8.286 Class 11
October 13, 2020

INTRODUCTION TO NON-EUCLIDEAN SPACES PART 3

(Modified 10/23/20 to add note about the meaning of $g_{ij}\left(x^k\right)$ at the bottom of slide 16, and to mark the end of slides reached in class.)

Announcements

Questions about quiz grading (or problem set grading):

Please ask either Bruno or me. We try to grade accurately, but sometimes we make mistakes. We are **always** happy to discuss this with you, and are happy to make changes when grading errors are found.

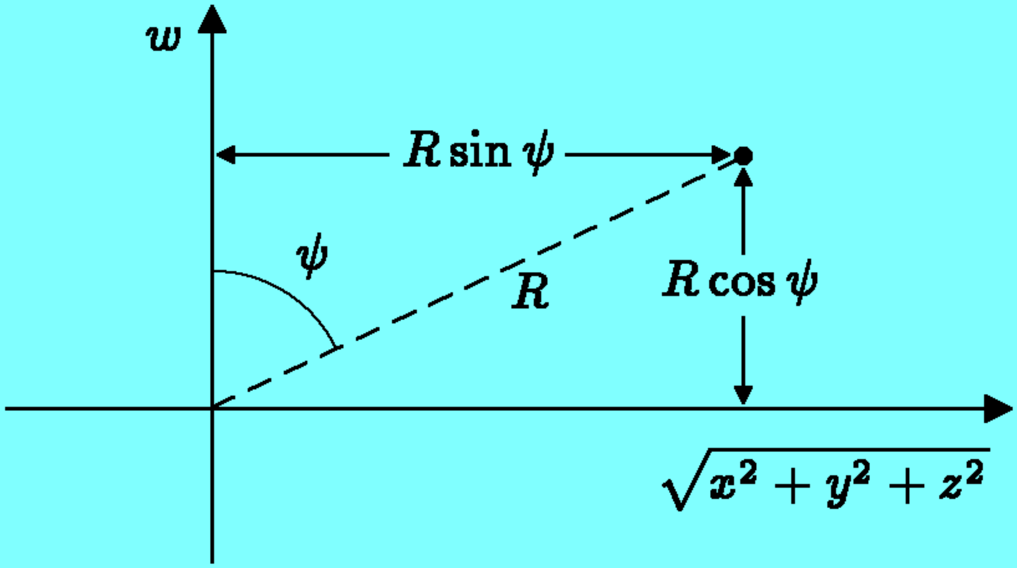
Review from previous class:

A Closed Three-Dimensional Space

$$x^2 + y^2 + z^2 + w^2 = R^2$$

$$x = R \sin\psi \sin\theta \cos\phi$$
$$y = R \sin\psi \sin\theta \sin\phi$$
$$z = R \sin\psi \cos\theta$$
$$w = R \cos\psi \ ,$$

$$\mathrm{d}s = R\, d\psi$$



Review from previous class:

Metric for the Closed 3D Space

Varying ψ: $\quad \mathrm{d}s = R\,d\psi$

Varying θ or ϕ: $\quad \mathrm{d}s^2 = R^2 \sin^2\psi(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)$

If the variations are orthogonal to each other, then

$$\mathrm{d}s^2 = R^2\left[\mathrm{d}\psi^2 + \sin^2\psi\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right]$$

Review from previous class, but enlarged:

Implications of General Relativity

- ☆ $ds^2 = R^2 \left[d\psi^2 + \sin^2\psi \left(d\theta^2 + \sin^2\theta \, d\phi^2\right)\right]$, where R is radius of curvature.
- ☆ According to GR, matter causes space to curve. So R, the curvature radius, should be determined by the matter.
- ☆ From the metric, or from the picture of a sphere of radius R in a 4D Euclidean embedding space, it is clear that R determines the size of the space. But $a(t)$, the scale factor, also determines the size of the space. So they must be proportional.
- ☆ But R is in meters, $a(t)$ in meters/notch. So dimensional consistency $\Longrightarrow$ $R \propto a(t)/\sqrt{k}$, since $[k] = \text{notch}^{-2}$.
- ☆ In fact,

$$R^2(t) = \frac{a^2(t)}{k} \; .$$

(I do not know any way to explain why the proportionality constant is 1, except by using the full equations of GR.)

