### 8.286 Class 24 December 7, 2020

GRAND UNIFIED THEORIES AND THE
MAGNETIC MONOPOLE PROBLEM, PART 2

## Calendar for the Home Stretch:

| NOVEMBER/DECEMBER |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: |
| MON | TUES | WED | THURS | FRI |
| 23 <br> Thanksgiving <br> Week | 24 | 25 | 26 | 27 |
| $30$ <br> Class 22 | December 1 | 2 <br> Class 23 <br> Quiz 3 | 3 | 4 |
| $\begin{aligned} & 7 \\ & \text { Class } 24 \end{aligned}$ | 8 | 9 <br> Class 25 <br> PS 9 due <br> Last Class | 10 | 11 |

## Announcements

is Quiz 3 grades are posted, and solutions are posted.
is Quiz 3 Results: Mostly good, a little scattered.
is Class Average: 77.3. Standard deviation: 15.5. For comparison, the previous two averages were 92.3 and 85.9.
is Top grades were great: two 98's, a 96 , a 95 , a 93 , a 91,

is Apparently the test was too long - one piece of evidence is that the scores on the last parts were very low: $52 \%$ on each. If these two parts were omitted, and the rest of the quiz was averaged, the class average would be 83.6 .
is If you have any questions about the grading of your paper, please contract Bruno and/or me. We are happy to reconsider any grade. We try to grade as accurately as we can, but I am sure that we sometimes make mistakes.

> Though, in reviewing the incidents of my administration, I am unconscious of intentional error, I am nevertheless too sensible of my defects not to think it probable that I may have committed many errors.

## - George Washington's Farewell Address

is There is one last problem set, Problem Set 9, due the last day of classes, this Wednesday December 9. No final exam!
is Special office hours this week:
Me: Today, Monday 12/7/20 at 5:00 pm.
Bruno: Tomorrow, Tuesday 12/8/20 at 6:00 pm.

## Exit Poll, Class 22 (Last Class)

| Polling 1: Exit poll | Edit |
| :--- | :--- |
| Polling is closed | 14 voted |
| 1. How well were you able to follow this lecture? |  |
| Very well | (0) $0 \%$ |
| Well | (4) $29 \%$ |
| Borderline | (8) $57 \%$ |
| Badly | (1) $7 \%$ |
| Was mostly lost | (1) $7 \%$ |
| 2. How was the pace of the lecture? | (1) $7 \%$ | | Too fast | (12) $86 \%$ |
| :--- | :--- | :--- |

## 

## Particle Content:



Wikimedia Commons. Source: PBS NOVA, Fermilab, Office of Science, United States Department of Energy, Particle Data Group.

## Quarks are Colored

is A quark is specified by its flavor $[\mathrm{u}(\mathrm{p}), \mathrm{d}(\mathrm{own}), \mathrm{c}$ (harmed), $\mathrm{s}($ trange $), \mathrm{t}(\mathrm{op})$, $\mathrm{b}($ ottom $)$ ], its spin [up or down, along any chosen $z$ axis], whether it is a quark or antiquark, AND ITS COLOR [three choices, often red, blue, or green].

## Gauge Theories: Electromagnetic Example

Fields and potentials: $\vec{E}=-\vec{\nabla} \phi-\frac{1}{c} \frac{\partial \vec{A}}{\partial t}, \quad \vec{B}=\vec{\nabla} \times \vec{A}$.
Four-vector notation: $A_{\mu}=\left(-\phi, A_{i}\right), F_{\mu \nu}=\partial_{\mu} A_{\nu}-\partial_{\nu} A_{\mu}$.
Gauge transformations, in four-vector notation:

$$
A_{\mu}^{\prime}(x)=A_{\mu}(x)+\frac{\partial \Lambda}{\partial x^{\mu}}, \text { where } x \equiv(t, \vec{x})
$$

Field configurations $A_{\mu}(x)$ that are related by a gauge transformation represent the SAME physical situation.

