac{4\pi}{3}\frac{G\rho\tilde{a}^3}{c^2} .$$

$[\alpha]$ = meter. α is constant, since ρa^3 is constant.

$$\left(\frac{d\tilde{a}}{d\tilde{t}}\right)^2 = \frac{2\alpha}{\tilde{a}} - 1 \quad \Longrightarrow \quad d\tilde{t} = \frac{\tilde{a}\, d\tilde{a}}{\sqrt{2\alpha\tilde{a} - \tilde{a}^2}} \ .$$

Then

$$\tilde{t}_f = \int_0^{\tilde{t}_f} d\tilde{t} = \int_0^{\tilde{a}_f} \frac{\tilde{a}\, d\tilde{a}}{\sqrt{2\alpha\tilde{a} - \tilde{a}^2}} \ ,$$

where $\tilde{t}_f$ is an arbitrary choice for a “final time” for the calculation, and $\tilde{a}_f$ is the value of $\tilde{a}$ at time $\tilde{t}_f$.

To carry out the integral, we first complete the square:

$$\tilde{t}_f = \int_0^{\tilde{a}_f} \frac{\tilde{a}\, d\tilde{a}}{\sqrt{\alpha^2 - (\tilde{a} - \alpha)^2}} \ .$$

Now simplify by defining $x \equiv \tilde{a} - \alpha$, so

$$\tilde{t}_f = \int_{-\alpha}^{\tilde{a}_f - \alpha} \frac{(x + \alpha)\, dx}{\sqrt{\alpha^2 - x^2}} \ .$$

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To simplify $\alpha^2 - x^2$, define θ so that $x = -\alpha\cos\theta$.

(Choice of the minus sign simplifies the final answer. Recall that x represents the scale factor, and θ will be replacing x. The minus sign leads to $\mathrm{d}x/\mathrm{d}\theta = \alpha\sin\theta$, which is positive for small positive θ, so both will be growing at the start of the universe.)

Substituting,

$$\mathrm{d}x = \alpha\sin\theta\,\mathrm{d}\theta \ .$$

$$\sqrt{\alpha^2 - x^2} = \alpha\sqrt{1-\cos^2\theta} = \alpha\sin\theta \ .$$

Then

$$\tilde{t}_f = \alpha\int_0^{\theta_f} (1-\cos\theta)d\theta = \alpha(\theta_f - \sin\theta_f) \ .$$

This equation relates t_f to θ_f, but we really want to relate the scale factor and time. But θ_f is related to the scale factor, if we trace back the definitions: $x_f = -\alpha\cos\theta_f = \tilde{a}_f - \alpha$, so

