8.286 Class 13<br>October 19, 2020

INTRODUCTION TO
NON-EUCLIDEAN SPACES
PART 5
BLACK-BODY RADIATION AND
THE EARLY HISTORY OF THE UNIVERSE

## Announcements

Reminder: Problem Set 6 is due this Friday at 5:00 pm.
Quiz 2 will be next Wednesday, October 28. Procedures will be the same as for Quiz 1. Precise coverage will be announced soon. Lecture Notes 6 will be included only through the Dynamics of a Flat Radiation-Dominated Universe, ending at the top of p. 12.

## BLACK HOLES (Fun!)

## The Schwarzschild Metric:

For any spherically symmetric distribution of mass, outside the mass the metric is given by the Schwarzschild metric,

$$
\begin{aligned}
\mathrm{d} s^{2}=-c^{2} \mathrm{~d} \tau^{2}=- & \left(1-\frac{2 G M}{r c^{2}}\right) c^{2} \mathrm{~d} t^{2}+\left(1-\frac{2 G M}{r c^{2}}\right)^{-1} \mathrm{~d} r^{2} \\
& +r^{2}\left(\mathrm{~d} \theta^{2}+\sin ^{2} \theta \mathrm{~d} \phi^{2}\right)
\end{aligned}
$$

where $M$ is the total mass, $G$ is Newton's gravitational constant, $c$ is the speed of light, and $\theta$ and $\phi$ have the usual polar-angle ranges.

## Schwarzschild Horizon

$$
\begin{aligned}
\mathrm{d} s^{2}=-c^{2} \mathrm{~d} \tau^{2}=- & \left(1-\frac{2 G M}{r c^{2}}\right) c^{2} \mathrm{~d} t^{2}+\left(1-\frac{2 G M}{r c^{2}}\right)^{-1} \mathrm{~d} r^{2} \\
& +r^{2}\left(\mathrm{~d} \theta^{2}+\sin ^{2} \theta \mathrm{~d} \phi^{2}\right)
\end{aligned}
$$

The metric is singular at

$$
r=R_{S} \equiv \frac{2 G M}{c^{2}}
$$

where the coefficient of $c^{2} \mathrm{~d} t^{2}$ vanishes, and the coefficient of $\mathrm{d} r^{2}$ is infinite.
Surprisingly, this singularity is not real - it is a coordinate artifact. There are other coordinate systems where the metric is smooth at $R_{S}$.
But $R_{S}$ is a horizon: If you fall past the horizon, there is no return, even if you are photon.

## Schwarzschild Radius of the Sun

$$
\begin{aligned}
R_{S, \odot} & =\frac{2 G M}{c^{2}} \\
& =\frac{2 \times 6.673 \times 10^{-11} \mathrm{~m}^{3}-\mathrm{kg}^{-1}-\mathrm{s}^{-2} \times 1.989 \times 10^{30} \mathrm{~kg}}{\left(2.998 \times 10^{8} \mathrm{~m}-\mathrm{s}^{-1}\right)^{2}} \\
& =2.95 \mathrm{~km} .
\end{aligned}
$$

is If the Sun were compressed to this radius, it would become a black hole. Since the Sun is much larger than $R_{S}$, and the Schwarzschild metric is only valid outside the matter, there is no Schwarzschild horizon in the Sun.
is At the center of our galaxy is a supermassive black hole, with $M=4.1 \times$ $10^{6} M_{\odot}$. This gives $R_{S}=1.2 \times 10^{10}$ meters $\approx 1 / 4$ of radius of orbit of Mercury $\approx 17$ times radius of Sun.

