

8.286 Class 20
November 16, 2020

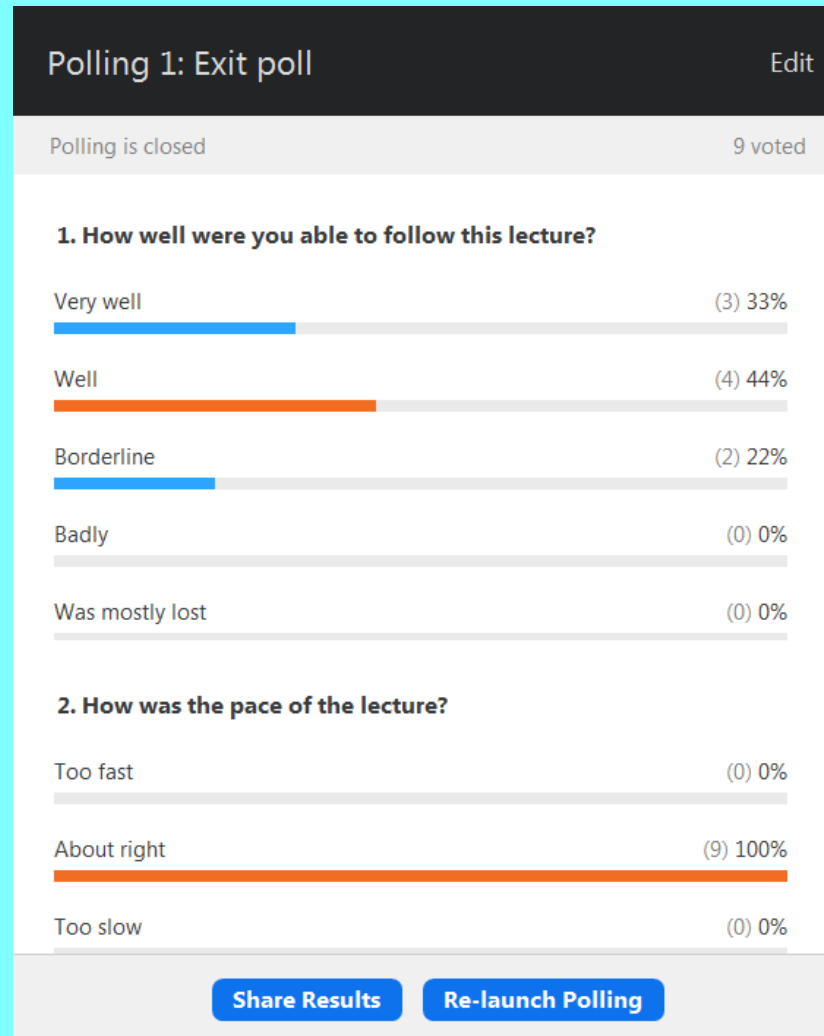
**THE
COSMOLOGICAL CONSTANT
PART 2**

Announcements

Problem Set 8 is due this Friday, November 20.



Exit Poll, Last Class



A Brief History of the Cosmological Constant

- ★ In 1917, Einstein applied his new GR to the universe, and discovered that a static universe would collapse.
- ★ Convinced that the universe was static, Einstein introduced the *cosmological constant* Λ into his field equations
— the equations that describe how matter affects the metric — to create a gravitational repulsion to oppose the collapse.
- ★ From a modern point of view, Λ represents a *vacuum energy density* u_{vac} , with

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because u_{vac} appears in the field equations exactly as a vacuum energy density would. To Einstein, however, it was simply a new term in the field equations. Before quantum theory, the vacuum was viewed as completely empty, so it was inconceivable that it could have a nonzero energy density.

★ Once the expansion of the universe was discovered by Hubble in 1929, Einstein abandoned Λ as being no longer needed or wanted.

★ In 1998, however, two (large) groups of astronomers, both using measurements of Type Ia supernova at redshifts $z \lesssim 1$, discovered evidence that the expansion of the universe is currently accelerating!

At the time, it was shocking! *Science* magazine proclaimed it (correctly!) as the “Breakthrough of the Year”.