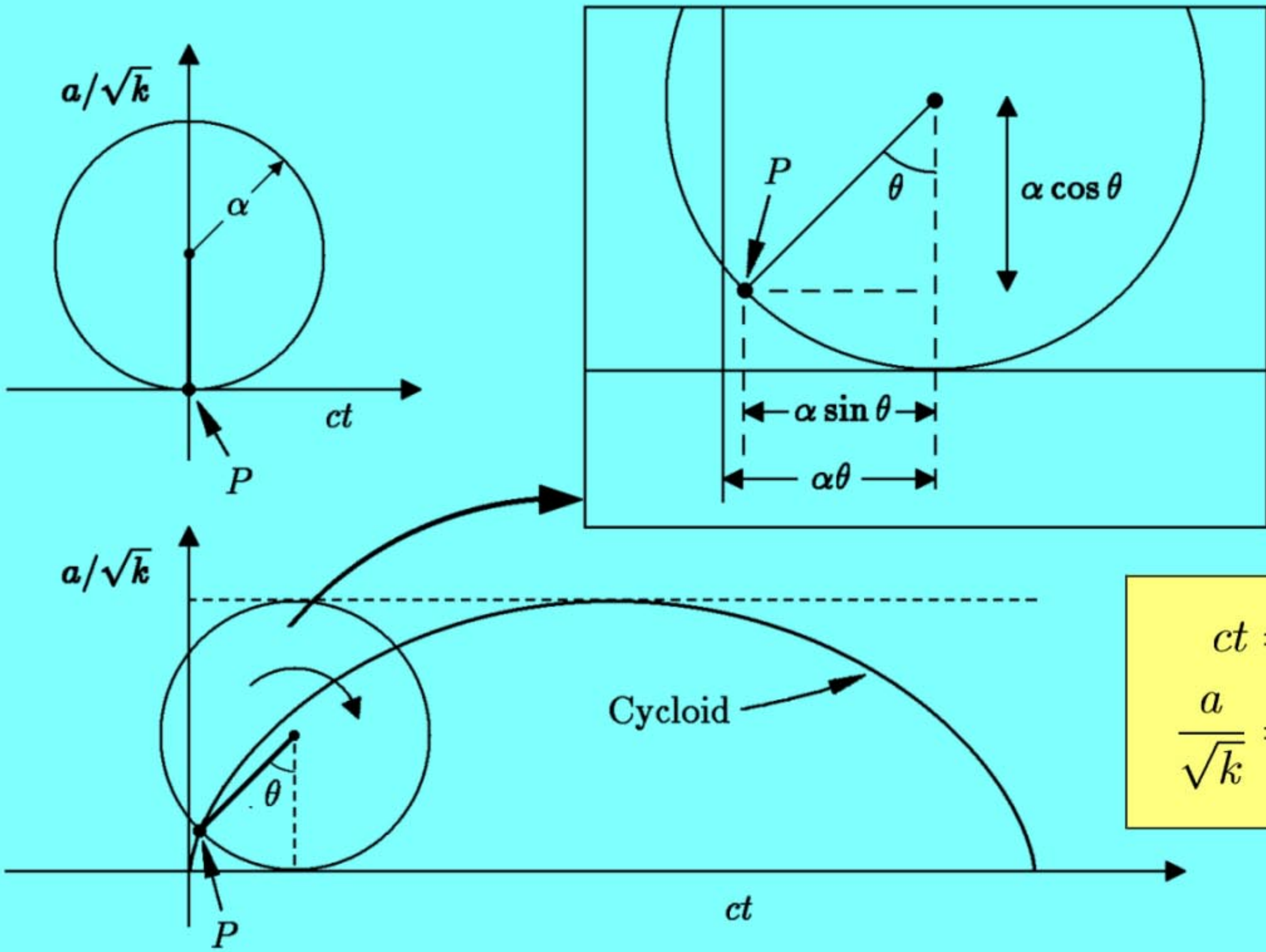
$$\tilde{a}_f = \alpha(1-\cos\theta_f) \ .$$

Parametric Solution for the Evolution of a Closed Matter-Dominated Universe

$$ct = \alpha(\theta - \sin\theta) \ ,$$

$$\frac{a}{\sqrt{k}} = \alpha(1 - \cos\theta) \ .$$

The angle θ is sometimes called the "development angle," because it describes the stage of development of the universe. The universe begins at $\theta = 0$, reaches its maximum expansion at $\theta = \pi$, and then is terminated by a big crunch at $\theta = 2\pi$.



$$ct = \alpha(\theta - \sin\theta) \ ,$$

$$\frac{a}{\sqrt{k}} = \alpha(1 - \cos\theta) \ .$$

Duration and Maximum Size

$$\frac{a}{\sqrt{k}} = \alpha(1 - \cos\theta) \quad \Longrightarrow \quad \boxed{\frac{a_{\rm max}}{\sqrt{k}} = 2\alpha\ ,}$$

where

$$\alpha = \frac{4\pi}{3}\frac{G\rho a^3}{k^{3/2}c^2}\ .$$

Similarly, $ct = \alpha(\theta - \sin\theta)$ implies that the total duration of the universe, from big bang to big crunch is

$$\boxed{t_{\rm total} = \frac{2\pi\alpha}{c} = \frac{\pi a_{\rm max}}{c\sqrt{k}}\ .}$$

Age of a Closed Matter-Dominated Universe

$$ct = \alpha(\theta - \sin\theta)$$

gives the age in terms of α and θ. But astronomers measure H and Ω. So we would like to express the age in terms of H and Ω.