## Radial Geodesics in the Schwarzschild Metric

$$
\begin{aligned}
\mathrm{d} s^{2}=-c^{2} \mathrm{~d} \tau^{2}=- & \left(1-\frac{2 G M}{r c^{2}}\right) c^{2} \mathrm{~d} t^{2}+\left(1-\frac{2 G M}{r c^{2}}\right)^{-1} \mathrm{~d} r^{2} \\
& +r^{2}\left(\mathrm{~d} \theta^{2}+\sin ^{2} \theta \mathrm{~d} \phi^{2}\right)
\end{aligned}
$$

Consider a particle released from rest at $r=r_{0}$.
$r$ is a "radial coordinate," but not the radius, since it is not the distance from some center. If $r$ is varied by $\mathrm{d} r$, the distance traveled is not $\mathrm{d} r$, but $\mathrm{d} r / \sqrt{1-2 G M / r c^{2}} . r$ can be called the "circumferential radius," since the term $r^{2}\left(\mathrm{~d} \theta^{2}+\sin ^{2} \theta \mathrm{~d} \phi^{2}\right)$ in the metric implies that the circumference of a circle about the origin is $2 \pi r$.

By symmetry, the particle will fall straight down, with no change in $\theta$ or $\phi$. Spherical symmetry implies that all directions in $\theta$ and $\phi$ are equivalent, so any motion in $\theta-\phi$ space would violate this symmetry.

## Particle Trajectories in Spacetime

Particle trajectories are timelike, so we use proper time $\tau$ to parameterize them, where $\mathrm{d} s^{2} \equiv-c^{2} \mathrm{~d} \tau^{2}$. This implies that $A=-c^{2}$, instead of $A=1$, but as long as $A$ is constant, it drops out of the geodesic equation.
By tradition, the spacetime indices in general relativity are denoted by Greek letters such as $\mu, \nu, \lambda, \sigma$, and are summed from 0 to 3 , where $x^{0} \equiv t$.
The geodesic equation

$$
\frac{\mathrm{d}}{\mathrm{~d} s}\left[g_{i j} \frac{\mathrm{~d} x^{j}}{\mathrm{~d} s}\right]=\frac{1}{2} \frac{\partial g_{j k}}{\partial x^{i}} \frac{\mathrm{~d} x^{j}}{\mathrm{~d} s} \frac{\mathrm{~d} x^{k}}{\mathrm{~d} s}
$$

is then rewritten as

$$
\frac{\mathrm{d}}{\mathrm{~d} \tau}\left[g_{\mu \nu} \frac{\mathrm{d} x^{\nu}}{\mathrm{d} \tau}\right]=\frac{1}{2} \frac{\partial g_{\lambda \sigma}}{\partial x^{\mu}} \frac{\mathrm{d} x^{\lambda}}{\mathrm{d} \tau} \frac{\mathrm{~d} x^{\sigma}}{\mathrm{d} \tau}
$$

## Radial Trajectory Equations

Only $\mathrm{d} r / \mathrm{d} \tau$ and $\mathrm{d} t / \mathrm{d} \tau$ are nonzero. But they are related by the metric:

$$
c^{2} \mathrm{~d} \tau^{2}=\left(1-\frac{2 G M}{r c^{2}}\right) c^{2} \mathrm{~d} t^{2}-\left(1-\frac{2 G M}{r c^{2}}\right)^{-1} \mathrm{~d} r^{2}
$$

implies that

$$
c^{2}=\left(1-\frac{2 G M}{r c^{2}}\right) c^{2}\left(\frac{\mathrm{~d} t}{\mathrm{~d} \tau}\right)^{2}-\left(1-\frac{2 G M}{r c^{2}}\right)^{-1}\left(\frac{\mathrm{~d} r}{\mathrm{~d} \tau}\right)^{2}
$$

Then, looking at the $\mu=r$ geodesic equation,

$$
\frac{\mathrm{d}}{\mathrm{~d} \tau}\left[g_{\mu \nu} \frac{\mathrm{d} x^{\nu}}{\mathrm{d} \tau}\right]=\frac{1}{2} \frac{\partial g_{\lambda \sigma}}{\partial x^{\mu}} \frac{\mathrm{d} x^{\lambda}}{\mathrm{d} \tau} \frac{\mathrm{~d} x^{\sigma}}{\mathrm{d} \tau}
$$

implies that

$$
\frac{\mathrm{d}}{\mathrm{~d} \tau}\left[g_{r r} \frac{\mathrm{~d} r}{\mathrm{~d} \tau}\right]=\frac{1}{2} \partial_{r} g_{r r}\left(\frac{\mathrm{~d} r}{\mathrm{~d} \tau}\right)^{2}+\frac{1}{2} \partial_{r} g_{t t}\left(\frac{\mathrm{~d} t}{\mathrm{~d} \tau}\right)^{2}
$$

where

$$
g_{r r}=\left(1-\frac{2 G M}{r c^{2}}\right)^{-1}, \quad g_{t t}=-c^{2}\left(1-\frac{2 G M}{r c^{2}}\right)
$$