- $ds^2 = R^2 \left[d\psi^2 + \sin^2\psi \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right) \right]$, where R is radius of curvature.
- In fact,

$$R^2(t) = \frac{a^2(t)}{k} \; .$$

- So,

$$ds^2 = \frac{a^2(t)}{k} \left[d\psi^2 + \sin^2\psi \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right) \right] \; .$$

- It is common to introduce a new radial variable $r \equiv \sin\psi/\sqrt{k}$, so $dr = \cos\psi \, d\psi/\sqrt{k} = \sqrt{1 - kr^2} \, d\psi/\sqrt{k}$. In terms of r,

$$ds^2 = a^2(t) \left\{ \frac{dr^2}{1 - kr^2} + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right) \right\} \; .$$

This is the spatial part of the Robertson-Walker metric.

Open Universes

- For $k > 0$ (closed universe),

$$ds^2 = a^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right\}$$

describes a homogeneous isotropic universe.

- For $k < 0$ (open universe),

$$ds^2 = a^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right\}$$

still describes a homogeneous isotropic universe.

- Properties are very different. The closed universe reaches its equator at $r = 1/\sqrt{k}$, which is a finite distance from the origin,

$$a(t)\int_0^{1/\sqrt{k}} \frac{dr}{\sqrt{1-kr^2}} = \frac{\pi a(t)}{2\sqrt{k}}\ .$$

The total volume is finite. For the open universe, r has no limit, and the volume is infinite.

From Space to Spacetime

In special relativity,

$$s^2_{AB} \equiv (x_A - x_B)^2 + (y_A - y_B)^2 + (z_A - z_B)^2 - c^2\,(t_A - t_B)^2 \quad .$$

s^2_{AB} is Lorentz-invariant — it has the same value for all inertial reference frames.

From Space to Spacetime

In special relativity,

$$s_{AB}^2 \equiv (x_A - x_B)^2 + (y_A - y_B)^2 + (z_A - z_B)^2 - c^2\,(t_A - t_B)^2 \;\; .$$

s_{AB}^2 is Lorentz-invariant — it has the same value for all inertial reference frames.

Meaning of s_{AB}^2:

If positive, it is the distance^2 between the two events in the inertial frame in which they are simultaneous. (Spacelike.)

If negative, then $s_{AB}^2 = -c^2\,\Delta\tau^2$, where $\Delta\tau$ is the time interval between the two events in the inertial frame in which they occur at the same place. (Timelike.)

If zero, it implies that a light pulse could travel from the earlier to the later event. (Lightlike.)

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☆ If you are interested, Lecture Notes 5 has an appendix which derives the Lorentz transformation from time dilation, Lorentz contraction, and the relativity of simultaneity, and shows that s_{AB}^2 is invariant.

Infinitesimal Separations and the Metric

- Following Gauss, we focus on the distance between infinitesimally separated points. So

$$s_{AB}^2 \equiv (x_A - x_B)^2 + (y_A - y_B)^2 + (z_A - z_B)^2 - c^2 (t_A - t_B)^2$$

is replaced by

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 - c^2\,\mathrm{d}t^2 \;,$$

which is called the *Minkowski metric.*

- The interpretation is the same as before: $\mathrm{d}s^2 > 0 \implies$ distance^2 in frame where events are simultaneous; $\mathrm{d}s^2 < 0 \implies \mathrm{d}s^2 = -c^2\,\mathrm{d}\tau^2$, where $\mathrm{d}\tau$ = time difference in frame where events are at same place; $\mathrm{d}s^2 = 0 \implies$ light can travel from one event to the other.

- This will be our springboard to metric used in general relativity.

Coordinates in Curves Spaces

- In Newtonian physics or special relativity, coordinates have a direct physical meaning: they directly measure distances or time intervals.
- In curves spaces, there is generally no way to construct coordinates that are directly connected to distances.
- For example, on the surface of the Earth we measure East-West position by longitude, but the distance for a longitude distance of 1 degree depends on the latitude.
- Bottom line: in general relativity (or in any curved space), coordinates are just arbitrary markers, with any set of coordinates in principle as good as any other.
- Distances are determined from the coordinates, using the metric.
- If one changes from one coordinate system to another, one changes the metric so that distances remain unchanged.

General Relativity: the Equivalence Principle and Free-Falling Observers

☆ Consider a person holding a rock inside an elevator, initially at rest.

