8.286 Lecture 4
September 19, 2022

THE KINEMATICS of a HOMOGENEOUSLY EXPANDING UNIVERSE

Hubble's Law

$$v = Hr\ .$$

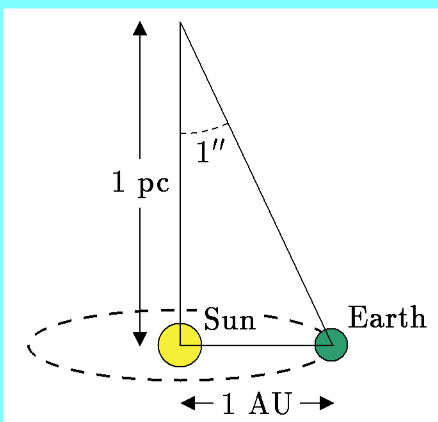
Here

$$v \equiv \text{recession velocity}\ ,$$
$$H \equiv \text{Hubble expansion rate}\ ,$$

and

$$r \equiv \text{distance to galaxy}\ .$$

The Parsec



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Units for the Hubble Expansion Rate

$v = Hr \quad \Longrightarrow \quad [H] = [v]/[r] = (L/T)/L = 1/T.$

Astronomers invariably think in terms of velocity/distance, which they measure in km-s^{-1}-Mpc^{-1}.

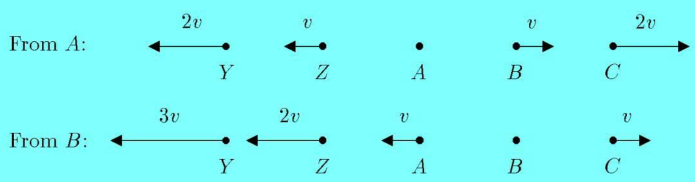
1 pc = 3.2616 light-yr; Mpc = megaparsec = 10^6 pc.

Relation to inverse time:

$$\frac{1}{10^{10}\ \text{yr}} = 97.8\ \text{km-s}^{-1}\text{-Mpc}^{-1}.$$

Homogeneity and Hubble's Law

- Does Hubble's law imply that we are in the center of the universe? No.
- As Weinberg explains it in *The First Three Minutes:*



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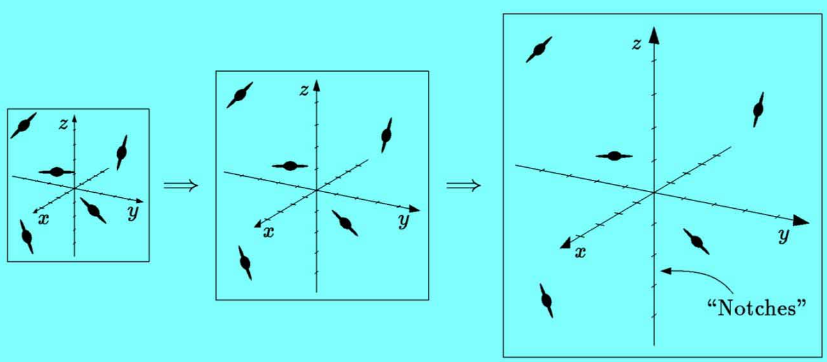
Comoving Coordinates

- If the Earth kept getting larger, uniformly, would be have to keep redrawing the map?
- No. Any map has a scale marked in the corner someplace: e.g., 1 inch = 1,000 miles. If the Earth kept getting larger, uniformly, we could keep the map and continuously change the scale. That is what we do in cosmology.
- We imagine a fixed 3D map of the universe, with distances marked in some arbitrary unit: I call them "notches," to make it clear that they have no fixed meaning in terms of any standard units of length. The "scale," or scale factor, is denoted by $a(t)$, where $a(t)$ is measured in meters (or light-year, or Mpc, or whatever) per notch. The relation is then

$$\ell_p(t) = a(t)\,\ell_c\ ,$$

where $\ell_p(t)$ is the **physical** distance, measured in meters (or light-years, etc.), and ℓ_c is the **coordinate** distance, measured in notches.



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Hubble's Law as a Consequence of Uniform Expansion

$$\ell_p(t) = a(t)\,\ell_c\ ,$$

So how fast does $\ell_p(t)$ change?

$$v = \frac{\mathrm{d}\ell_p}{\mathrm{d}t} = \frac{\mathrm{d}a}{\mathrm{d}t}\ell_c = \left[\frac{1}{a(t)}\,\frac{\mathrm{d}a(t)}{\mathrm{d}t}\right] a(t)\ell_c\ .$$

Note that this can be rewritten as

$$v = \frac{\mathrm{d}\ell_p}{\mathrm{d}t} = H\ell_p\ ,\qquad \text{where}\qquad H(t) = \frac{1}{a(t)}\,\frac{\mathrm{d}a(t)}{\mathrm{d}t}\ .$$

Light Rays in an Expanding Universe

- How do we describe light rays in the comoving coordinate system?
- The answer is simple:
  Light rays travel on a straight line, with a speed that would be measured by each local observer, as the light ray passes, at the standard value $c = 299{,}792{,}458$ m/s.
- Consider a light pulse moving along the x-axis. If the speed of light in m/s is c, and the number of meters per notch is $a(t)$, then the speed in notches per second is given by

$$\frac{dx}{dt} = \frac{c}{a(t)} .$$

- Justification: the above formula can be derived in general relativity by considering hypothetical point particles that travel at the speed of light, or by incorporating Maxwell's equations into general relativity.