Repeating,

$$
\begin{gathered}
c^{2}=\left(1-\frac{2 G M}{r c^{2}}\right) c^{2}\left(\frac{\mathrm{~d} t}{\mathrm{~d} \tau}\right)^{2}-\left(1-\frac{2 G M}{r c^{2}}\right)^{-1}\left(\frac{\mathrm{~d} r}{\mathrm{~d} \tau}\right)^{2} . \\
\frac{\mathrm{d}}{\mathrm{~d} \tau}\left[g_{r r} \frac{\mathrm{~d} r}{\mathrm{~d} \tau}\right]=\frac{1}{2} \partial_{r} g_{r r}\left(\frac{\mathrm{~d} r}{\mathrm{~d} \tau}\right)^{2}+\frac{1}{2} \partial_{r} g_{t t}\left(\frac{\mathrm{~d} t}{\mathrm{~d} \tau}\right)^{2},
\end{gathered}
$$

where

$$
g_{r r}=\left(1-\frac{2 G M}{r c^{2}}\right)^{-1}, \quad g_{t t}=-c^{2}\left(1-\frac{2 G M}{r c^{2}}\right)
$$

Expand

$$
\frac{\mathrm{d}}{\mathrm{~d} \tau}\left[g_{r r} \frac{\mathrm{~d} r}{\mathrm{~d} \tau}\right]
$$

with the product rule, replace $(\mathrm{d} t / \mathrm{d} \tau)^{2}$ using the equation above, and simplify. Result:

$$
\frac{\mathrm{d}^{2} r}{\mathrm{~d} \tau^{2}}=-\frac{G M}{r^{2}},
$$

which looks just like Newton, but it is not really the same. Here $\tau$ is the proper time as measured by the infalling object, and $r$ is not the radial distance.

## Solving the Equation

$$
\frac{\mathrm{d}^{2} r}{\mathrm{~d} \tau^{2}}=-\frac{G M}{r^{2}}
$$

Like Newton's equation, multiply by $\mathrm{d} r / \mathrm{d} \tau$, and it can then be written as

$$
\frac{\mathrm{d}}{\mathrm{~d} \tau}\left\{\frac{1}{2}\left(\frac{\mathrm{~d} r}{\mathrm{~d} \tau}\right)^{2}-\frac{G M}{r}\right\}=0
$$

Quantity in curly brackets is conserved. Initial value (on release from rest at $\left.r_{0}\right)$ is $-G M / r_{0}$, so it always has this value. Then

$$
\frac{\mathrm{d} r}{\mathrm{~d} \tau}=-\sqrt{2 G M\left(\frac{1}{r}-\frac{1}{r_{0}}\right)}=-\sqrt{\frac{2 G M\left(r_{0}-r\right)}{r r_{0}}} .
$$

Repeating,

$$
\frac{\mathrm{d} r}{\mathrm{~d} \tau}=-\sqrt{2 G M\left(\frac{1}{r}-\frac{1}{r_{0}}\right)}=-\sqrt{\frac{2 G M\left(r_{0}-r\right)}{r r_{0}}}
$$

Bring all $r$-dependent factors to one side, and bring $\mathrm{d} \tau$ to the other side, and integrate:

$$
\begin{aligned}
\tau\left(r_{f}\right) & =-\int_{r_{0}}^{r_{f}} \mathrm{~d} r \sqrt{\frac{r r_{0}}{2 G M\left(r_{0}-r\right)}} \\
& =\sqrt{\frac{r_{0}}{2 G M}}\left\{r_{0} \tan ^{-1}\left(\sqrt{\frac{r_{0}-r_{f}}{r_{f}}}\right)+\sqrt{r_{f}\left(r_{0}-r_{f}\right)}\right\}
\end{aligned}
$$

where $\tan ^{-1} \equiv \arctan$.
Conclusion: object will reach $r=0$ in a finite proper time $\tau$.