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- ☆ Now imagine that the elevator cable is cut, so the elevator falls — we assume that there is no friction or air resistance. The elevator, person, and rock all accelerate together. The person no longer feels his feet pressed to the floor; if he lets go of the rock, it floats. The effects of gravity have disappeared.

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- Now imagine that the elevator cable is cut, so the elevator falls — we assume that there is no friction or air resistance. The elevator, person, and rock all accelerate together. The person no longer feels his feet pressed to the floor; if he lets go of the rock, it floats. The effects of gravity have disappeared.
- The *Equivalence Principle* says that the disappearance of gravity is precise: as long as the elevator is small enough so that the gravitational field is uniform, then there is absolutely no way that the person in the free-falling elevator can detect the gravitational field of the Earth.

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- The person in the elevator is called a *free-falling observer*, and the local coordinate system that he would construct in his immediate vicinity is called a free-falling coordinate system. The metric for the free-falling coordinates, in the immediate vicinity of the person, is described by the Minkowski metric. It is called *locally Minkowskian*.

- We mentioned earlier that any quadratic metric for space (i.e., a positive definite metric) is locally Euclidean. If the metric is negative for one direction, then it is always locally Minkowskian.

Adding Time to the Robertson-Walker Metric

$$ds^2 = -c^2\,dt^2 + a^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\} .$$

Why does dt^2 term look like it does:

- The coefficient of dt^2 term must be independent of position, due to homogeneity.
- Terms such as $dt\,dr$ or $dt\,d\phi$ cannot appear, due to isotropy. That is, a term $dt\,dr$ would behave differently for $dr > 0$ and $dr < 0$, creating an asymmetry between the $+r$ and -r directions.
- The coefficient must be negative, to match the sign in Minkowski space for a locally free-falling coordinate system.

Adding Time to the Robertson-Walker Metric

$$ds^2 = -c^2\,dt^2 + a^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\} .$$

Meaning:

- If $ds^2 > 0$, it is the square of the spatial separation measured by a local free-falling observer for whom the two events happen at the same time.
- If $ds^2 < 0$, it is $-c^2$ times the square of the time separation measured by a local free-falling observer for whom the two events happen at the same location.
- If $ds^2 = 0$, then the two events can be joined by a light pulse.

Summary: Metrics of Interest

Minkowski Metric: (Special relativity)

$$\begin{aligned} ds^2 &= -c^2 dt^2 + dx^2 + dy^2 + dz^2 \\ &= -c^2 dt^2 + dr^2 + r^2 (d\theta^2 + \sin^2\theta\, d\phi^2) \ . \end{aligned}$$

Robertson–Walker Metric:

$$ds^2 = -c^2\, dt^2 + a^2(t) \left\{ \frac{dr^2}{1 - kr^2} + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2 \right) \right\} \ .$$

Meaning: If $ds^2 > 0$, ds is distance in freely falling frame in which events are simultaneous. If $ds^2 < 0$, $ds^2 = -c^2 d\tau^2$, where $d\tau$ is time interval in freely falling frame in which events occur at same point. If $ds^2 = 0$, events are lightlike separated.

Geodesics in General Relativity

A geodesic is a path connecting two points in spacetime, with the property that the length of the curve is stationary with respect to small changes in the path. It can be a maximum, minimum, or saddle point.

In a curved spacetime, a geodesic is the closest thing to a straight line that exists.

In general relativity, if no forces act on a particle other than gravity, the particle travels on a geodesic.

Geodesics in Two Spatial Dimensions

Metric:

$$ds^2 = g_{xx}dx^2 + g_{xy}dx\,dy + g_{yx}dy\,dx + g_{yy}dy^2 \ .$$

Let $x^1 \equiv x$, $x^2 \equiv y$, so x^i is either, as $i = 1$ or 2.

$$ds^2 = \sum_{i=1}^{2}\sum_{j=1}^{2} g_{ij}(x^k)\,dx^i\,dx^j$$

$$= g_{ij}(x^k)\,dx^i\,dx^j \ .$$

Einstein summation convention: repeated indices within one term are summed over coordinate indices (1 and 2), unless otherwise specified.

The sum is always over one upper index and one lower, but we will not discuss why some indices are written as upper and some as lower.

$g_{ij}(x^k)$ indicates that g_{ij} is a function of all the components of x^k, i.e., x^1 and x^2.

The Length of Path

Consider a path from A to B.