## Electromagnetism as a $U(1)$ Gauge Theory

$\Lambda(x)$ is an element of the real numbers.
To construct the gauge transformation, it is sufficient to know

$$
u \equiv e^{i e_{0} \Lambda(x)}
$$

where $e_{0}$ is the charge of a proton and $e=2.71828 \ldots$.
This is LESS information, since we only have to know $\Lambda(x)$ modulo $2 \pi / e_{0}$.
$u$ is an element of the group $U(1)$, the group of complex phases $u=e^{i \chi}$, where
$\chi$ is real. So $\mathrm{E} \& \mathrm{M}$ is a $U(1)$ gauge theory.

## Gauge Groups of the Standard Model

$U(1)$ is abelian (commutative), but Yang and Mills showed in 1954 how to construct a nonabelian gauge theory. The standard model contains the following gauge symmetries:
$S U(3)$ : This is the group of $3 \times 3$ complex matrices that are
$S \equiv$ Special: they have determinant 1.
$\mathrm{U} \equiv$ Unitary: they obey $u^{\dagger} u=1$, which means that when they multiply a $1 \times 3$ column vector $v$, they preserve the norm $|v| \equiv \sqrt{v_{i}^{*} v_{i}}$.
$S U(2)$ : The group of $2 \times 2$ complex matrices that are special (S) and unitary $(\mathrm{U})$. As you may have learned in quantum mechanics, $S U(2)$ is almost the same as the rotation group in 3 D , with a $2: 1$ group-preserving mapping between $S U(2)$ and the rotation group.
$U(1)$ : The group of complex phases. The $U(1)$ of the standard model is not the $U(1)$ of $\mathrm{E} \& \mathrm{M}$; instead $U(1)_{\mathrm{E} \& \mathrm{M}}$ is a linear combination of the $U(1)$ of the standard model and a rotation about one fixed direction in $S U(2)$.

Combining the groups: the gauge symmetry group of the standard model is usually described as $S U(3) \times S U(2) \times U(1)$. An element of this group is an ordered triplet $\left(u_{3}, u_{2}, u_{1}\right)$, where $u_{3} \in S U(3), u_{2} \in S U(2)$, and $u_{1} \in U(1)$, so $S U(3) \times S U(2) \times U(1)$ is really no different from thinking of the 3 symmetries separately.
$S U(3)$ describes the strong interactions, and $S U(2) \times U(1)$ together describe the electromagnetic and weak interactions in a unified way, called the electroweak interactions.

Gauge theories always have one gauge boson for each parameter of the gauge group:
$S U(3): 8$ parameters $\Longrightarrow 8$ gluons.
$S U(2) \times U(1): 3+1$ parameters $\Longrightarrow \quad$ photon and $W^{+}, W^{-}$, and $Z$.
The gauge symmetry dictates how these particles interact. If the gauge symmetry is not spontaneously broken (to be discussed shortly), the gauge boson is massless, like the photon.

## The Higgs Field and Spontaneous Symmetry Breaking

The Higgs field is a complex doublet:

$$
H(x) \equiv\binom{H_{1}(x)}{H_{2}(x)}
$$

Under $S U(2)$ transformations, $H^{\prime}(x)=u_{2}(x) H(x)$, where $u_{2}(x)$ is the complex
$2 \times 2$ matrix that defines the $S U(2)$ gauge transformation at $x$.

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Toy Theory (easier to understand): Consider a "vector Higgs" $\vec{\phi}(x)$, with 3 real components:

$$
\vec{\phi}(x) \equiv\left(\begin{array}{l}
\phi_{1}(x) \\
\phi_{2}(x) \\
\phi_{3}(x)
\end{array}\right)
$$

Recall that $S U(2)$ is closely related to the 3 D rotation group: there are 2 elements of $S U(2)$ for every element of the rotation group. $\vec{\phi}$ transforms just like any vector under these rotations.
Since the gauge symmetry implies that the potential energy density of the Higgs field $V(H)$ must be gauge-invariant, $V$ can depend only on $|H| \equiv \sqrt{\left|H_{1}\right|^{2}+\left|H_{2}\right|^{2}}$, or in the toy theory, $|\vec{\phi}| \equiv \sqrt{\phi_{1}^{2}+\phi_{2}^{2}+\phi_{3}^{2}}$, which is unchanged by $S U(2)$ transformations.

