8.286 Class 20
November 16, 2020

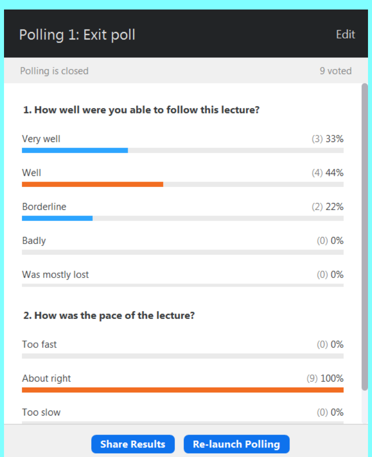
THE COSMOLOGICAL CONSTANT PART 2

(Modified 12/27/20 to fix a minor typo on p. 5.)

Announcements

Problem Set 8 is due this Friday, November 20.

Exit Poll, Last Class



Review from last class:

A Brief History of the Cosmological Constant

☆ In 1917, Einstein applied his new GR to the universe, and discovered that a static universe would collapse.

☆ Convinced that the universe was static, Einstein introduced the *cosmological constant* Λ into his field equations — the equations that describe how matter affects the metric — to create a gravitational repulsion to oppose the collapse.

☆ From a modern point of view, Λ represents a *vacuum energy density* u_{vac}, with

$$u_{\text{vac}} = \rho_{\text{vac}} c^2 = \frac{\Lambda c^4}{8\pi G} ,$$

☆ **From a modern point of view, Λ represents a *vacuum energy density* $u_{\rm vac}$, with**

$$u_{\rm vac} = \rho_{\rm vac} c^2 = \frac{\Lambda c^4}{8\pi G} ,$$

because $u_{\rm vac}$ appears in the field equations exactly as a vacuum energy density would. To Einstein, however, it was simply a new term in the field equations. Before quantum theory, the vacuum was viewed as completely empty, so it was inconceivable that it could have a nonzero energy density.

Review from last class:

☆ **Once the expansion of the universe was discovered by Hubble in 1929, Einstein abandoned Λ as being no longer needed or wanted.**

☆ **In 1998, however, two (large) groups of astronomers, both using measurements of Type Ia supernovae at redshifts $z \lesssim 1$, discovered evidence that the expansion of the universe is currently accelerating!**

At the time, it was shocking! *Science* magazine proclaimed it (correctly!) as the "Breakthough of the Year".

☆ In 2011 the Nobel Prize in Physics was awarded to Saul Permutter, Brian Schmidt, and Adam Riess for this discovery. In 2015 the Breakthrough Prize in Fundamental Physics was awarded to these three, and also the two entire teams.

Review from last class:

Gravitational Effect of Pressure

$$\frac{d^2a}{dt^2} = -\frac{4\pi}{3} G \left(\rho + \frac{3p}{c^2}\right) a .$$

Vacuum Energy and the Cosmological Constant:

$$u_{\rm vac} = \rho_{\rm vac} c^2 = \frac{\Lambda c^4}{8\pi G} .$$

Recall that

$$\dot\rho = -3\frac{\dot a}{a}\left(\rho + \frac{p}{c^2}\right) ,$$

where the overdot indicates a time derivative. So

$$\dot\rho_{\rm vac} = 0 \quad \Longrightarrow \quad p_{\rm vac} = -\rho_{\rm vac} c^2 = -\frac{\Lambda c^4}{8\pi G} .$$

Review from last class:

Defining $\rho = \rho_n + \rho_{\rm vac}$ and $p = p_n + p_{\rm vac}$, the Friedmann equations become:

$$\ddot a = -\frac{4\pi}{3} G \left(\rho_n + \frac{3p_n}{c^2} - 2\rho_{\rm vac}\right) a .$$

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi}{3} G(\rho_n + \rho_{\rm vac}) - \frac{kc^2}{a^2} ,$$

where an overdot ($\dot{}$) is a derivative with respect to t. At late times, $\rho_n \propto 1/a^3$ or $1/a^4$, $\rho_{\rm vac}$ = constant, so $\rho_{\rm vac}$ dominates. Then

$$a(t) \propto e^{H_{\rm vac} t} ,$$

$$H \to H_{\rm vac} = \sqrt{\frac{8\pi}{3} G \rho_{\rm vac}} .$$

Review from last class:

Age of the Universe with Λ

The first order Friedmann equation

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi}{3}G\,(\underbrace{\rho_m}_{\propto \frac{1}{a^3(t)}} + \underbrace{\rho_{\rm rad}}_{\propto \frac{1}{a^4(t)}} + \rho_{\rm vac}) - \frac{kc^2}{a^2} .$$

can be rewritten as

$$\left(\frac{\dot a}{a}\right)^2 = H_0^2\left(\frac{\Omega_{m,0}}{x^3} + \frac{\Omega_{\rm rad,0}}{x^4} + \Omega_{\rm vac}\right) - \frac{kc^2}{a^2} ,$$

where $x \equiv a(t)/a(t_0)$.

and where we used

$$\Omega_{X,0} = \frac{\rho_{X,0}}{\rho_{c,0}} = \frac{8\pi G\rho_{X,0}}{3H_0^2} .$$

Review from last class:

$$\left(\frac{\dot a}{a}\right)^2 = H_0^2\left(\frac{\Omega_{m,0}}{x^3} + \frac{\Omega_{\rm rad,0}}{x^4} + \Omega_{\rm vac}\right) - \frac{kc^2}{a^2} ,$$

where $x \equiv a(t)/a(t_0)$.

Define

$$\Omega_{k,0} \equiv -\frac{kc^2}{a^2(t_0)H_0^2} .$$

So

$$\left(\frac{\dot a}{a}\right)^2 = \left(\frac{\dot x}{x}\right)^2 = \frac{H_0^2}{x^4}\left(\Omega_{m,0}x + \Omega_{\rm rad,0} + \Omega_{\rm vac,0}x^4 + \Omega_{k,0}x^2\right) .$$

Review from last class:

$$\left(\frac{\dot a}{a}\right)^2 = \left(\frac{\dot x}{x}\right)^2 = \frac{H_0^2}{x^4}\left(\Omega_{m,0}x + \Omega_{\rm rad,0} + \Omega_{\rm vac,0}x^4 + \Omega_{k,0}x^2\right) .$$

At present time, $\dot a/a = H_0$ and $x = 1$, so the sum of the Ω's must equal 1. Thus, $\Omega_{k,0}$ can be evaluated from

$$\Omega_{k,0} = 1 - \Omega_{m,0} - \Omega_{\rm rad,0} - \Omega_{\rm vac,0} .$$

Observationally, $\Omega_{k,0}$ is consistent with zero, but we can still allow for it in our final formula for the age:

$$t_0 = \frac{1}{H_0}\int_0^1 \frac{x\,dx}{\sqrt{\Omega_{m,0}x + \Omega_{\rm rad,0} + \Omega_{\rm vac,0}x^4 + \Omega_{k,0}x^2}} .$$

Review from last class:

Numerical Integration with Mathematica

```
IN: t0[H0_,Ωm0_,Ωrad0_,Ωvac0_,Ωk0_] := (1/H0) *
      NIntegrate[x/Sqrt[Ωm0 x + Ωrad0 + Ωvac0 x^4 + Ωk0 x^2], {x,0,1}]
IN: PlanckH0 := Quantity[67.66,"km/sec/Mpc"]
IN: PlanckΩm0 := 0.311
IN: PlanckΩvac0 := 0.689
IN: UnitConvert[t0[PlanckH0,PlanckΩm0,0,PlanckΩvac0,0],"Years"]
OUT: 1.38022 × 10^10 years
```

Numerical Integration with Mathematica
Newer Data

Reference: N. Aghanim et al. (Planck Collaboration), "Planck 2018 results, VI: Cosmological parameters," Table 2, Column 6, arXiv:1807.06209.