★ In 2011 the Nobel Prize in Physics was awarded to Saul Perlmutter, Brian Schmidt, and Adam Riess for this discovery. In 2015 the Breakthrough Prize in Fundamental Physics was awarded to these three, and also the two entire teams.

Gravitational Effect of Pressure

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Vacuum Energy and the Cosmological Constant:

$$u_{\text{vac}} = \rho_{\text{vac}} c^2 = \frac{\Lambda c^4}{8\pi G} .$$

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Vacuum Energy and the Cosmological Constant:

$$u_{\text{vac}} = \rho_{\text{vac}} c^2 = \frac{\Lambda c^4}{8\pi G} .$$

Recall that

$$\dot{\rho} = -3\frac{\dot{a}}{a} \left(\rho + \frac{p}{c^2} \right) ,$$

where the overdot indicates a time derivative. So

$$\dot{\rho}_{\text{vac}} = 0 \quad \Longrightarrow \quad p_{\text{vac}} = -\rho_{\text{vac}} c^2 = -\frac{\Lambda c^4}{8\pi G} .$$

Defining $\rho = \rho_n + \rho_{\text{vac}}$ and $p = p_n + p_{\text{vac}}$, the Friedmann equations become:

$$\ddot{a} = -\frac{4\pi}{3}G \left(\rho_n + \frac{3p_n}{c^2} - 2\rho_{\text{vac}} \right) a .$$

$$\left(\frac{\dot{a}}{a} \right)^2 = \frac{8\pi}{3}G(\rho_n + \rho_{\text{vac}}) - \frac{kc^2}{a^2} ,$$

where an overdot ($\dot{}$) is a derivative with respect to t .

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where an overdot ($\dot{}$) is a derivative with respect to t . At late times, $\rho_n \propto 1/a^3$ or $1/a^4$, $\rho_{\text{vac}} = \text{constant}$, so ρ_{vac} dominates. Then

$$a(t) \propto e^{H_{\text{vac}} t} ,$$

$$H \rightarrow H_{\text{vac}} = \sqrt{\frac{8\pi}{3}G\rho_{\text{vac}}} .$$

Age of the Universe with Λ

The first order Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G \left(\underbrace{\rho_m}_{\propto \frac{1}{a^3(t)}} + \underbrace{\rho_{\text{rad}}}_{\propto \frac{1}{a^4(t)}} + \rho_{\text{vac}} \right) - \frac{kc^2}{a^2} .$$

can be rewritten as

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2 \left(\frac{\Omega_{m,0}}{x^3} + \frac{\Omega_{\text{rad},0}}{x^4} + \Omega_{\text{vac}} \right) - \frac{kc^2}{a^2} ,$$

$$\text{where } x \equiv a(t)/a(t_0) .$$

and where we used

$$\Omega_{X,0} = \frac{\rho_{X,0}}{\rho_{c,0}} = \frac{8\pi G \rho_{X,0}}{3H_0^2} .$$

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2 \left(\frac{\Omega_{m,0}}{x^3} + \frac{\Omega_{\text{rad},0}}{x^4} + \Omega_{\text{vac}} \right) - \frac{kc^2}{a^2} ,$$

where $x \equiv a(t)/a(t_0)$.

Define

$$\Omega_{k,0} \equiv -\frac{kc^2}{a^2(t_0)H_0^2} .$$

So

$$\left(\frac{\dot{a}}{a}\right)^2 = \left(\frac{\dot{x}}{x}\right)^2 = \frac{H_0^2}{x^4} \left(\Omega_{m,0}x + \Omega_{\text{rad},0} + \Omega_{\text{vac},0}x^4 + \Omega_{k,0}x^2 \right) .$$

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At present time, $\dot{a}/a = H_0$ and $x = 1$, so the sum of the Ω 's must equal 1. Thus, $\Omega_{k,0}$ can be evaluated from

$$\Omega_{k,0} = 1 - \Omega_{m,0} - \Omega_{\text{rad},0} - \Omega_{\text{vac},0} .$$

Observationally, $\Omega_{k,0}$ is consistent with zero, but we can still allow for it in our final formula for the age:

$$t_0 = \frac{1}{H_0} \int_0^1 \frac{xdx}{\sqrt{\Omega_{m,0}x + \Omega_{\text{rad},0} + \Omega_{\text{vac},0}x^4 + \Omega_{k,0}x^2}} .$$

