8.286 Class 14
October 21, 2020

BLACK-BODY RADIATION AND THE EARLY HISTORY OF THE UNIVERSE, PART 2

Announcements

Reminder: Problem Set 6 is due this Friday at 5:00 pm.

Quiz 2 will be next Wednesday, October 28. Procedures will be the same as for Quiz 1. Review Problems for Quiz 2 have been posted, and they contain a complete description of what will be covered on the quiz.

WARNING: don't let your wonderful success on Quiz 1 cause you to become complacent. The material has gotten harder. The last time I taught this course, in 2018, the class average was 85.0 on the first quiz, and fell to 69.7 on the second. Don't let that happen this year!

Review session by Bruno Scheihing: Monday, 10/26/20, at 7:30 pm.

Special office hours next week:

Bruno: Monday 10/26/20 at 4:00 pm

Me: Tuesday 10/27/20 at 6:00 pm

To be posted today: Compiled course documents (lecture notes, problem sets, solutions, quiz review problems, Quiz 1), on Lecture Notes page. Compiled lecture slides, on main web page.

$$E = mc^2$$

⭐ THE most famous equation in physics. But I was not able to find any actual surveys.

⭐ **Meaning: Mass and energy are equivalent.** They are just two different ways of expressing exactly the same thing. The total energy of any system is equal to the total mass of the system — sometimes called the relativistic mass — times c^2, the square of the speed of light.

⭐ One can imagine measuring the mass/energy of an object in either kilograms, joules, or kilowatt-hours, with

$$1 \text{ kg} = 8.9876 \times 10^{16} \text{ joule} = 2.497 \times 10^{10} \text{ kW-hr.}$$

$E = mc^2$ and the World Power Supply

⭐ The total amount of power produced in the world, on average, is about 1.89×10^{10} kW, according to the International Energy Agency.

⭐ This amounts to about 2.5 kW per person.

The total world power output is about 2.5 kW per person.

⭐ **If a 15 gallon tank of gasoline could be converted *entirely* into usable energy, it would power the world for $2\frac{1}{2}$ days.**

⭐ **However, it is not so easy! Even with nuclear power, when a uranium-235 nucleus undergoes fission, only about 0.09% of its mass is converted to energy.**

$$E = mc^2 \text{ and Particle Masses}$$

⭐ Nuclear and particle physicists tend to measure the mass of elementary particles in energy units, usually using either MeV (10^6 eV) or GeV (10^9 eV) as the unit of energy, where

$$1\,\text{eV} = 1 \text{ electron volt} = 1.6022 \times 10^{-19}\,\text{J},$$

and then

$$1\,\text{GeV} = 1.7827 \times 10^{-27}\,\text{kg}.$$

The mass of a proton is 0.938 GeV,

the mass of an electron is 0.511 MeV.

Energy and Momentum in Special Relativity

If energy and momentum are to be conserved in all inertial reference frames, then the Newtonian definitions must be modified.

Energy, Momentum, and the Energy-Momentum Four-Vector

★ The energy-momentum four-vector is defined by starting with the momentum three-vector $(p^1, p^2, p^3) \equiv (p^x, p^y, p^z)$, and appending a fourth component

$$p^0 = \frac{E}{c} \ ,$$

so the four-vector can be written as

$$p^\mu = \left(\frac{E}{c} , \vec{p} \right) \ .$$

As with the three-vector momentum, the energy-momentum four-vector can be defined for a system of particles as the sum of the vectors for the individual particles.

⭐ The 4-vector p^μ transforms, when we change frames of reference, according to the Lorentz transformation, exactly like the 4-vector $x^\mu = (ct, \vec{x})$.

⭐ Furthermore, the total energy-momentum 4-vector is conserved — in any inertial frame of reference.

Relation of Energy and Momentum to Rest Mass and Velocity

The mass of a particle in its own rest frame is called its *rest mass*, which we denote by m_0. At velocity $\vec{v}$

$$\vec{p} = \gamma m_0 \vec{v} \ ,$$

$$E = \gamma m_0 c^2 = \sqrt{(m_0 c^2)^2 + |\vec{p}|^2 c^2} \ ,$$

where as usual γ is defined by

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \ .$$

Lorentz Invariance of p^2

$$\vec{p} = \gamma m_0 \vec{v} \ ,$$

$$E = \gamma m_0 c^2 = \sqrt{(m_0 c^2)^2 + |\vec{p}|^2 c^2} \ ,$$

Like the Lorentz-invariant interval that we discussed as $\mathrm{d}s^2 = |\mathrm{d}\vec{x}^2| - c^2 \mathrm{d}t^2$, the energy-momentum four-vector has a Lorentz-invariant square:

$$p^2 \equiv |\vec{p}|^2 - \left(p^0\right)^2 = |\vec{p}|^2 - \frac{E^2}{c^2} = -\left(m_0 c\right)^2 \ .$$

For a particle at rest,

$$E = m_0 c^2 \ .$$

Energy Exchange in a Simple Chemical Reaction

Consider the reaction

$$p + e^- \longrightarrow H + \gamma \ .$$
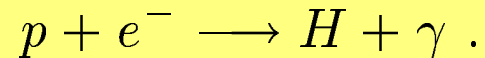
Assuming that the proton and electron begin at rest, and ignoring the very small kinetic energy of the hydrogen atom when it recoils from the emitted photon, conservation of energy implies that

$$m_H = m_p + m_e - E_\gamma/c^2 \ .$$

The energy given off when the proton and electron bind is called the **binding energy** of the hydrogen atom. It is 13.6 eV.

