8.286 Class 9
October 5, 2020

THE DYNAMICS OF NEWTONIAN COSMOLOGY, PART 4

Announcements

- Quiz 1 came off smoothly, and the class did extremely well. Class average was 92.3, which is amazing. There were 4 perfect papers, 3 99's, 1 98, 2 97's, and 2 96's. I should have your grades, solutions, and a grade histogram with estimated letter-grade cuts all posted this afternoon.
- One significant cause for delay was Problem 1(e), "Why is the night sky not uniformly bright?". Bruno and I exchanged many emails about this one. The answer we intended was (iii), referring to

  (C) The universe is not infinitely old.

  (E) The cosmological redshift makes stars look dimmer and dimmer as they are further away from us.

  Actually, we view (E) as the most important factor for our universe. The surface brightness of a star at redshift z falls off as $1/(1+z)^4$. (You've derived the pieces: total radiation flux $\propto 1/(1+z)^2$ – one power from loss of energy of each photon, and one power from rate of arrival of photons. In addition, angular size $\theta \propto (1+z)$, so solid angle $\propto (1+z)^2$.) So stars at high z contribute little to the night sky brightness.

For comparison, the finite age: The finite age means that we don't see any stars further than the horizon distance — the present distance of the most distant objects for which light has had time to reach us. What is the redshift at the horizon?

Ans: infinite. Time of emission $t_e = 0$, so $a(t_e) = 0$, and $1 + z = a(t_0)/a(t_e) = \infty$.

We finally decided that, depending on how one interpreted the question, any of the answers can arguably be true, so in the end we decided to give credit for any answer.

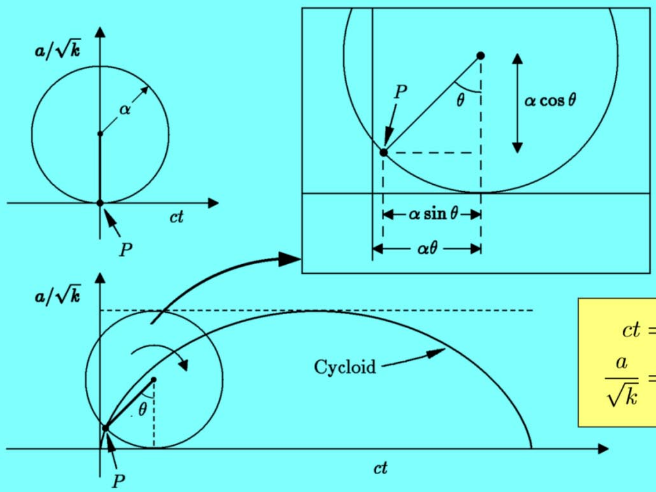
Review from Class 7

Parametric Solution for the Evolution of a Closed Matter-Dominated Universe

$$ct = \alpha(\theta - \sin\theta) \; ,$$
$$\frac{a}{\sqrt{k}} = \alpha(1 - \cos\theta) \; .$$

The angle θ is sometimes called the "development angle," because it describes the stage of development of the universe. The universe begins at $\theta = 0$, reaches its maximum expansion at $\theta = \pi$, and then is terminated by a big crunch at $\theta = 2\pi$.

Review from Class 7



$$ct = \alpha(\theta - \sin\theta) \ ,$$
$$\frac{a}{\sqrt{k}} = \alpha(1 - \cos\theta) \ .$$

Review from Class 7

Duration and Maximum Size

$$\frac{a}{\sqrt{k}} = \alpha(1-\cos\theta) \quad \Longrightarrow \quad \frac{a_{\rm max}}{\sqrt{k}} = 2\alpha \ ,$$

where

$$\alpha = \frac{4\pi}{3}\frac{G\rho a^3}{k^{3/2}c^2} \ .$$

Similarly, $ct = \alpha(\theta - \sin\theta)$ implies that the total duration of the universe, from big bang to big crunch is

$$t_{\rm total} = \frac{2\pi\alpha}{c} = \frac{\pi a_{\rm max}}{c\sqrt{k}} \ .$$

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Age of a Closed Matter-Dominated Universe

$$ct = \alpha(\theta - \sin\theta)$$

gives the age in terms of α and θ. But astronomers measure H and Ω. So we would like to express the age in terms of H and Ω.

Start with ρ:

$$\rho = \Omega\rho_c = \left(\frac{3H^2}{8\pi G}\right)\Omega \ .$$

The first-order Friedmann equation can then be rewritten as

$$H^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{a^2} \quad \Longrightarrow \quad H^2 = H^2\Omega - \frac{kc^2}{a^2} \ ,$$

so

$$\tilde{a} = \frac{a}{\sqrt{k}} = \frac{c}{|H|\sqrt{\Omega - 1}} \ .$$

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$$\tilde{a} = \frac{a}{\sqrt{k}} = \frac{c}{|H|\sqrt{\Omega - 1}} \ .$$

In taking the square root, recall that $a > 0$, $k > 0$, while H changes sign — it is positive during the expansion phase, and negative during the collapse phase. So we need $|H|$, not just H, for the equation to be valid. Then

$$\alpha = \frac{4\pi}{3}\frac{G\rho\tilde{a}^3}{c^2} = \frac{c}{2|H|}\frac{\Omega}{(\Omega-1)^{3/2}} \ .$$

To find age, we need to express α and θ in terms of H and Ω. To express θ, use expression for $\tilde{a}$ above, and 2nd parametric equation

$$\tilde{a} = \frac{a}{\sqrt{k}} = \alpha(1 - \cos\theta) \ .$$

Then

$$\frac{c}{|H|\sqrt{\Omega - 1}} = \frac{c}{2|H|}\frac{\Omega}{(\Omega-1)^{3/2}}(1 - \cos\theta) \ ,$$

Then

$$\frac{c}{|H|\sqrt{\Omega-1}} = \frac{c}{2|H|}\,\frac{\Omega}{(\Omega-1)^{3/2}}(1-\cos\theta)\ ,$$

which can be solved for either $\cos\theta$ or for Ω:

$$\cos\theta = \frac{2-\Omega}{\Omega}\ ,\quad \Omega = \frac{2}{1+\cos\theta}\ .$$

Evolution of Ω: At $t=0$, $\theta=0$, so $\Omega=1$. Any (matter-dominated) closed universe begins with $\Omega=1$.