Numerical Integration with Mathematica

IN: $t_0[H_0, \Omega_{m0}, \Omega_{rad0}, \Omega_{vac0}, \Omega_{k0}] := (1/H_0) *$

$NIntegrate[x/\text{Sqrt}[\Omega_{m0} x + \Omega_{rad0} + \Omega_{vac0} x^4 + \Omega_{k0} x^2], \{x, 0, 1\}]$

IN: $\text{PlanckH0} := \text{Quantity}[67.66, \text{"km/sec/Mpc"}]$

IN: $\text{Planck}\Omega_{m0} := 0.311$

IN: $\text{Planck}\Omega_{vac0} := 0.689$

IN: $\text{UnitConvert}[t_0[\text{PlanckH0}, \text{Planck}\Omega_{m0}, 0, \text{Planck}\Omega_{vac0}, 0], \text{"Years"}]$

OUT: 1.38022×10^{10} years

Numerical Integration with Mathematica

Newer Data

Reference: N. Aghanim et al. (Planck Collaboration), “Planck 2018 results, VI: Cosmological parameters,” Table 2, Column 6, arXiv:1807.06209.

```
IN: t0[H0_,Ωm0_,Ωrad0_,Ωvac0_,Ωk0_] := (1/H0) *
```

```
  NIntegrate[x/Sqrt[Ωm0 x + Ωrad0 + Ωvac0 x^4 + Ωk0 x^2], {x,0,1}]
```

```
IN: PlanckH0 := Quantity[67.66,"km/sec/Mpc"]
```

```
IN: PlanckΩm0 := 0.3111
```

```
IN: PlanckΩvac0 := 0.6889
```

```
IN: Ωrad0 :=  $4.15 \times 10^{-5} h_0^{-2} = 9.07 \times 10^{-5}$ 
```

```
IN: UnitConvert[t0[PlanckH0, PlanckΩm0 - Ωrad0/2, Ωrad0, PlanckΩvac0 -  
  Ωrad0/2 ,0], "Years"]
```

```
OUT:  $1.3796 \times 10^{10}$  years
```

The Planck paper gives 13.787 ± 0.020 Gyr. The difference is about 9 million years, 0.06%, or 0.45σ .

Look-Back Time

Question: If we observe a distant galaxy at redshift z , how long has it been since the light left the galaxy? The answer is called the *look-back time*.

To answer, recall that we wrote t_0 as an integral over $x = a(t)/a(t_0)$. We can change variables to

$$1 + z = \frac{a(t_0)}{a(t)} = \frac{1}{x} ,$$

which gives

$$t_0 = \frac{1}{H_0} \int_0^\infty \frac{dz}{(1+z) \sqrt{\Omega_{m,0}(1+z)^3 + \Omega_{\text{rad},0}(1+z)^4 + \Omega_{\text{vac},0} + \Omega_{k,0}(1+z)^2}} .$$

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The integral over any interval of z gives the corresponding time interval, so the look-back time is just the integral from 0 to z :