Relativistic Mass

★ Since $E = mc^2$, we can define the *relativistic mass* of any particle or system as simply

$$m_{\text{rel}} \equiv \frac{E}{c^2} \ .$$

★ Some authors avoid using the concept of relativistic mass, reserving the word "mass" to mean rest mass m_0. Relativistic mass is certainly a redundant concept, since anything that can be described in terms of m_{rel} can also be described in terms of E.

★ For cosmology the concept of relativistic mass will be helpful, since relativistic mass is the source of gravity. By calling E/c^2 a mass, we are indicating our recognition that it is the source of gravity.

This is beyond the level of what we need, but for those who are interested, I mention that the Einstein field equations imply that the source of gravitational fields is the *energy-momentum tensor* $T^{\mu\nu}$, where μ and ν are 4-vector indices that take on values from 0 to 3.

$T^{00} = u =$ energy density,

$T^{0i} = T^{i0}$ is $\frac{1}{c}$ times the flow of energy in the i'th direction (i=1,2,3) and is also c times the density of the i'th component of momentum,

$T^{ij} = T^{ji}$ is the flow in the j'th direction of the i'th component of momentum. T^{ij} is often diagonal, with $T^{ij} = p\,\delta^{ij}$, where p is the pressure.

For a homogeneous, isotropic universe model, only u and p will serve as sources for gravity.

Mass of Radiation

⭐ Electromagnetic radiation has energy. The energy density is given by

$$u = \frac{1}{2}\left[\epsilon_0 \left|\vec{E}\right|^2 + \frac{1}{\mu_0}\left|\vec{B}\right|^2\right] \ .$$

We won't need this equation, but we need to know that electromagnetic radiation **has** an energy density u.

⭐ Energy density implies a (relativistic) mass density

$$\rho = u/c^2 \ .$$

(*Relativistic mass* is defined to be the energy divided by c^2.)

Energy and Momentum of Photons

Photons have zero rest mass.

In general,

$$p^2 = |\vec{p}|^2 - \frac{E^2}{c^2} = -\left(m_0 c\right)^2 \ ,$$

but for photons, $m_0 = 0$, so

$$|\vec{p}|^2 - \frac{E^2}{c^2} = 0 \ , \quad \text{or} \quad E = c|\vec{p}| \ .$$

Radiation in an Expanding Universe

⭐ From the end of inflation (maybe about 10^{-35} second, to be discussed later) until stars form, the number of photons is almost exactly conserved.

⭐ Therefore,

$$n_\gamma \propto \frac{1}{a^3(t)} \ .$$

⭐ Bu1t the frequency of each photon redshifts:

$$\nu \propto \frac{1}{a(t)} \ .$$

$$n_\gamma \propto \frac{1}{a^3(t)} \ , \qquad \nu \propto \frac{1}{a(t)} \ .$$

⭐ But according to quantum mechanics, the energy of each photon is

$$E = h\nu \ ,$$

so the energy of each photon is proportional to $1/a(t)$.

⭐ Finally,

$$n_\gamma \propto \frac{1}{a^3(t)} \quad , \quad E_\gamma \propto \frac{1}{a(t)} \quad \Longrightarrow \quad \rho_\gamma = \frac{u_\gamma}{c^2} \propto \frac{1}{a^4(t)} \ .$$

The Radiation Dominated Era

Radiation energy density today (including photons and neutrinos):

$$u_r = 7.01 \times 10^{-14} \text{ J/m}^3 \, , \qquad \rho_r = u_r/c^2 = 7.80 \times 10^{-34} \text{ g/cm}^3 \, .$$

Total mass density today, ρ_0, is equal to within uncertainties to the critical density,

$$\rho_c = \frac{3H_0^2}{8\pi G} = 1.88 \, h_0^2 \times 10^{-29} \text{ g/cm}^3 \, ,$$

where

$$H_0 = 100 \, h_0 \text{ km-s}^{-1}\text{-Mpc}^{-1} \, , \qquad h_0 \approx 67 \, ,$$

which gives the present value of Ω_r as $\boxed{\Omega_r \approx 9.2 \times 10^{-5}}$.

Since $\rho_r \propto 1/a^4(t)$, while $\rho_m \propto 1/a^3(t)$.