Path description: $x^i(\lambda)$, where λ is parameter running from 0 to λ_f.

$$x^i(0) = x^i_A , \qquad x^i(\lambda_f) = x^i_B \ .$$

Between λ and $\lambda + \mathrm{d}\lambda$,

$$\mathrm{d}x^i = \frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\mathrm{d}\lambda \ ,$$

so

$$\mathrm{d}s^2 = g_{ij}\left(x^k(\lambda)\right) \frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\,\mathrm{d}\lambda^2 \ ,$$

and then

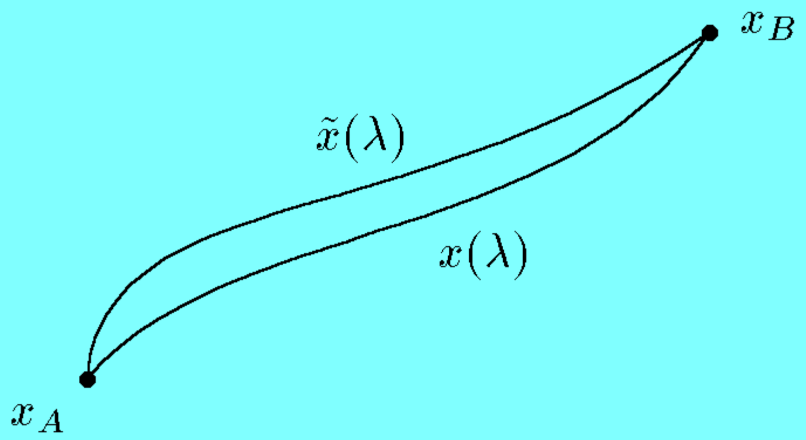
$$\mathrm{d}s = \sqrt{g_{ij}\left(x^k(\lambda)\right) \frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}}\,\mathrm{d}\lambda \ ,$$

and

$$S[x^i(\lambda)] = \int_0^{\lambda_f} \sqrt{g_{ij}\left(x^k(\lambda)\right) \frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}}\,\mathrm{d}\lambda \ .$$

This and the following slides were not reached, but will be discussed in the next class.

Varying the Path



$$\tilde{x}^i(\lambda) = x^i(\lambda) + \alpha w^i(\lambda) \ ,$$

where

$$w^i(0) = 0 \ , \qquad w^i(\lambda_f) = 0 \ .$$

Geodesic condition:

$$\left. \frac{\mathrm{d}\, S\left[\tilde{x}^i(\lambda)\right]}{\mathrm{d}\alpha} \right|_{\alpha=0} = 0 \qquad \text{for all } w^i(\lambda) \quad .$$

Define

$$A(\lambda, \alpha) = g_{ij}\left(\tilde{x}^k(\lambda)\right) \frac{\mathrm{d}\tilde{x}^i}{\mathrm{d}\lambda} \frac{\mathrm{d}\tilde{x}^j}{\mathrm{d}\lambda} \ ,$$

so we can write

$$\begin{aligned} S\left[\tilde{x}^i(\lambda)\right] &= \int_0^{\lambda_f} \sqrt{g_{ij}\left(x^k(\lambda)\right) \frac{\mathrm{d}x^i}{\mathrm{d}\lambda} \frac{\mathrm{d}x^j}{\mathrm{d}\lambda}}\, \mathrm{d}\lambda \\ &= \int_0^{\lambda_f} \sqrt{A(\lambda, \alpha)}\, \mathrm{d}\lambda \ . \end{aligned}$$

Using chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} g_{ij}\left(\tilde{x}^k(\lambda)\right)\bigg|_{\alpha=0} = \frac{\partial g_{ij}}{\partial x^k}\bigg|_{x^k=x^k(\lambda)} \frac{\partial \tilde{x}^k}{\partial \alpha}\bigg|_{\alpha=0} = \frac{\partial g_{ij}}{\partial x^k}\left(x^\ell(\lambda)\right) w^k \ ,$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\left(\frac{\partial \tilde{x}^i}{\partial \lambda}\right) = \frac{\mathrm{d}}{\mathrm{d}\alpha}\left[\frac{\partial x^i(\lambda)}{\partial \lambda} + \alpha \frac{\partial w^i(\lambda)}{\partial \lambda}\right] = \frac{\partial w^i(\lambda)}{\partial \lambda} \ .$$

$$S\left[\tilde{x}^i(\lambda)\right] = \int_0^{\lambda_f} \sqrt{A(\lambda,\alpha)}\,\mathrm{d}\lambda\ ,$$