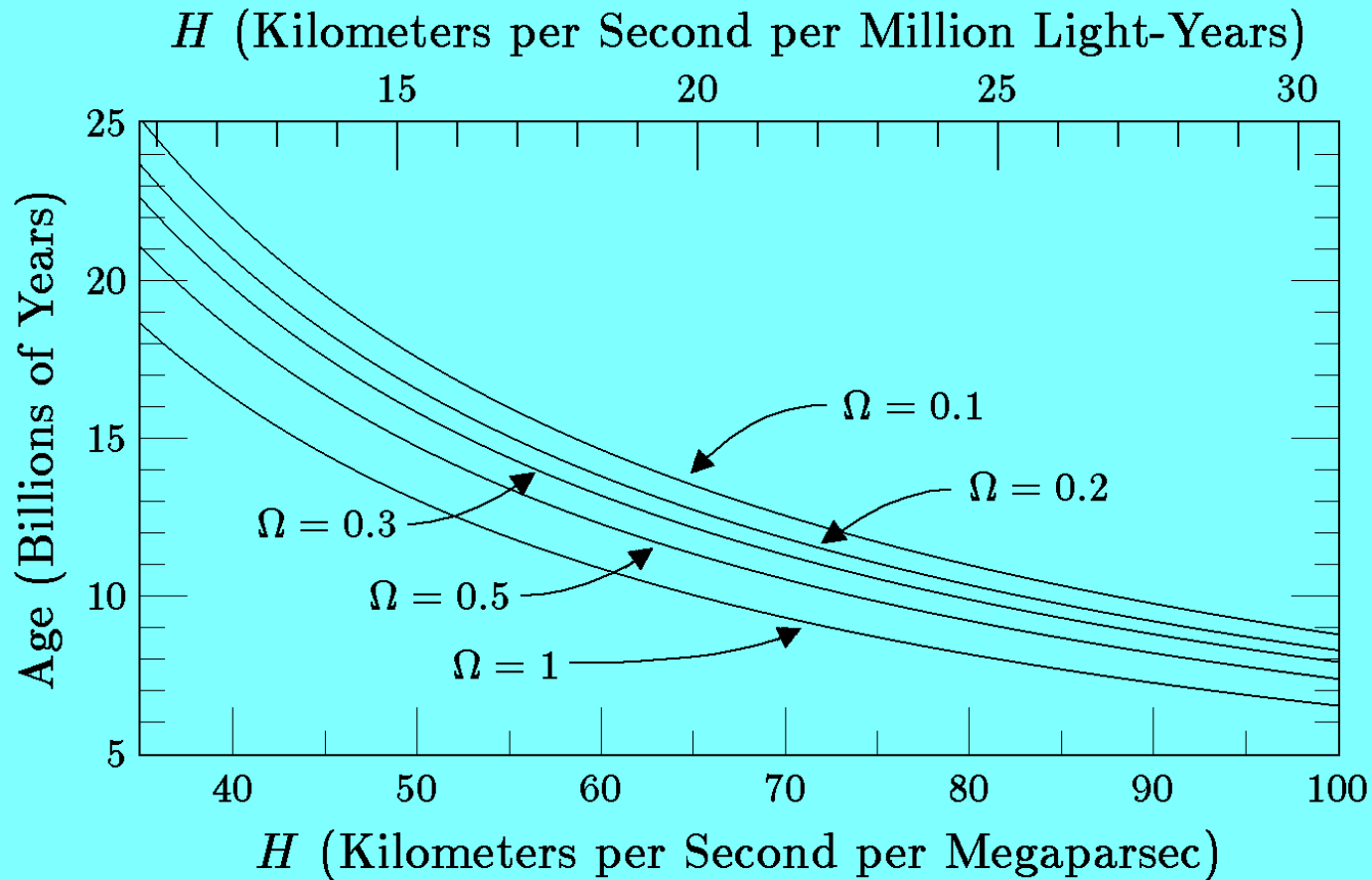
$$t_{\text{look-back}}(z) = \frac{1}{H_0} \int_0^z \frac{dz'}{(1+z') \sqrt{\Omega_{m,0}(1+z')^3 + \Omega_{\text{rad},0}(1+z')^4 + \Omega_{\text{vac},0} + \Omega_{k,0}(1+z')^2}} .$$