IN: t0[H0_,Ωm0_,Ωrad0_,Ωvac0_,Ωk0_] := (1/H0) *
NIntegrate[x/Sqrt[Ωm0 x + Ωrad0 + Ωvac0 x^4 + Ωk0 x^2], {x,0,1}]

IN: PlanckH0 := Quantity[67.66,"km/sec/Mpc"]

IN: PlanckΩm0 := 0.3111

IN: PlanckΩvac0 := 0.6889

IN: Ωrad0 := $4.15 \times 10^{-5}\, h_0^{-2} = 9.07 \times 10^{-5}$

IN: UnitConvert[t0[PlanckH0, PlanckΩm0 − Ωrad0/2, Ωrad0, PlanckΩvac0 − Ωrad0/2 ,0], "Years"]

OUT: 1.3796×10^{10} years

The Planck paper gives 13.787 ± 0.020 Gyr. The difference is about 9 million years, 0.06%, or 0.45 σ.

Look-Back Time

Question: If we observe a distant galaxy at redshift z, how long has it been since the light left the galaxy? The answer is called the *look-back time.*

To answer, recall that we wrote t_0 as an integral over $x = a(t)/a(t_0)$. We can change variables to

$$1 + z = \frac{a(t_0)}{a(t)} = \frac{1}{x} \ ,$$

which gives

$$t_0 = \frac{1}{H_0}\int_0^\infty \frac{dz}{(1+z)\sqrt{\Omega_{m,0}(1+z)^3 + \Omega_{\rm rad,0}(1+z)^4 + \Omega_{\rm vac,0} + \Omega_{k,0}(1+z)^2}} \ .$$

$$t_0 = \frac{1}{H_0}\int_0^\infty \frac{dz}{(1+z)\sqrt{\Omega_{m,0}(1+z)^3 + \Omega_{\rm rad,0}(1+z)^4 + \Omega_{\rm vac,0} + \Omega_{k,0}(1+z)^2}} \ .$$

The integral over any interval of z gives the corresponding time interval, so the look-back time is just the integral from 0 to z:

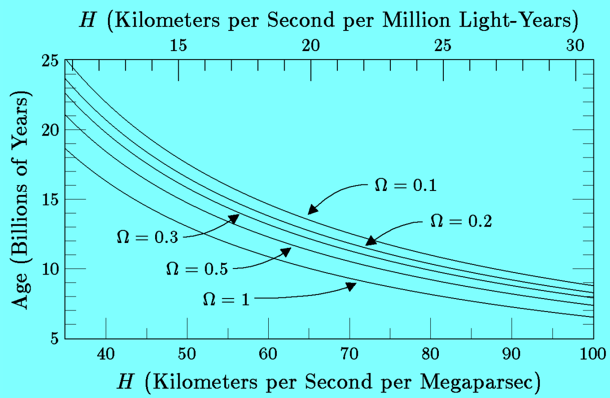
$$t_{\text{look-back}}(z) = \frac{1}{H_0}\int_0^z \frac{dz'}{(1+z')\sqrt{\Omega_{m,0}(1+z')^3 + \Omega_{\rm rad,0}(1+z')^4 + \Omega_{\rm vac,0} + \Omega_{k,0}(1+z')^2}} \ .$$

Age of a Flat Universe with Λ and Matter Only

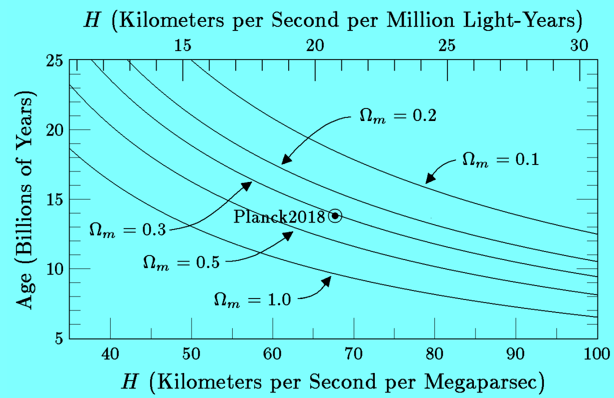
If $\Omega_{\rm rad} = \Omega_k = 0$, then it is possible to carry out the integral for the age analytically:

$$t_0 = \begin{cases} \dfrac{2}{3H_0}\dfrac{\tan^{-1}\sqrt{\Omega_{m,0}-1}}{\sqrt{\Omega_{m,0}-1}} & \text{if } \ \Omega_{m,0} > 1,\ \Omega_{\rm vac} < 0 \\ \dfrac{2}{3H_0} & \text{if } \ \Omega_{m,0} = 1,\ \Omega_{\rm vac} = 0 \\ \dfrac{2}{3H_0}\dfrac{\tanh^{-1}\sqrt{1-\Omega_{m,0}}}{\sqrt{1-\Omega_{m,0}}} & \text{if } \ \Omega_{m,0} < 1,\ \Omega_{\rm vac} > 0 \ . \end{cases}$$

The Age Problem with Only Nonrelativistic Matter



Age of a Flat Universe with Λ and Matter Only



Ryden Benchmark and Planck 2018 Best Fit

Parameters	**Ryden Benchmark**	**Planck 2018 Best Fit**
$\boldsymbol{H_0}$	68	67.7 ± 0.4 km·s^{-1}·Mpc^{-1}
Baryonic matter $\boldsymbol{\Omega_b}$	0.048	$0.0490 \pm 0.0007^*$
Dark matter $\boldsymbol{\Omega_{\rm dm}}$	0.262	$0.261 \pm 0.004^*$
Total matter $\boldsymbol{\Omega_m}$	0.31	0.311 ± 0.006
Vacuum energy $\boldsymbol{\Omega_{\rm vac}}$	0.69	0.689 ± 0.006

Controversy in Parameters: "Hubble Tension"

- ☆ From the CMB, the best number is from

  Planck 2018: $H_0 = 67.66 \pm 0.42$ km sec^{-1} Mpc^{-1}

- ☆ From standard candles and Cepheid variables,

  SH0ES (Supernovae, H0, for the Equation of State of dark energy, group led by Adam Riess): $H_0 = 74.03 \pm 1.42$ km sec^{-1} Mpc^{-1}.

- ☆ The difference is about 4.3 σ. If the discrepancy is random and the normal probability distribution applies, the probability of such a large deviation is about 1 in 50,000.

- ☆ From the "tip of the red giant branch",

  Wendy Freedman's group: $H_0 = 69.6 \pm 0.8(\text{stat}) \pm 1.7(\text{sys})$ km sec^{-1} Mpc^{-1}

References: A. Riess et al., Astrophys. J. 876 (2019) 85 [arXiv:1903.07603].
W. Freedman et al., arXiv:2002.01550 (2020).

The Hubble Diagram: Radiation Flux vs. Redshift

If we live in a universe like we have described, what do we expect to find if we measure the energy flux from a "standard candle" as a function of its redshift?

Consider closed universe:

$$ds^2 = -c^2\,dt^2 + a^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\} .$$

We will be interested in tracing radial trajectories, so we can simplify the radial metric by a change of variables

$$\sin\psi \equiv \sqrt{k}\,r .$$

Then

$$d\psi = \frac{\sqrt{k}\,dr}{\cos\psi} = \frac{\sqrt{k}\,dr}{\sqrt{1-kr^2}} ,$$

$$ds^2 = -c^2\,dt^2 + a^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\} .$$

$$d\psi = \frac{\sqrt{k}\,dr}{\cos\psi} = \frac{\sqrt{k}\,dr}{\sqrt{1-kr^2}} ,$$

and the metric simplifies to

$$ds^2 = -c^2\,dt^2 + \tilde{a}^2(t)\left\{d\psi^2 + \sin^2\psi\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\} ,$$

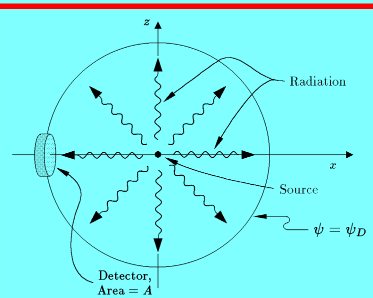
where

$$\tilde{a}(t) \equiv \frac{a(t)}{\sqrt{k}} .$$

Note: ψ is in fact the same angle ψ that we used in our construction of the closed-universe metric: it is the angle from the w-axis.