where

$$A(\lambda,\alpha) = g_{ij}\left(\tilde{x}^k(\lambda)\right)\frac{\mathrm{d}\tilde{x}^i}{\mathrm{d}\lambda}\frac{\mathrm{d}\tilde{x}^j}{\mathrm{d}\lambda}\ ,$$

with

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} g_{ij}\left(\tilde{x}^k(\lambda)\right)\bigg|_{\alpha=0} = \frac{\partial g_{ij}}{\partial x^k}\left(x^\ell(\lambda)\right) w^k\ , \qquad \frac{\mathrm{d}}{\mathrm{d}\alpha}\left(\frac{\partial \tilde{x}^i}{\partial\lambda}\right) = \frac{\partial w^i(\lambda)}{\partial\lambda}\ .$$

Then

$$\frac{\mathrm{d}S\left[\tilde{x}^i(\lambda)\right]}{\mathrm{d}\alpha}\bigg|_{\alpha=0} = \frac{1}{2}\int_0^{\lambda_f}\frac{1}{\sqrt{A(\lambda,0)}}\left\{\frac{\partial g_{ij}}{\partial x^k} w^k \frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda} + \right.$$

$$\left. + g_{ij}\frac{\mathrm{d}w^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda} + g_{ij}\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}w^j}{\mathrm{d}\lambda}\right\}\mathrm{d}\lambda\ ,$$

where the metric g_{ij} is to be evaluated at $x^\ell(\lambda)$.

$$\left.\frac{\mathrm{d}S\left[\tilde{x}^i(\lambda)\right]}{\mathrm{d}\alpha}\right|_{\alpha=0} = \frac{1}{2}\int_0^{\lambda_f} \frac{1}{\sqrt{A(\lambda,0)}}\left\{\frac{\partial g_{ij}}{\partial x^k} w^k \frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda} + \right.$$

$$\left. + g_{ij}\frac{\mathrm{d}w^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda} + g_{ij}\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}w^j}{\mathrm{d}\lambda}\right\}\mathrm{d}\lambda \; .$$

Manipulating "dummy" indices: in third term, replace $i \to j$ and $j \to i$, and recall that $g_{ij} = g_{ji}$. Then 2nd & 3rd term are equal:

$$\left.\frac{\mathrm{d}S\left[\tilde{x}^i(\lambda)\right]}{\mathrm{d}\alpha}\right|_{\alpha=0} = \frac{1}{2}\int_0^{\lambda_f} \frac{1}{\sqrt{A(\lambda,0)}}\left\{\frac{\partial g_{ij}}{\partial x^k} w^k \frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda} + 2g_{ij}\frac{\mathrm{d}w^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\right\}\mathrm{d}\lambda \; .$$

Repeating,

$$\left.\frac{\mathrm{d}S\left[\tilde{x}^i(\lambda)\right]}{\mathrm{d}\alpha}\right|_{\alpha=0} = \frac{1}{2}\int_0^{\lambda_f} \frac{1}{\sqrt{A(\lambda,0)}}\left\{\frac{\partial g_{ij}}{\partial x^k} w^k \frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda} + 2 g_{ij}\frac{\mathrm{d}w^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\right\}\mathrm{d}\lambda \, .$$

Integration by Parts: Integral depends on both w^k and $\mathrm{d}w^i/\mathrm{d}\lambda$. Can eliminate $\mathrm{d}w^i/\mathrm{d}\lambda$ by integrating by parts:

$$\int_0^{\lambda_f}\left[\frac{1}{\sqrt{A}} g_{ij}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\right]\frac{\mathrm{d}w^i}{\mathrm{d}\lambda}\,\mathrm{d}\lambda = \int_0^{\lambda_f}\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\frac{1}{\sqrt{A}} g_{ij}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda} w^i\right]\mathrm{d}\lambda - \int_0^{\lambda_f}\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\frac{1}{\sqrt{A}} g_{ij}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\right] w^i\,\mathrm{d}\lambda \, .$$

But

$$\int_0^{\lambda_f}\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\frac{1}{\sqrt{A}} g_{ij}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda} w^i\right]\mathrm{d}\lambda = \left.\left[\frac{1}{\sqrt{A}} g_{ij}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda} w^i\right]\right|_{\lambda=0}^{\lambda=\lambda_f} = 0 \, ,$$

since $w^i(\lambda)$ vanishes at $\lambda = 0$ and $\lambda = \lambda_f$.