Age of a Flat Universe with Λ and Matter Only

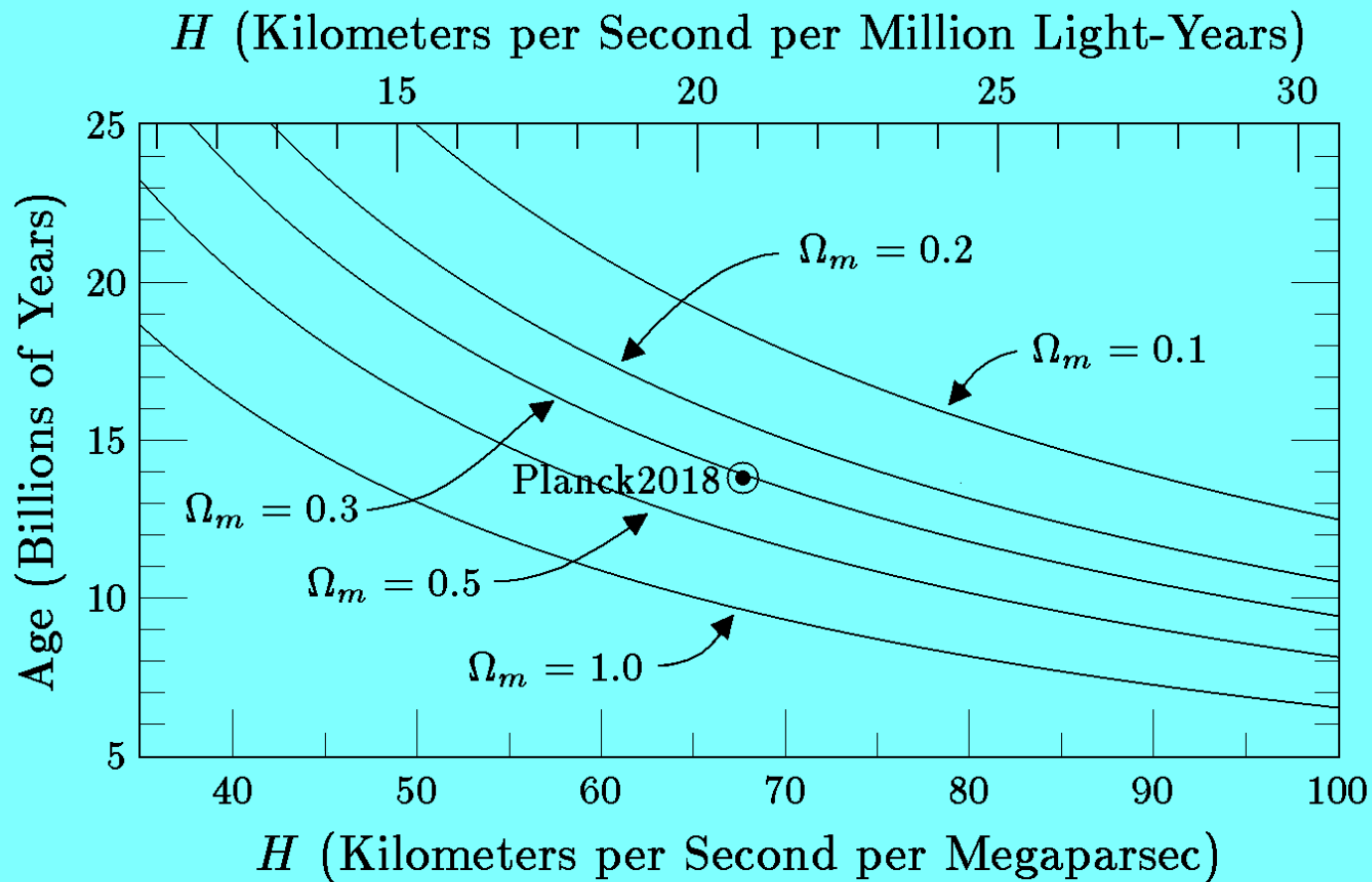
If $\Omega_{\text{rad}} = \Omega_k = 0$, then it is possible to carry out the integral for the age analytically:

$$t_0 = \begin{cases} \frac{2}{3H_0} \frac{\tan^{-1} \sqrt{\Omega_{m,0} - 1}}{\sqrt{\Omega_{m,0} - 1}} & \text{if } \Omega_{m,0} > 1, \Omega_{\text{vac}} < 0 \\ \frac{2}{3H_0} & \text{if } \Omega_{m,0} = 1, \Omega_{\text{vac}} = 0 \\ \frac{2}{3H_0} \frac{\tanh^{-1} \sqrt{1 - \Omega_{m,0}}}{\sqrt{1 - \Omega_{m,0}}} & \text{if } \Omega_{m,0} < 1, \Omega_{\text{vac}} > 0 \end{cases} .$$

The Age Problem with Only Nonrelativistic Matter



Age of a Flat Universe with Λ and Matter Only



Ryden Benchmark and Planck 2018 Best Fit

Parameters	Ryden Benchmark	Planck 2018 Best Fit
H_0	68	$67.7 \pm 0.4 \text{ km}\cdot\text{s}^{-1}\cdot\text{Mpc}^{-1}$
Baryonic matter Ω_b	0.048	$0.0490 \pm 0.0007^*$
Dark matter Ω_{dm}	0.262	$0.261 \pm 0.004^*$
Total matter Ω_m	0.31	0.311 ± 0.006
Vacuum energy Ω_{vac}	0.69	0.689 ± 0.006