As θ increases from 0 to π, Ω grows from 1 to infinity. At $\theta=\pi$, a reaches its maximum size, and $H=0$. So $\rho_c=0$ and $\Omega=\infty$.

During the collapse phase, $\pi<\theta<2\pi$, Ω falls from ∞ to 1.

What about $\sin\theta$?

$$\sin\theta = \pm\sqrt{1-\cos^2\theta} = \pm\frac{2\sqrt{\Omega-1}}{\Omega}\ .$$

$\sin\theta$ is positive during the expansion phase (while $0<\theta<\pi$), and negative during the collapse phase (while $\pi<\theta<2\pi$).

Evolution of a Closed Universe

$$ct = \alpha(\theta - \sin\theta)\ ,$$
$$\frac{a}{\sqrt{k}} = \alpha(1-\cos\theta)\ .$$

$$t = \frac{\Omega}{2|H|(\Omega-1)^{3/2}}\left\{\sin^{-1}\left(\pm\frac{2\sqrt{\Omega-1}}{\Omega}\right) \mp \frac{2\sqrt{\Omega-1}}{\Omega}\right\}\ .$$

$$t = \frac{\Omega}{2|H|(\Omega-1)^{3/2}}\left\{\sin^{-1}\left(\pm\frac{2\sqrt{\Omega-1}}{\Omega}\right) \mp \frac{2\sqrt{\Omega-1}}{\Omega}\right\}\ .$$

Quadrant	θ	Phase	Ω	Sign Choice
1	0 to $\frac{\pi}{2}$	Expanding	1 to 2	Upper
2	$\frac{\pi}{2}$ to π	Expanding	2 to ∞	Upper
3	π to $\frac{3\pi}{2}$	Contracting	∞ to 2	Lower
4	$\frac{3\pi}{2}$ to 2π	Contracting	2 to 1	Lower

Evolution of Open Matter-Dominated Universes

$$ct = \alpha(\sinh\theta - \theta) \ ,$$
$$\frac{a}{\sqrt{\kappa}} = \alpha(\cosh\theta - 1) \ .$$

where $\kappa = -k$, and

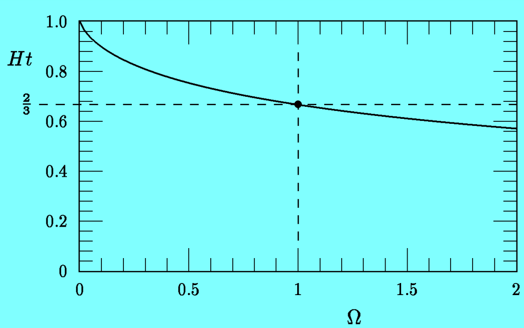
$$\tilde{a}(t) = \frac{a(t)}{\sqrt{\kappa}} \ , \qquad \alpha \equiv \frac{4\pi}{3}\frac{G\rho\tilde{a}^3}{c^2} \ .$$

θ evolves from 0 to ∞.

Age for Open, Flat, and Closed Matter-Dominated Universes

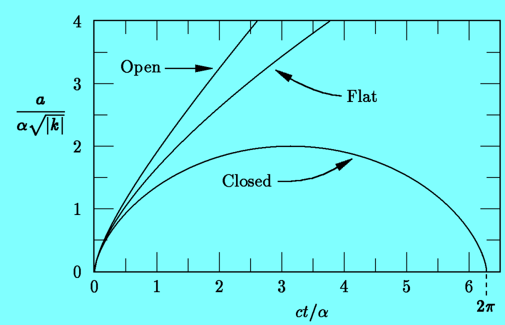
$$|H|\,t = \begin{cases} \dfrac{\Omega}{2(1-\Omega)^{3/2}}\left[\dfrac{2\sqrt{1-\Omega}}{\Omega} - \sinh^{-1}\left(\dfrac{2\sqrt{1-\Omega}}{\Omega}\right)\right] & \text{if } \Omega < 1 \\ 2/3 & \text{if } \Omega = 1 \\ \dfrac{\Omega}{2(\Omega-1)^{3/2}}\left[\sin^{-1}\left(\pm\dfrac{2\sqrt{\Omega-1}}{\Omega}\right) \mp \dfrac{2\sqrt{\Omega-1}}{\Omega}\right] & \text{if } \Omega > 1 \end{cases}$$

The Age of a Matter-Dominated Universe



The age of a matter-dominated universe, expressed as Ht (where t is the age and H is the Hubble expansion rate), as a function of Ω. The curve describes all three cases of an open ($\Omega < 1$), flat ($\Omega = 1$), and closed ($\Omega > 1$) universe.

Evolution of a Matter-Dominated Universe



The evolution of a matter-dominated universe. Closed and open universes can be characterized by a single parameter α. With the scalings shown on the axis labels, the evolution of a matter-dominated universe is described in all cases by the curves shown in this graph.