So

$$\left.\frac{\mathrm{d}S}{\mathrm{d}\alpha}\right|_{\alpha=0} = \frac{1}{2}\int_0^{\lambda_f} \left\{ \frac{1}{\sqrt{A}} \frac{\partial g_{ij}}{\partial x^k} \frac{\mathrm{d}x^i}{\mathrm{d}\lambda} \frac{\mathrm{d}x^j}{\mathrm{d}\lambda} w^k - 2\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\frac{1}{\sqrt{A}} g_{ij} \frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\right] w^i \right\} \mathrm{d}\lambda \ .$$

More index juggling: in 1st term replace $i \to j, j \to k, k \to i$:

$$\left.\frac{\mathrm{d}S}{\mathrm{d}\alpha}\right|_{\alpha=0} = \int_0^{\lambda_f} \left\{ \frac{1}{2\sqrt{A}} \frac{\partial g_{jk}}{\partial x^i} \frac{\mathrm{d}x^j}{\mathrm{d}\lambda} \frac{\mathrm{d}x^k}{\mathrm{d}\lambda} - \frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\frac{1}{\sqrt{A}} g_{ij} \frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\right] \right\} w^i(\lambda)\, \mathrm{d}\lambda \ .$$

To vanish ***for all $w^i(\lambda)$*** which vanish at $\lambda = 0$ and $\lambda = \lambda_f$, the quantity in curly brackets must vanish. If not, then suppose that $\{\,\} \neq 0$ at some $\lambda = \lambda_0$. By continuity, $\{\,\} \neq 0$ in some neighborhood of λ_0. Choose $w^i(\lambda)$ to be positive in this neighborhood, and zero everywhere else, and one has a contradiction.

So

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\frac{1}{\sqrt{A}} g_{ij} \frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\right] = \frac{1}{2\sqrt{A}} \frac{\partial g_{jk}}{\partial x^i} \frac{\mathrm{d}x^j}{\mathrm{d}\lambda} \frac{\mathrm{d}x^k}{\mathrm{d}\lambda} \quad .$$

Repeating,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\frac{1}{\sqrt{A}} g_{ij} \frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\right] = \frac{1}{2\sqrt{A}} \frac{\partial g_{jk}}{\partial x^i} \frac{\mathrm{d}x^j}{\mathrm{d}\lambda} \frac{\mathrm{d}x^k}{\mathrm{d}\lambda} \quad .$$

This is ***complicated,*** since A is complicated.

Simplify by choice of parameterization: This result is valid for any parameterization. We don't need that! We can choose λ to be the path length. Since

$$\mathrm{d}s = \sqrt{g_{ij}\big(x^k(\lambda)\big) \frac{\mathrm{d}x^i}{\mathrm{d}\lambda} \frac{\mathrm{d}x^j}{\mathrm{d}\lambda}}\, \mathrm{d}\lambda = \sqrt{A}\, \mathrm{d}\lambda \ ,$$

we see that $\mathrm{d}\lambda = \mathrm{d}s$ implies

$$A = 1 \qquad (\text{for } \lambda = \text{path length}).$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[g_{ij} \frac{\mathrm{d}x^j}{\mathrm{d}s}\right] = \frac{1}{2} \frac{\partial g_{jk}}{\partial x^i} \frac{\mathrm{d}x^j}{\mathrm{d}s} \frac{\mathrm{d}x^k}{\mathrm{d}s} \quad .$$

Alternative Form of Geodesic Equation

Most books write the geodesic equation differently, as

$$\frac{d^2x^i}{ds^2} = -\Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} ,$$

where

$$\Gamma^i_{jk} = \frac{1}{2}g^{i\ell}\left(\partial_j g_{\ell k} + \partial_k g_{\ell j} - \partial_\ell g_{jk}\right)$$

and $g^{i\ell}$ is the matrix inverse of g_{ij}. The quantity Γ^i_{jk} is called the affine connection.

If you are interested, see the lecture notes. If you are not interested, you can skip this.

BLACK HOLES (Fun!)

The Schwarzschild Metric:

For any spherically symmetric distribution of mass, outside the mass the metric is given by the Schwarzschild metric,

$$\mathrm{d}s^2 = -c^2\mathrm{d}\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right) c^2\mathrm{d}t^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2 \sin^2\theta \, \mathrm{d}\phi^2 \; ,$$

where M is the total mass, G is Newton's gravitational constant, c is the speed of light, and θ and ϕ have the usual polar-angle ranges.