Controversy in Parameters: "Hubble Tension"

★ From the CMB, the best number is from

$$\text{Planck 2018: } H_0 = 67.66 \pm 0.42 \text{ km sec}^{-1} \text{ Mpc}^{-1}$$

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SH0ES (Supernovae, H0, for the Equation of State of dark energy, group led by Adam Riess): $H_0 = 74.03 \pm 1.42 \text{ km sec}^{-1} \text{ Mpc}^{-1}$.

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- ★ From the “tip of the red giant branch”,

$$\text{Wendy Freedman's group: } H_0 = 69.6 \pm 0.8(\text{stat}) \pm 1.7(\text{sys}) \text{ km sec}^{-1} \text{ Mpc}^{-1}$$

References: A. Riess et al., *Astrophys. J.* 876 (2019) 85 [arXiv:1903.07603].

W. Freedman et al., arXiv:2002.01550 (2020).

The Hubble Diagram: Radiation Flux vs. Redshift

If we live in a universe like we have described, what do we expect to find if we measure the energy flux from a “standard candle” as a function of its redshift?

Consider closed universe:

$$ds^2 = -c^2 dt^2 + a^2(t) \left\{ \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right\} .$$

We will be interested in tracing radial trajectories, so we can simplify the radial metric by a change of variables

$$\sin \psi \equiv \sqrt{k} r .$$

Then

$$d\psi = \frac{\sqrt{k} dr}{\cos \psi} = \frac{\sqrt{k} dr}{\sqrt{1 - kr^2}} ,$$

$$ds^2 = -c^2 dt^2 + a^2(t) \left\{ \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right\} .$$

$$d\psi = \frac{\sqrt{k} dr}{\cos \psi} = \frac{\sqrt{k} dr}{\sqrt{1 - kr^2}} ,$$

and the metric simplifies to

$$ds^2 = -c^2 dt^2 + \tilde{a}^2(t) \{ d\psi^2 + \sin^2 \psi (d\theta^2 + \sin^2 \theta d\phi^2) \} ,$$

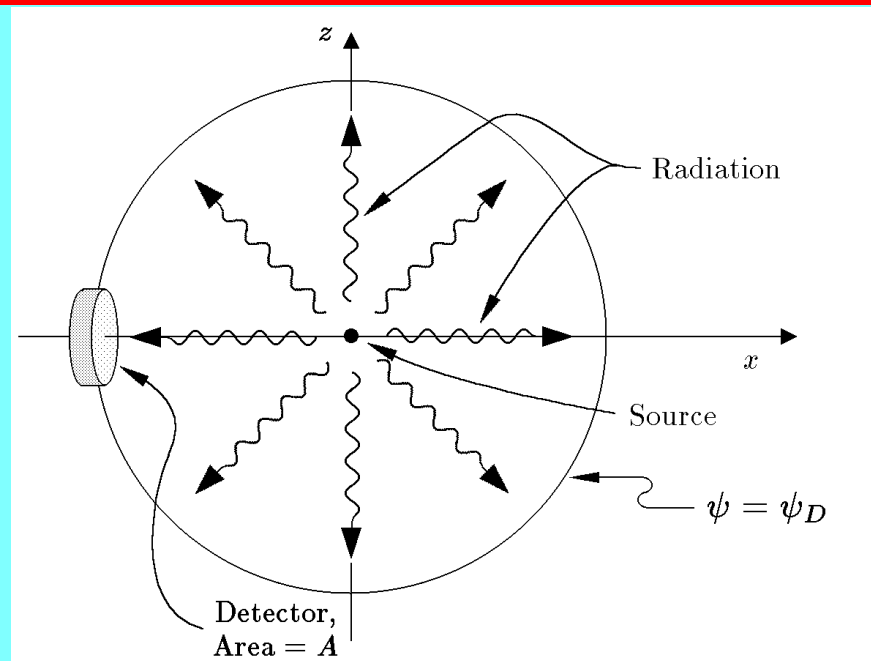
where

$$\tilde{a}(t) \equiv \frac{a(t)}{\sqrt{k}} .$$

Note: ψ is in fact the same angle ψ that we used in our construction of the closed-universe metric: it is the angle from the w -axis.