$$
\tau\left(r_{f}\right)=\sqrt{\frac{r_{0}}{2 G M}}\left\{r_{0} \tan ^{-1}\left(\sqrt{\frac{r_{0}-r_{f}}{r_{f}}}\right)+\sqrt{r_{f}\left(r_{0}-r_{f}\right)}\right\} .
$$

Setting $r_{f}=0$ to find the proper time when the object reaches $r=0$,

$$
\tau(0)=\sqrt{\frac{r_{0}}{2 G M}}\left\{r_{0} \tan ^{-1}(\infty)+0\right\}
$$

$$
=\sqrt{\frac{\pi}{2} \sqrt{\frac{r_{0}^{3}}{2 G M}}} .
$$

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## Falling from the Schwarzschild Horizon to $r=0$

Recall,

$$
\tau(0)=\frac{\pi}{2} \sqrt{\frac{r_{0}^{3}}{2 G M}} .
$$

For $r_{0}=R_{S}$,

$$
\tau=\frac{\pi G M}{c^{3}}
$$

For $r_{0}=R_{S}$,

$$
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$$

For the Sun, this gives

$$
\tau=1.55 \times 10^{-5} \mathrm{~s}
$$

For the black hole in the center of our galaxy,

$$
\tau=6.34 \mathrm{~s}
$$

Note that inside the black hole,

$$
\begin{aligned}
\mathrm{d} s^{2}=-c^{2} \mathrm{~d} \tau^{2}=- & \left(1-\frac{2 G M}{r c^{2}}\right) c^{2} \mathrm{~d} t^{2}+\left(1-\frac{2 G M}{r c^{2}}\right)^{-1} \mathrm{~d} r^{2} \\
& +r^{2}\left(\mathrm{~d} \theta^{2}+\sin ^{2} \theta \mathrm{~d} \phi^{2}\right)
\end{aligned}
$$

but

$$
\left(1-\frac{2 G M}{r c^{2}}\right)<0
$$

which implies that $t$ is spacelike, and $r$ is timelike! The calculation that we just did is still correct. The singularity at $r=0$ cannot be avoided for the same reason that we cannot prevent ourselves from reaching tomorrow!

## But Coordinate Time $t$ is Different!

$$
\begin{gathered}
\frac{\mathrm{d} r}{\mathrm{~d} \tau}=-\sqrt{2 G M\left(\frac{1}{r}-\frac{1}{r_{0}}\right)}=-\sqrt{\frac{2 G M\left(r_{0}-r\right)}{r r_{0}}} . \\
c^{2}=\left(1-\frac{2 G M}{r c^{2}}\right) c^{2}\left(\frac{\mathrm{~d} t}{\mathrm{~d} \tau}\right)^{2}-\left(1-\frac{2 G M}{r c^{2}}\right)^{-1}\left(\frac{\mathrm{~d} r}{\mathrm{~d} \tau}\right)^{2} .
\end{gathered}
$$

$$
\begin{aligned}
\frac{\mathrm{d} r}{\mathrm{~d} t}=\frac{\mathrm{d} r}{\mathrm{~d} \tau} \frac{\mathrm{~d} \tau}{\mathrm{~d} t} & =\frac{\mathrm{d} r / \mathrm{d} \tau}{\mathrm{~d} t / \mathrm{d} \tau} \\
& =\frac{\mathrm{d} r / \mathrm{d} \tau}{\sqrt{h^{-1}(r)+c^{-2} h^{-2}(r)\left(\frac{\mathrm{d} r}{\mathrm{~d} \tau}\right)^{2}}},
\end{aligned}
$$