Schwarzschild Horizon

$$ds^2 = -c^2 d\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2$$
$$+ r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \; .$$

The metric is singular at

$$r = R_S \equiv \frac{2GM}{c^2} \; ,$$

where the coefficient of $c^2 dt^2$ vanishes, and the coefficient of dr^2 is infinite.

Surprisingly, this singularity is not real — it is a coordinate artifact. There are other coordinate systems where the metric is smooth at R_S.

But R_S is a ***horizon:*** If you fall past the horizon, there is no return, even if you are photon.

Schwarzschild Radius of the Sun

$$R_{S,\odot} = \frac{2GM}{c^2}$$

$$= \frac{2 \times 6.673 \times 10^{-11}\ \text{m}^3\text{-kg}^{-1}\text{-s}^{-2} \times 1.989 \times 10^{30}\ \text{kg}}{(2.998 \times 10^8\ \text{m-s}^{-1})^2}$$

$$= 2.95\ \text{km}\ .$$

- ☆ If the Sun were compressed to this radius, it would become a black hole. Since the Sun is much larger than R_S, and the Schwarzschild metric is only valid outside the matter, there is no Schwarzschild horizon in the Sun.
- ☆ At the center of our galaxy is a supermassive black hole, with $M = 4.1 \times 10^6\ M_\odot$. This gives $R_S = 1.2 \times 10^{10}$ meters $\approx 1/4$ of radius of orbit of Mercury ≈ 17 times radius of Sun.

Radial Geodesics in the Schwarzschild Metric

$$\mathrm{d}s^2 = -c^2\mathrm{d}\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right)c^2\mathrm{d}t^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1}\mathrm{d}r^2$$
$$+ r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2\ .$$

Consider a particle released from rest at $r = r_0$.

r is a "radial coordinate," but not the radius, since it is not the distance from some center. If r is varied by $\mathrm{d}r$, the distance traveled is not $\mathrm{d}r$, but $\mathrm{d}r/\sqrt{1 - 2GM/rc^2}$. r can be called the "circumferential radius," since the term $r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)$ in the metric implies that the circumference of a circle about the origin is $2\pi r$.

By symmetry, the particle will fall straight down, with no change in θ or ϕ. Spherical symmetry implies that all directions in θ and ϕ are equivalent, so any motion in θ–ϕ space would violate this symmetry.

Particle Trajectories in Spacetime

Particle trajectories are timelike, so we use proper time τ to parameterize them, where $\mathrm{d}s^2 \equiv -c^2\mathrm{d}\tau^2$. This implies that $A = -c^2$, instead of $A = 1$, but as long as A is constant, it drops out of the geodesic equation.

By tradition, the spacetime indices in general relativity are denoted by Greek letters such as μ, ν, λ, σ, and are summed from 0 to 3, where $x^0 \equiv t$.

The geodesic equation

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[g_{ij}\frac{\mathrm{d}x^j}{\mathrm{d}s}\right] = \frac{1}{2}\frac{\partial g_{jk}}{\partial x^i}\frac{\mathrm{d}x^j}{\mathrm{d}s}\frac{\mathrm{d}x^k}{\mathrm{d}s}$$

is then rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[g_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right] = \frac{1}{2}\frac{\partial g_{\lambda\sigma}}{\partial x^\mu}\frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau} \; .$$

Radial Trajectory Equations

Only $\mathrm{d}r/\mathrm{d}\tau$ and $\mathrm{d}t/\mathrm{d}\tau$ are nonzero. But they are related by the metric:

$$c^2 \mathrm{d}\tau^2 = \left(1 - \frac{2GM}{rc^2}\right) c^2\, \mathrm{d}t^2 - \left(1 - \frac{2GM}{rc^2}\right)^{-1} \mathrm{d}r^2$$

implies that

$$c^2 = \left(1 - \frac{2GM}{rc^2}\right) c^2 \left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 - \left(1 - \frac{2GM}{rc^2}\right)^{-1} \left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 .$$