Geometry of Flux Calculation

$$ds^2 = -c^2\,dt^2 + \tilde{a}^2(t)\left\{d\psi^2 + \sin^2\psi\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\}$$



The fraction of the photons hitting the sphere that hit the detector is just the ratio of the areas:

$$\text{fraction} = \frac{\text{area of detector}}{\text{area of sphere}} = \frac{A}{4\pi\tilde{a}^2(t_0)\sin^2\psi_D} .$$

☆ The power hitting the sphere is less than the power P emitted by the source by two factors of $(1+z_S)$, where z_S is the redshift of the source: one factor due to redshift of each photon, and one factor due to the redshift of the rate of arrival of photons.

$$P_{\text{received}} = \frac{P}{(1+z_S)^2}\,\frac{A}{4\pi\tilde{a}^2(t_0)\sin^2\psi_D} .$$

Flux $J = P_{\text{received}}/A$.

Expressing the Result in Terms of Astronomical Quantities

$$\Omega_{k,0} \equiv -\frac{kc^2}{a^2(t_0)H_0^2} \quad \Longrightarrow \quad \tilde{a}(t_0) = \frac{cH_0^{-1}}{\sqrt{|\Omega_{k,0}|}} .$$

But we must still express ψ_D in terms of z_S. Since

$$ds^2 = -c^2\,dt^2 + \tilde{a}^2(t)\left\{d\psi^2 + \sin^2\psi\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\} ,$$

the equation for a null trajectory is

$$0 = -c^2\,dt^2 + \tilde{a}^2(t)\,d\psi^2 \quad \Longrightarrow \quad \frac{d\psi}{dt} = \frac{c}{\tilde{a}(t)} .$$

$$0 = -c^2\,dt^2 + \tilde{a}^2(t)\,d\psi^2 \quad \Longrightarrow \quad \frac{d\psi}{dt} = \frac{c}{\tilde{a}(t)} .$$

The first-order Friedmann equation implies

$$H^2 = \left(\frac{\dot{\tilde{a}}}{\tilde{a}}\right)^2 = \frac{H_0^2}{x^4}\left(\Omega_{m,0}x + \Omega_{\text{rad},0} + \Omega_{\text{vac},0}x^4 + \Omega_{k,0}x^2\right) ,$$

where

$$x = \frac{a(t)}{a(t_0)} = \frac{\tilde{a}(t)}{\tilde{a}(t_0)} .$$

The coordinate distance that the light pulse can travel between t_S (when it left the source) and t_0 (now) is

$$\psi(z_S) = \int_{t_S}^{t_0} \frac{c}{\tilde{a}(t)}\,dt .$$

Changing variables to z, with

$$1 + z = \frac{\tilde{a}(t_0)}{\tilde{a}(t)} .$$

Then

$$dz = -\frac{\tilde{a}(t_0)}{\tilde{a}(t)^2}\dot{\tilde{a}}(t)\,dt = -\tilde{a}(t_0)H(t)\,\frac{dt}{\tilde{a}(t)} .$$

The integration becomes

$$\psi(z_S) = \frac{1}{\tilde{a}(t_0)}\int_0^{z_S} \frac{c}{H(z)}\,dz .$$

$$\psi(z_S) = \frac{1}{\tilde{a}(t_0)}\int_0^{z_S} \frac{c}{H(z)}\,dz .$$

In this expression we can replace $\tilde{a}(t_0)\,H(z)$ using our previous equations. This gives our final expression for $\psi(z_S)$:

$$\psi(z_S) = \sqrt{|\Omega_{k,0}|} \times \int_0^{z_S} \frac{dz}{\sqrt{\Omega_{m,0}(1+z)^3 + \Omega_{\text{rad},0}(1+z)^4 + \Omega_{\text{vac},0} + \Omega_{k,0}(1+z)^2}} .$$

Using this in our previous expression for J

$$J = \frac{PH_0^2|\Omega_{k,0}|}{4\pi(1+z_S)^2c^2\sin^2\psi(z_S)} ,$$