Potential energy function $V(|H|)$ or $V(\vec{\phi})$ :


The minimum is not at $|H|=0$, but instead at $|H|=H_{v}$.
$|H|=0$ is $S U(2)$ gauge-invariant, but $|H|=H_{v}$ is not. $H$ randomly picks out some direction in the space of 2 D complex vectors.
In the toy vector Higgs theory, $\vec{\phi}=0$ is rotationally invariant, but $\vec{\phi} \neq 0$ must pick out some direction. $\vec{\phi}$ is invariant under rotations about $\vec{\phi}$, but not under other rotations. So the vector Higgs "breaks" the 3D rotation group symmetry down to 1 D rotations (which is the same as $U(1)$.

## Spontaneous Symmetry Breaking

Definition: Whenever the ground state of a system has less symmetry than the underlying laws, it is called spontaneous symmetry breaking. Other examples: crystals, ferromagnetism.

## Higgs Fields Give Mass to Other Particles

When $H=0$, all the fundamental particles of the standard model are massless. (Protons, however, are not massless.) Furthermore, there is no distinction between the electron $e$ and the electron neutrino $\nu_{e}$, or between $\mu$ and $\nu_{\mu}$, or between $\tau$ and $\nu_{\tau}$.

To describe how $H$ gives mass to the other particles, consider the toy vector Higgs theory. For $|\vec{\phi}| \neq 0, \vec{\phi}$ randomly picks out a direction in the 3D space of $\left(\phi_{1}, \phi_{2}, \phi_{3}\right)$. Since all directions are otherwise equivalent, we can assume that in the vacuum,

$$
\vec{\phi}=\left(\begin{array}{c}
0 \\
0 \\
\phi_{v}
\end{array}\right)
$$

Components of other fields that interact with $\phi_{3}$ then start to behave differently from fields that interact with other components of $\vec{\phi}$.

Mass: $m c^{2}$ of a particle is the state of lowest energy above the ground state. In a field theory, this corresponds to a homogeneous oscillation of the field, which in turn corresponds to a particle with zero momentum.
If we ignore the interactions between fields, the field acts exactly like a harmonic oscillator. The first excited state has energy $h \nu=\hbar \omega$ above the ground state. So, $m c^{2}=\hbar \omega$.
$\omega$ is determined by inertia and the restoring force. When $\vec{\phi}=0$, the standard model interactions provide no restoring forces. Any such restoring force would break gauge invariance.
When $\vec{\phi}=\left(\begin{array}{c}0 \\ 0 \\ \phi_{v}\end{array}\right)$, the interactions with $\vec{\phi}$ create a restoring force for some components of other fields, giving them a mass. (That is, the energy density can contain terms such as $\phi_{3} \psi^{2}$, creating a restoring force for the field $\psi$.) This "Higgs mechanism" creates the distinction between electrons and neutrinos - the electrons are the particles that get a mass, and the neutrinos do not. (Neutrinos are exactly massless in the Standard Model of Particle Physics. There are various ways to modify the model to account for neutrino masses.)

The Higgs mechanism, through the nonzero components of $\vec{\phi}$, also gives a mass to some of the gauge bosons. The gauge bosons that correspond to broken symmetries are given a mass, while the gauge bosons that

## Beyond the Standard Model

With neutrino masses added, the standard model is spectacularly successful: it agrees with all reliable particle experiments.
Nonetheless, most physicists regard it as incomplete, for at least two types of reasons:

1) It does not include gravity, and it does not include any particle to account for the dark matter. (Maybe black holes can be the dark matter, but that requires a mass distribution that we cannot explain.)
2) The theory appears too inelegant to be the final theory. It contains more arbitrary features and free parameters than one would hope for in a final theory. Why $S U(3) \times S U(2) \times U(1)$ ? Why three generations of fermions? The original theory had 19 free parameters, with more needed for neutrino masses and even more if supersymmetry is added.