Start with ρ:

$$\rho = \Omega \rho_c = \left(\frac{3H^2}{8\pi G}\right)\Omega \ .$$

The first-order Friedmann equation can then be rewritten as

$$H^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{a^2} \quad \Longrightarrow \quad H^2 = H^2\Omega - \frac{kc^2}{a^2} \ ,$$

so

$$\boxed{\tilde{a} = \frac{a}{\sqrt{k}} - \frac{c}{|H|\sqrt{\Omega - 1}} \ .}$$

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In taking the square root, recall that $a > 0$, $k > 0$, while H changes sign — it is positive during the expansion phase, and negative during the collapse phase. So we need $|H|$, not just H, for the equation to be valid. Then

$$\alpha = \frac{4\pi}{3}\frac{G\rho\tilde{a}^3}{c^2} = \frac{c}{2|H|}\frac{\Omega}{(\Omega-1)^{3/2}}\ .$$

To find age, we need to express α and θ in terms of H and Ω. To express θ, use expression for $\tilde{a}$ above, and 2nd parametric equation

$$\tilde{a} = \frac{a}{\sqrt{k}} = \alpha(1 - \cos\theta)\ .$$

Then

$$\frac{c}{|H|\sqrt{\Omega - 1}} = \frac{c}{2|H|}\frac{\Omega}{(\Omega-1)^{3/2}}(1 - \cos\theta)\ ,$$

Then

$$\frac{c}{|H|\sqrt{\Omega - 1}} = \frac{c}{2|H|} \frac{\Omega}{(\Omega - 1)^{3/2}} (1 - \cos\theta) \ ,$$

which can be solved for either $\cos\theta$ or for Ω:

$$\boxed{\cos\theta = \frac{2 - \Omega}{\Omega} \ , \quad \Omega = \frac{2}{1 + \cos\theta} \ .}$$

Evolution of Ω: At $t = 0$, $\theta = 0$, so $\Omega = 1$. Any (matter-dominated) closed universe begins with $\Omega = 1$.

As θ increases from 0 to π, Ω grows from 1 to infinity. At $\theta = \pi$, a reaches its maximum size, and $H = 0$. So $\rho_c = 0$ and $\Omega = \infty$.

During the collapse phase, $\pi < \theta < 2\pi$, Ω falls from ∞ to 1.

What about $\sin\theta$?

$$\sin\theta = \pm\sqrt{1-\cos^2\theta} = \pm\frac{2\sqrt{\Omega-1}}{\Omega} \ .$$

$\sin\theta$ is positive during the expansion phase (while $0 < \theta < \pi$), and negative during the collapse phase (while $\pi < \theta < 2\pi$).

Evolution of a Closed Universe

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$$t = \frac{\Omega}{2|H|(\Omega - 1)^{3/2}} \left\{ \arcsin\left(\pm \frac{2\sqrt{\Omega - 1}}{\Omega} \right) \mp \frac{2\sqrt{\Omega - 1}}{\Omega} \right\} \ .$$

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Quadrant	Phase	Ω	Sign Choice	$\sin^{-1}()$
1	Expanding	1 to 2	Upper	0 to $\frac{\pi}{2}$
2	Expanding	2 to ∞	Upper	$\frac{\pi}{2}$ to π
3	Contracting	∞ to 2	Lower	π to $\frac{3\pi}{2}$
4	Contracting	2 to 1	Lower	$\frac{3\pi}{2}$ to 2π