Then, looking at the $\mu = r$ geodesic equation,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[g_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right] = \frac{1}{2}\frac{\partial g_{\lambda\sigma}}{\partial x^\mu}\frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau}$$

implies that

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[g_{rr}\frac{\mathrm{d}r}{\mathrm{d}\tau}\right] = \frac{1}{2}\partial_r g_{rr}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + \frac{1}{2}\partial_r g_{tt}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 ,$$

where

$$g_{rr} = \left(1 - \frac{2GM}{rc^2}\right)^{-1} , \quad g_{tt} = -c^2\left(1 - \frac{2GM}{rc^2}\right) .$$

Repeating,

$$c^2 = \left(1 - \frac{2GM}{rc^2}\right) c^2 \left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 - \left(1 - \frac{2GM}{rc^2}\right)^{-1} \left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 .$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[g_{rr}\frac{\mathrm{d}r}{\mathrm{d}\tau}\right] = \frac{1}{2}\partial_r g_{rr}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + \frac{1}{2}\partial_r g_{tt}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 ,$$

where

$$g_{rr} = \left(1 - \frac{2GM}{rc^2}\right)^{-1} , \quad g_{tt} = -c^2\left(1 - \frac{2GM}{rc^2}\right) .$$

Expand

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[g_{rr}\frac{\mathrm{d}r}{\mathrm{d}\tau}\right]$$

with the product rule, replace $(\mathrm{d}t/\mathrm{d}\tau)^2$ using the equation above, and simplify. Result:

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} = -\frac{GM}{r^2} ,$$

which looks just like Newton, but it is not really the same. Here τ is the proper time as measured by the infalling object, and r is not the radial distance.

Solving the Equation

$$\frac{d^2 r}{d\tau^2} = -\frac{GM}{r^2} \quad .$$

Like Newton's equation, multiply by $dr/d\tau$, and it can then be written as

$$\frac{d}{d\tau}\left\{\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 - \frac{GM}{r}\right\} = 0 \ .$$

Quantity in curly brackets is conserved. Initial value (on release from rest at r_0) is $-GM/r_0$, so it always has this value. Then

$$\frac{dr}{d\tau} = -\sqrt{2GM\left(\frac{1}{r} - \frac{1}{r_0}\right)} = -\sqrt{\frac{2GM(r_0 - r)}{r r_0}} \ .$$

Repeating,

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = -\sqrt{2GM\left(\frac{1}{r} - \frac{1}{r_0}\right)} = -\sqrt{\frac{2GM(r_0 - r)}{r r_0}} \ .$$

Bring all r-dependent factors to one side, and bring $\mathrm{d}\tau$ to the other side, and integrate:

$$\tau(r_f) = -\int_{r_0}^{r_f} \mathrm{d}r \sqrt{\frac{r r_0}{2GM(r_0 - r)}}$$

$$= \sqrt{\frac{r_0}{2GM}} \left\{ r_0 \tan^{-1}\left(\sqrt{\frac{r_0 - r_f}{r_f}}\right) + \sqrt{r_f(r_0 - r_f)} \right\} \ .$$

Conclusion: object will reach $r = 0$ in a finite proper time τ.

But Coordinate Time t is Different!

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t} = \frac{\mathrm{d}r/\mathrm{d}\tau}{\mathrm{d}t/\mathrm{d}\tau}$$

$$= \frac{\mathrm{d}r/\mathrm{d}\tau}{\sqrt{h^{-1}(r) + c^{-2}h^{-2}(r)\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2}} \ ,$$

where

$$h(r) \equiv 1 - \frac{R_S}{r} = 1 - \frac{2GM}{rc^2} \ .$$

Look at behavior near horizon; $h^{-1}(r)$ blows up:

$$h^{-1}(r) = \frac{r}{r - R_S} \approx \frac{R_S}{r - R_S} \ .$$

Denominator is dominated by 2nd term, which gives

$$\frac{\mathrm{d}r}{\mathrm{d}t} \approx c\left(\frac{r - R_S}{R_S}\right) \ .$$

Repeating,

$$\frac{\mathrm{d}r}{\mathrm{d}t} \approx c\left(\frac{r - R_S}{R_S}\right) \; .$$

Rearranging and integrating to some final $r = r_f$, one finds

$$t(r_f) \approx -\frac{R_S}{c}\int^{r_f} \frac{\mathrm{d}r'}{r' - R_S} \approx \boxed{-\frac{R_S}{c}\ln(r_f - R_S) \; .}$$

Thus t diverges logarithmically as $r_f \to R_S$, so the object does not reach R_S for any finite value of t.