Result: BSM (Beyond the Standard Model) particle physics has become a major industry.

Alan Guth

## Grand Unified Theories

Goal: Unify $S U(3) \times S U(2) \times U(1)$ by embedding all three into a single, larger group. (Gravity is left for another day.)

The breaking of the symmetry to $S U(3) \times S U(2) \times U(1)$ is accomplished by introducing new Higgs fields to spontaneously break the symmetry.

In the fundamental theory, before spontaneous symmetry breaking, there is no distinction between an electron, an electron neutrino, or an up or down quark.

## The SU(5) Grand Unified Theory

In 1974, Howard Georgi and Sheldon Glashow of Harvard proposed the original grand unified theory, based on $S U(5)$. They pointed out that $S U(3) \times S U(2) \times U(1)$ fits elegantly into $S U(5)$.

To start, let the $S U(3)$ subgroup be matrices of the form

$$
u_{3}=\left(\begin{array}{lllll}
x & x & x & 0 & 0 \\
x & x & x & 0 & 0 \\
x & x & x & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{array}\right)
$$

where the $3 \times 3$ block of $x$ 's represents an arbitrary $\mathrm{SU}(3)$ matrix.

Alan Guth

Similarly let the $S U(2)$ subgroup be matrices of the form

$$
u_{2}=\left(\begin{array}{lllll}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & x & x \\
0 & 0 & 0 & x & x
\end{array}\right)
$$

where this time the $2 \times 2$ block of $x$ 's represents an arbitrary $\mathrm{SU}(2)$ matrix.
Note that $u_{3}$ and $u_{2}$ commute, since each acts like the identity matrix in the space in which the other is nontrivial.
Finally, the $U(1)$ subgroup can be written as

$$
u_{1}=\left(\begin{array}{ccccc}
e^{2 i \theta} & 0 & 0 & 0 & 0 \\
0 & e^{2 i \theta} & 0 & 0 & 0 \\
0 & 0 & e^{2 i \theta} & 0 & 0 \\
0 & 0 & 0 & e^{-3 i \theta} & 0 \\
0 & 0 & 0 & 0 & e^{-3 i \theta}
\end{array}\right)
$$

where the factors of 2 and 3 in the exponents were chosen so that the determinant - in this case the product of the diagonal entries - is equal to 1.

Repeating, the $U(1)$ subgroup can be written as

$$
u_{1}=\left(\begin{array}{ccccc}
e^{2 i \theta} & 0 & 0 & 0 & 0 \\
0 & e^{2 i \theta} & 0 & 0 & 0 \\
0 & 0 & e^{2 i \theta} & 0 & 0 \\
0 & 0 & 0 & e^{-3 i \theta} & 0 \\
0 & 0 & 0 & 0 & e^{-3 i \theta}
\end{array}\right)
$$

$u_{1}$ commutes with any matrix of the form of $u_{2}$ or $u_{3}$, since within either the upper $3 \times 3$ block or within the lower $2 \times 2$ block, $u_{1}$ is proportional to the identity matrix.

Thus, any element $\left(u_{3}, u_{2}, u_{1}\right)$ of $S U(3) \times S U(2) \times U(1)$ can be written as an element $u_{5}$ of $S U(5)$, just by setting $u_{5}=u_{3} u_{2} u_{1}$.

## How Can Three Different Types of Interaction Look Like One?

In the standard model, each type of gauge interaction - $S U(3), S U(2)$, and $U(1)$ - has its own interaction strength, described by "coupling constants" $g_{3}, g_{2}$, and $g_{1}$. Their values of are different from each other! How can they be one interaction?

BUT: the interaction strength varies with energy in a calculable way. When the calculations are extended to superhigh energies, of the order of $10^{16} \mathrm{GeV}$, the three interaction strengths become about equal!