where $h^{-1}(r) \equiv 1 / h(r)$, not the inverse function, and

$$
h(r) \equiv 1-\frac{R_{S}}{r}=1-\frac{2 G M}{r c^{2}} .
$$

$$
\begin{gathered}
\frac{\mathrm{d} r}{\mathrm{~d} \tau}=-\sqrt{2 G M\left(\frac{1}{r}-\frac{1}{r_{0}}\right)}=-\sqrt{\frac{2 G M\left(r_{0}-r\right)}{r r_{0}}} \\
\frac{\mathrm{~d} r}{\mathrm{~d} t}=\frac{\mathrm{d} r / \mathrm{d} \tau}{\sqrt{h^{-1}(r)+c^{-2} h^{-2}(r)\left(\frac{\mathrm{d} r}{\mathrm{~d} \tau}\right)^{2}}}
\end{gathered}
$$

where

$$
h(r) \equiv 1-\frac{R_{S}}{r}=1-\frac{2 G M}{r c^{2}}
$$

Look at behavior near horizon; $h^{-1}(r)$ blows up:

$$
h^{-1}(r)=\frac{r}{r-R_{S}} \approx \frac{R_{S}}{r-R_{S}}
$$

Denominator of $\mathrm{d} r / \mathrm{d} t$ is dominated by 2 nd term, which gives

$$
\frac{\mathrm{d} r}{\mathrm{~d} t} \approx-\operatorname{ch}(r)=-c\left(\frac{r-R_{S}}{R_{S}}\right)
$$

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Repeating,

$$
\frac{\mathrm{d} r}{\mathrm{~d} t} \approx-c\left(\frac{r-R_{S}}{R_{S}}\right) .
$$

Rearranging,

$$
\mathrm{d} t=-\frac{R_{S}}{c} \frac{\mathrm{~d} r}{r-R_{S}} .
$$

We can find the time needed to fall from some $r_{i}$ near the horizon, to a smaller $r_{f}$ which is nearer to the horizon:

$$
t\left(r_{f}\right) \approx-\frac{R_{S}}{c} \int_{r_{i}}^{r_{f}} \frac{\mathrm{~d} r^{\prime}}{r^{\prime}-R_{S}} \approx \frac{R_{S}}{c} \ln \left(\frac{r_{i}-R_{S}}{r_{f}-R_{S}}\right)
$$

Thus $t$ diverges logarithmically as $r_{f} \rightarrow R_{S}$, so the object does not reach $R_{S}$ for any finite value of $t$.

### 8.286 Class 13 <br> October 19, 2020

## BLACK-BODY RADIATION

 AND
## THE EARLY HISTORY OF THE UNIVERSE



Alan Guth

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## $E=m c^{2}$

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is Meaning: Mass and energy are equivalent. They are just two different ways of expressing exactly the same thing. The total energy of any system is equal to the total mass of the system - sometimes called the relativistic mass - times $c^{2}$, the square of the speed of light.

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is Meaning: Mass and energy are equivalent. They are just two different ways of expressing exactly the same thing. The total energy of any system is equal to the total mass of the system - sometimes called the relativistic mass - times $c^{2}$, the square of the speed of light.
is One can imagine measuring the mass/energy of an object in either kilograms, joules, or kilowatt-hours, with

$$
1 \mathrm{~kg}=8.9876 \times 10^{16} \text { joule }=2.497 \times 10^{10} \mathrm{~kW}-\mathrm{hr} .
$$

## $E=m c^{2}$ and the World Power Supply

is The total amount of power produced in the world, on average, is about $1.89 \times 10^{10} \mathrm{~kW}$, according to the International Energy Agency.
is This amounts to about 2.5 kW per person.
is If a 15 gallon tank of gasoline could be converted entirely into usable energy, it would power the world for $2 \frac{1}{2}$ days.
is However, it is not so easy! Even with nuclear power, when a uranium-235 nucleus undergoes fission, only about $0.09 \%$ of its mass is converted to energy.