Geometry of Flux Calculation

$$ds^2 = -c^2 dt^2 + \tilde{a}^2(t) \{ d\psi^2 + \sin^2 \psi (d\theta^2 + \sin^2 \theta d\phi^2) \}$$



The fraction of the photons hitting the sphere that hit the detector is just the ratio of the areas:

$$\text{fraction} = \frac{\text{area of detector}}{\text{area of sphere}} = \frac{A}{4\pi \tilde{a}^2(t_0) \sin^2 \psi_D}.$$

- ★ The power hitting the sphere is less than the power P emitted by the source by two factors of $(1 + z_S)$, where z_S is the redshift of the source: one factor due to redshift of each photon, and one factor due to the redshift of the rate of arrival of photons.

$$P_{\text{received}} = \frac{P}{(1 + z_S)^2} \frac{A}{4\pi \tilde{a}^2(t_0) \sin^2 \psi_D} .$$

$$\text{Flux } J = P_{\text{received}}/A.$$

Expressing the Result in Terms of Astronomical Quantities

$$\Omega_{k,0} \equiv -\frac{kc^2}{a^2(t_0)H_0^2} \implies \tilde{a}(t_0) = \frac{cH_0^{-1}}{\sqrt{|\Omega_{k,0}|}} .$$

But we must still express ψ_D in terms of z_S . Since

$$ds^2 = -c^2 dt^2 + \tilde{a}^2(t) \{ d\psi^2 + \sin^2 \psi (d\theta^2 + \sin^2 \theta d\phi^2) \} ,$$

the equation for a null trajectory is

$$0 = -c^2 dt^2 + \tilde{a}^2(t) d\psi^2 \implies \frac{d\psi}{dt} = \frac{c}{\tilde{a}(t)} .$$

$$0 = -c^2 dt^2 + \tilde{a}^2(t) d\psi^2 \quad \Longrightarrow \quad \frac{d\psi}{dt} = \frac{c}{\tilde{a}(t)} .$$

The first-order Friedmann equation implies

$$H^2 = \left(\frac{\dot{\tilde{a}}}{\tilde{a}} \right)^2 = \frac{H_0^2}{x^4} \left(\Omega_{m,0} x + \Omega_{\text{rad},0} + \Omega_{\text{vac},0} x^4 + \Omega_{k,0} x^2 \right) ,$$

where

$$x = \frac{a(t)}{a(t_0)} = \frac{\tilde{a}(t)}{\tilde{a}(t_0)} .$$

The coordinate distance that the light pulse can travel between t_S (when it left the source) and t_0 (now) is

$$\psi(z_S) = \int_{t_S}^{t_0} \frac{c}{\tilde{a}(t)} dt .$$

Changing variables to z , with

$$1 + z = \frac{\tilde{a}(t_0)}{\tilde{a}(t)} .$$

Then

$$dz = -\frac{\tilde{a}(t_0)}{\tilde{a}(t)^2} \dot{\tilde{a}}(t) dt = -\tilde{a}(t_0) H(t) \frac{dt}{\tilde{a}(t)} .$$

The integration becomes

$$\psi(z_S) = \frac{1}{\tilde{a}(t_0)} \int_0^{z_S} \frac{c}{H(z)} dz .$$

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In this expression we can replace $\tilde{a}(t_0) H(z)$ using our previous equations. This gives our final expression for $\psi(z_S)$:

$$\psi(z_S) = \sqrt{|\Omega_{k,0}|} \times \int_0^{z_S} \frac{dz}{\sqrt{\Omega_{m,0}(1+z)^3 + \Omega_{\text{rad},0}(1+z)^4 + \Omega_{\text{vac},0} + \Omega_{k,0}(1+z)^2}} .$$

Using this in our previous expression for J

$$J = \frac{PH_0^2 |\Omega_{k,0}|}{4\pi(1+z_S)^2 c^2 \sin^2 \psi(z_S)} ,$$