## Running Coupling Constants



MSSM: $\mathrm{m}_{0}=\mathrm{M}_{1 / 2}=2 \mathrm{TeV}, \mathrm{A}_{0}=0, \tan \beta=30$


The top graph shows the running of coupling constants in the standard model, showing that the three coupling constants do not quite meet. The bottom graph shows the running of coupling constants in the MSSM - the Minimal Supersymmetric Standard Model, in which the meeting of the couplings is almost perfect. $\alpha_{i}=g_{i}^{2} / 4 \pi$.

Source: Particle Data Group 2016 Review of Particle Physics, Chapter 16, Grand Unified Theories, Revised January 2016 by A. Hebecker and J. Hisano.

## Running Couplings Minimal Supersymmetric Standard Model



Bottom line: An $S U(5)$ grand unified theory can be constructed by introducing a Higgs field that breaks the $S U(5)$ symmetry to $S U(3) \times S U(2) \times U(1)$ at an energy of about $10^{16} \mathrm{GeV}$. At energies above $10^{16} \mathrm{GeV}$, the theory behaves like a fully unified $S U(5)$ gauge theory. At lower energies, it behaves like the standard model. The gauge particles that are part of $S U(5)$ but not part of $S U(3) \times S U(2) \times U(1)$ acquire masses of order $10^{16} \mathrm{GeV}$.

GUTs (Grand Unified Theories) allow two unique phenomena at low energies, neither of which have been seen:

1) Proton decay. The superheavy gauge particles can mediate proton decay. The minimal $S U(5)$ model - with the simplest conceivable particle content - predicts a proton lifetime of about $10^{31}$ years, which is ruled out by experiments, which imply a lifetime $\gtrsim 10^{34}$ years.
2) Magnetic monopoles. All grand unified theories imply that magnetic monopoles should be a possible kind of particle. None have been seen.

The absence of evidence does not imply that GUTs are wrong, but we don't know.

## The Grand Unified Theory Phase Transition

When $k T \gg 10^{16} \mathrm{GeV}$, the Higgs fields of the GUT undergo large fluctuations, and average to zero. The GUT symmetry is unbroken, and the theory behaves as an $\operatorname{SU}(5)$ gauge theory.
As $k T$ falls to about $10^{16} \mathrm{GeV}$, the matter filling the universe would go through a phase transition, in which some of the components of the GUT Higgs field acquire nonzero values in the thermal equilibrium state, breaking the GUT symmetry. The breaking to $S U(3) \times S U(2) \times U(1)$ might occur in one phase transition, or in a series of phase transitions. We'll assume a single phase transition.
The Higgs fields start to randomly acquire nonzero values, but the nonzero values that form in one region may not align with those in another.
The expression for the energy density contains a term proportional to $|\nabla H|^{2}$, so the fields tend to fall into low energy states with small gradients. But sometimes the fields in one region acquire a pattern that cannot be smoothly joined with the pattern in a neighboring region, so the smoothing is imperfect, leaving "defects".

## Topological Defects

There are three types of defects:

1) Domain walls. For example, imagine a single real scalar field $\phi$ for which the potential energy function has two local minima, at $\phi_{1}$ and $\phi_{2}$ :


Then, as the system cools, some regions will have $\phi \approx \phi_{1}$ and others will have $\phi \approx \phi_{2}$. The boundaries between these regions will be surfaces of high energy density: domain walls. Some GUTS allow domain walls, others do not. The energy density of a domain wall is so high that none can exist in the visible universe.

1) Domain walls.
2) Cosmic strings. Linelike defects, which exist in some GUTs but not all.
3) Magnetic monopoles: Pointlike defects, which exist in all GUTs.

## Magnetic Monopoles

We'll consider the simplest theory in which monopoles arise, which is exactly the toy vector Higgs model that we have been discussing. It has a 3 -component (real) Higgs field, $\phi_{a}$, where $a=1,2$ or 3 . Gauge symmetry acting on $\phi_{a}$ has the same mathematical form as the rotations of an ordinary Cartesian 3 -vector.

To be gauge-invariant, the energy density function can depend only on

$$
|\phi| \equiv \sqrt{\phi_{1}^{2}+\phi_{2}^{2}+\phi_{3}^{2}},
$$

and we assume that it looks qualitatively like the graph for the standard model, with a minimum at $H_{v}$.