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Importance of Comoving Coordinates

Any problem involving an expanding (homogeneous and isotropic) universe should be described in **comoving coordinates.**

Why?

Because the paths of light rays are simple in comoving coordinates.

If instead you tried to use coordinates that directly measure physical distances, the path of a light ray would be **complicated** for any trajectory other than a radial one.

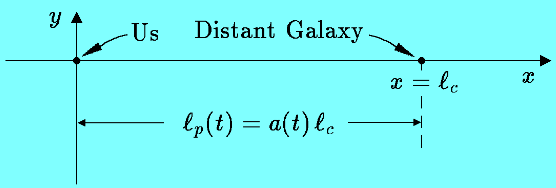
Cosmic Time and the Synchronization of Clocks

- In special relativity, clocks can be synchronized by sending time signals from a central clock. Other clocks, when using these time signals, use their distances from the central clock to take into account the light travel time.
- In an expanding universe, this does not work!
  1) Because the clocks are moving relative to each other, time dilation would have to be taken into account.
  2) Because the distances are changing with time, one can't know the distance until one knows the time, so the light travel time cannot be taken into account.

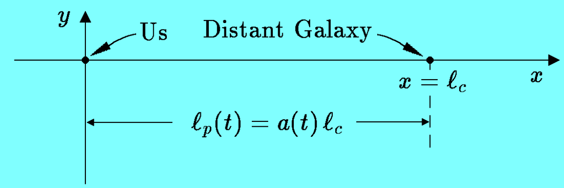
- In cosmology, we can imagine that "cosmic time" t is measured locally, on comoving clocks that tick in seconds defined by atomic standards. But they need to be synchronized somehow. Instead of using a central clock, one needs to find a clock that is available everywhere.
- In a simple, HOMOGENEOUS model of the universe, there are three possibilities:
  1) The Hubble expansion rate H. It can be measured anywhere, so can be used to define the $t = 0$ of cosmic time.
  2) The temperature T of the cosmic background radiation.
  3) If the universe starts with the scale factor $a = 0$, this starting time can be taken as $t = 0$.
- Will these three methods agree?
- Yes, they must, by the assumption of homogeneity. Homogeneity implies that the relation between H and T must be the same everywhere. So if you and I, in far away galaxies, measure the same value of H, we must also measure the same value of T.

Cosmological Redshift

- Use comoving coordinates!



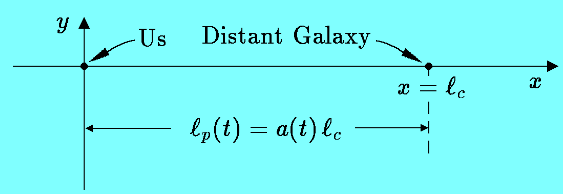
- Let Δt_S be the time between wave crests, as measured at the source.
- Since cosmic time t is measured on local clocks, Δt_S is the separation in cosmic time between the emission of crests.
- The physical wavelength at the source is $\lambda_S = c\Delta t_S$. When the 2nd crest is emitted, the first crest will be a physical distance λ_S from the source.
- When the 2nd crest is emitted, the first crest will be a coordinate distance $\Delta x = \lambda_S / a(t_S)$ from the source.



- When the 2nd crest is emitted, the first crest will be a coordinate distance $\Delta x = \lambda_S / a(t_S)$ from the source.
- As the first and second crests travel from source to us, they both travel at coordinate speed

$$\frac{dx}{dt} = \frac{c}{a(t)} \; .$$

- The speed depends on time, but not position: so the crests remain the same coordinate distance apart.



- When the crests reach us, at cosmic time t_O, they still have a coordinate separation $\Delta x = \lambda_S / a(t_S)$. The physical distance at the observer (us) is therefore

$$\lambda_O = a(t_O)\,\Delta x = \left[\frac{a(t_O)}{a(t_S)}\right] \lambda_S \; ,$$

so the wavelength is simply stretched with the expansion of the universe.

- The period of a light wave is proportional to its wavelength, so

$$1 + z \equiv \frac{\Delta t_O}{\Delta t_S} = \frac{\lambda_O}{\lambda_S} = \frac{a(t_O)}{a(t_S)} \; .$$