ρ_m = mass density of nonrelativistic matter, baryonic matter plus dark matter.
It follows that

$$\boxed{\rho_r/\rho_m \propto 1/a(t) \ .}$$

Today $\rho_m \approx 0.30\rho_c$, so $\rho_r/\rho_m \approx 9.2 \times 10^{-5}/0.30 \approx 3.1 \times 10^{-4}$. Thus

$$\frac{\rho_r(t)}{\rho_m(t)} = \frac{a(t_0)}{a(t)} \times 3.1 \times 10^{-4} \ .$$

t_{eq} is defined to be the time of matter-radiation equality. Thus

$$\frac{\rho_r(t_{\mathrm{eq}})}{\rho_m(t_{\mathrm{eq}})} \equiv 1 = \frac{a(t_0)}{a(t_{\mathrm{eq}})} \times 3.1 \times 10^{-4} \ .$$

Since $a(t_0)/a(t_{\mathrm{eq}}) = 1 + z_{\mathrm{eq}}$,

$$\boxed{z_{\mathrm{eq}} = \frac{1}{3.1 \times 10^{-4}} - 1 \approx 3200 \ .}$$

Since $a(t_0)/a(t_{eq}) = 1 + z_{eq}$,

$$z_{eq} = \frac{1}{3.1 \times 10^{-4}} - 1 \approx 3200 \ .$$

Time of matter-radiation equality:

We are not ready to calculate this accurately, but for now we can estimate it by assuming that between t_{eq} and now, $a(t) \propto t^{2/3}$, as in a matter-dominated flat universe. Then

$$(t_{eq}/t_0)^{2/3} = 3.1 \times 10^{-4} \ ,$$

so

$$t_{eq} = 5.5 \times 10^{-6} \, t_0 = 5.5 \times 10^{-6} \times 13.8 \text{ Gyr} \approx 75,000 \text{ years.}$$

Ryden (p. 96) gives 50,000 years, which is more accurate.

Dynamics of the Radiation-Dominated Era

$$\rho \propto \frac{1}{a^3} \implies \dot{\rho} = -3\frac{\dot{a}}{a}\,\rho \ , \quad \rho(t) \propto \frac{1}{a^4(t)} \implies \dot{\rho} = -4\frac{\dot{a}}{a}\,\rho \ .$$

$\dot{\rho}$ and pressure p: (Problem 4, Problem Set 6)

$$dU = -p\,dV \quad \implies \quad \frac{d}{dt}\left(a^3 \rho c^2\right) = -p\frac{d}{dt}(a^3)$$

$$\implies \quad \dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) \ .$$

Friedmann equations:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{a^2}$$

$$\ddot{a} = -\frac{4\pi}{3}G\rho a \ ,$$

$$\dot{\rho} = -3\frac{\dot{a}}{a}\rho$$

$$\left(\begin{array}{c} \text{matter-dominated} \\ \text{universe} \end{array}\right)$$

Any two of the above equations implies the third. So they become inconsistent if

$$\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) \ .$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{a^2} \ , \qquad \ddot{a} = -\frac{4\pi}{3}G\rho a \ .$$

Any two of the above equations implies the third. So they become inconsistent if

$$\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) \ .$$

So, if we believe the equation for $\dot{\rho}$, we must modify one of the two Friedmann equations. First order equation represents conservation of energy: pressure does not belong! (Pressures can change suddenly, as when dynamite explodes, so it does not make sense to have pressure in a conservation equation.) So modify the 2nd order equation, deriving it from the first order equation and the $\dot{\rho}$ equation:

$$\ddot{a} = -\frac{4\pi}{3}G\left(\rho + \frac{3p}{c^2}\right)a \ .$$

$$H^2 = \frac{8\pi G}{3}\rho \ , \ \rho \propto 1/a^4 \quad \Longrightarrow \quad \left(\frac{\dot{a}}{a}\right)^2 = \frac{\text{const}}{a^4} \ .$$

Then

$$a \, \mathrm{d}a = \sqrt{\text{const}} \, \mathrm{d}t \quad \Longrightarrow \quad \frac{1}{2}a^2 = \sqrt{\text{const}} \, t + \text{const}' \ .$$

So, setting our clocks so that $\text{const}' = 0$,

$$a(t) \propto \sqrt{t} \qquad \text{(flat radiation-dominated)} \ .$$

$$H(t) = \frac{\dot{a}}{a} = \frac{1}{2t} \qquad \text{(flat radiation-dominated)} \ .$$

$$\ell_{p,\text{horizon}}(t) = a(t) \int_0^t \frac{c}{a(t')} dt'$$

$$= \boxed{2ct \qquad \text{(flat radiation-dominated)} \ .}$$

$$H^2 = \frac{8\pi G}{3} \rho \quad \Longrightarrow \quad \boxed{\rho = \frac{3}{32\pi G t^2} \ .}$$