Now consider the following static configuration,

$$
\phi_{a}(\vec{r})=f(r) \hat{r}_{a}
$$

where $r \equiv|\vec{r}|, \hat{r}_{a}$ denotes the $a$-component of the unit vector $\hat{r}=\vec{r} / r$, and $f(r)$ is a function which vanishes when $r=0$ and approaches $H_{v}$ as $r \rightarrow \infty$.

Pictorially,

where the 3 components of the arrow at each point describe the 3 Higgs field components.

Repeating,

where the 3 components of the arrow at each point describe the 3 Higgs field components.

The directions in gauge space $\phi_{a}$ really have nothing to do with directions in physical space, but there is nothing that prevents the fields from existing in this configuration.
The configuration is topologically stable in the following sense: if the boundary conditions at infinity are fixed, and the fields are continuous, then there is always at least one point where $\phi_{1}=\phi_{2}=\phi_{3}=0$.
Thus, the monopoles are topologically stable knots in the Higgs field.

## Why Are These Things Magnetic Monopoles?

Definition: A magnetic monopole is an object with a net magnetic charge, north or south, with a radial magnetic field of the same form as the electric field of a point charge.
Known magnets are always dipoles, with a north end and a south end. If such a magnet is cut in half, one gets two dipoles, each with a north and south end.
Energy of the configuration: the energy density contains a term $\sum_{a} \vec{\nabla} \phi_{a} \cdot \vec{\nabla} \phi_{a}$, but the changing direction of $\phi_{a}$ (always radially outward) implies $\left|\nabla \phi_{a}\right| \propto 1 / r$. The total energy in a sphere of radius $R$ is proportional to

$$
4 \pi \int^{R} r^{2} \mathrm{~d} r\left(\frac{1}{r}\right)^{2}
$$

which diverges as $R$ for large $R$.
With the vector gauge fields, however, the energy density is more complicated. It can be made finite only if the gauge field configuration corresponds to a net magnetic charge.

Alan Guth

## Prediction of Magnetic Charge

The magnetic charge is uniquely determined, and is equal to $1 /(2 \alpha)$ times the electric charge of an electron, where $\alpha \simeq 1 / 137(\alpha=$ fine-structure constant $=e^{2} / \hbar c$ in Gaussian units, or $e^{2} /\left(4 \pi \epsilon_{0} \hbar c\right)$ in SI.)

The force between two monopoles is therefore $(68.5)^{2}$ times as large as the force between two electrons at the same distance. I.e., large!

## Kibble Estimate of Magnetic Monopole Production

Since magnetic monopoles are knots in the GUT Higgs fields, they form at the GUT phase transition, when the Higgs fields acquire nonzero mean values. ("Mean" = average over time, not space.)

The density of these knots will be related to the misalignment of the Higgs field in different regions.

Define a correlation length $\xi$, crudely, as the minimum distance such that the Higgs field at point is almost uncorrelated with the Higgs field a distance $\xi$ away.
T.W.B. Kibble of Imperial College (London) proposed that the number density of magnetic monopoles (and antimonopoles) can be estimated as

$$
n_{M} \approx 1 / \xi^{3}
$$

## Estimate of Correlation Length $\xi$

In the context of conventional (non-inflationary) cosmology, we can assume

1) that the Higgs field well before the GUT phase transition is in a thermal state, with no long-range correlations.
2) that the universe before the phase transition is well-approximated by a flat radiation-dominated Friedman-Robertson-Walker description.
3) phase transition happens promptly when the temperature of the GUT phase transition is reached, at $k T \approx 10^{16} \mathrm{GeV}$.

Under these assumptions, we are confident that the correlation length $\xi$ must be less than or equal to the horizon length at the time of the phase transition. This seemingly mild limit turns out to have huge implications.

On Problem Set 10, you will calculate the contribution to $\Omega$ today, from the monopoles. I won't give away the answer, but you should find that it is greater than $10^{20}$.