## $E=m c^{2}$ and Particle Masses

is Nuclear and particle physicists tend to measure the mass of elementary particles in energy units, usually using either $\mathrm{MeV}\left(10^{6} \mathrm{eV}\right)$ or $\mathrm{GeV}\left(10^{9}\right.$ eV ) as the unit of energy, where

$$
1 \mathrm{eV}=1 \text { electron volt }=1.6022 \times 10^{-19} \mathrm{~J}
$$

and then

$$
1 \mathrm{GeV}=1.7827 \times 10^{-27} \mathrm{~kg} .
$$

The mass of a proton is 0.938 GeV , and the mass of an electron is 0.511 MeV .

## Energy and Momentum in Special Relativity

is We have talked about the kinematic consequences of special relativity (time dilation, Lorentz contraction, and the relativity of simultaneity), but now we need to bring in the dynamical consequences, involving energy and momentum.
is In special relativity, the definitions of energy and momentum are different from those in Newtonian mechanics.
is Why? Because special relativity is based on the principle that the laws of physics in any inertial reference frame are the same, and furthermore, in order for the speed of light be the same in any inertial reference frame, these frames cannot be related to each other as in Newtonian physics. They must instead be related by Lorentz transformations, which take into account the kinematic effects mentioned above.
is Two important laws of physics are the conservation of energy and momentum.
is If energy and momentum kept their Newtonian definitions, then, if they were conserved in one frame, they would not be conserved in other frames.
is The requirement that the conservation equations hold in all frames requires the standard special relativity definitions.

## Energy, Momentum, and the Energy-Momentum Four-Vector

is The energy-momentum four-vector is defined by starting with the momentum three-vector $\left(p^{1}, p^{2}, p^{3}\right) \equiv\left(p^{x}, p^{y}, p^{z}\right)$, and appending a fourth component

$$
p^{0}=\frac{E}{c},
$$

so the four-vector can be written as

$$
p^{\mu}=\left(\frac{E}{c}, \vec{p}\right) .
$$

As with the three-vector momentum, the energy-momentum four-vector can be defined for a system of particles as the sum of the vectors for the individual particles.
is The 4 -vector $p^{\mu}$ transforms, when we change frames of reference, according to the Lorentz transformation, exactly like the 4 -vector $x^{\mu}=(c t, \vec{x})$.
is Furthermore, the total energy-momentum 4 -vector is conserved - in any inertial frame of reference.

## Relation of Energy and Momentum to Rest Mass and Velocity

The mass of a particle in its own rest frame is called its rest mass, which we denote by $m_{0}$. At velocity $\vec{v}$

$$
\begin{aligned}
\vec{p} & =\gamma m_{0} \vec{v} \\
E & =\gamma m_{0} c^{2}=\sqrt{\left(m_{0} c^{2}\right)^{2}+|\vec{p}|^{2} c^{2}}
\end{aligned}
$$

where as usual $\gamma$ is defined by

$$
\gamma=\frac{1}{\sqrt{1-\frac{v^{2}}{c^{2}}}}
$$

## Lorentz Invariance of $p^{2}$

$$
\begin{aligned}
\vec{p} & =\gamma m_{0} \vec{v} \\
E & =\gamma m_{0} c^{2}=\sqrt{\left(m_{0} c^{2}\right)^{2}+|\vec{p}|^{2} c^{2}}
\end{aligned}
$$

Like the Lorentz-invariant interval that we discussed as $\mathrm{d} s^{2}=-c^{2} \mathrm{~d} t^{2}+\mathrm{d} \vec{x}^{2}$, the energy-momentum four-vector has a Lorentz-invariant square:

$$
p^{2} \equiv|\vec{p}|^{2}-\left(p^{0}\right)^{2}=|\vec{p}|^{2}-\frac{E^{2}}{c^{2}}=-\left(m_{0} c\right)^{2}
$$

For a particle at rest,

$$
E=m_{0} c^{2}
$$

## Energy Exchange in a Simple Chemical Reaction

Consider the reaction

$$
p+e^{-} \longrightarrow H+\gamma
$$

Assuming that the proton and electron begin at rest, and ignoring the very small kinetic energy of the hydrogen atom when it recoils from the emitted photon, conservation of energy implies that

$$
m_{H}=m_{p}+m_{e}-E_{\gamma} / c^{2}
$$

The energy given off when the proton and electron bind is called the binding energy of the hydrogen atom.

