8.286 Lecture 5
September 21, 2022

THE DYNAMICS OF NEWTONIAN COSMOLOGY, PART 1

Isaac Newton to Richard Bentley, Letter 1
Newton on the Infinite Universe

As to your first query, it seems to me that if the matter of our sun and planets and all the matter of the universe were evenly scattered throughout all the heavens, and every particle had an innate gravity toward all the rest, and the whole space throughout which this matter was scattered was but finite, the matter on the outside of this space would, by its gravity, tend toward all the matter on the inside and, by consequence, fall down into the middle of the whole space and there compose one great spherical mass. But if the matter was evenly disposed throughout an infinite space, it could never convene into one mass; but some of it would convene into one mass and some into another, so as to make an infinite number of great masses, scattered at great distances from one to another throughout all that infinite space. And thus might the sun and fixed stars be formed, supposing the matter were of a lucid nature.

— December 10, 1692

But how the matter should divide itself into two sorts, and that part of it which is to compose a shining body should fall down into one mass and make a sun and the rest which is fit to compose an opaque body should coalesce, not into one great body, like the shining matter, but into many little ones; or if the sun at first were an opaque body like the planets or the planets lucid bodies like the sun, how he alone should be changed into a shining body whilst all they continue opaque, or all they be changed into opaque ones whilst he remains unchanged, I do not think explicable by mere natural causes, but am forced to ascribe it to the counsel and contrivance of a voluntary Agent.

— December 10, 1692

Web references: http://www.newtonproject.sussex.ac.uk/view/texts/normalized/THEM00254
http://books.google.com/books?id=8DkCAAAAQAAJ&pg=PA201

Isaac Newton to Richard Bentley, Letter 2
Newton on Infinities

But you argue, in the next paragraph of your letter, that every particle of matter in an infinite space has an infinite quantity of matter on all sides, and, by consequence, an infinite attraction every way, and therefore must rest in equilibrio, because all infinites are equal. Yet you suspect a paralogism in this argument; and I conceive the paralogism lies in the position, that all infinites are equal. The generality of mankind consider infinites no other ways than indefinitely; and in this sense they say all infinites are equal; though they would speak more truly if they should say, they are neither equal nor unequal, nor have any certain difference or proportion one to another. In this sense, therefore, no conclusions can be drawn from them about the equality, proportions, or differences of things; and they that attempt to do it usually fall into paralogisms.

— January 17, 1693

So, when men argue against the infinite divisibility of magnitude, by saying, that if an inch may be divided into an infinite number of parts, the sum of those parts will be an inch; and if a foot may be divided into an infinite number of parts, the sum of those parts must be a foot; and therefore, since all infinites are equal, those sums must be equal, that is, an inch equal to a foot. The falseness of the conclusion shews an error in the premises ; and the error lies in the position, that all infinites are equal.

— January 17, 1693

Web references: http://www.newtonproject.sussex.ac.uk/view/texts/normalized/THEM00255
http://books.google.com/books?id=8DkCAAAAQAAJ&pg=PA201

Can a Uniform Infinite Distribution of Mass Be Stable?

Gauss's Law of Gravity:

$$\vec{g} = -\frac{GM}{r^2}\hat{r} \quad \Longrightarrow \quad \boxed{\oint \vec{g} \cdot d\vec{a} = -4\pi G M_{\text{enclosed}}}$$

Poisson's Equation:

$$\boxed{\nabla^2 \phi = 4\pi G \rho \ ,} \quad \text{where } \vec{g} = -\vec{\nabla}\phi \ .$$

where ρ is the mass density, $\vec{\nabla}\phi$ is the gradient of ϕ:

$$\vec{\nabla}\phi \equiv \hat{\imath}\,\frac{\partial \phi}{\partial x} + \hat{\jmath}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z} \ ,$$

and $\nabla^2\phi$ is the Laplacian of ϕ:

$$\nabla^2\phi \equiv \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \ .$$

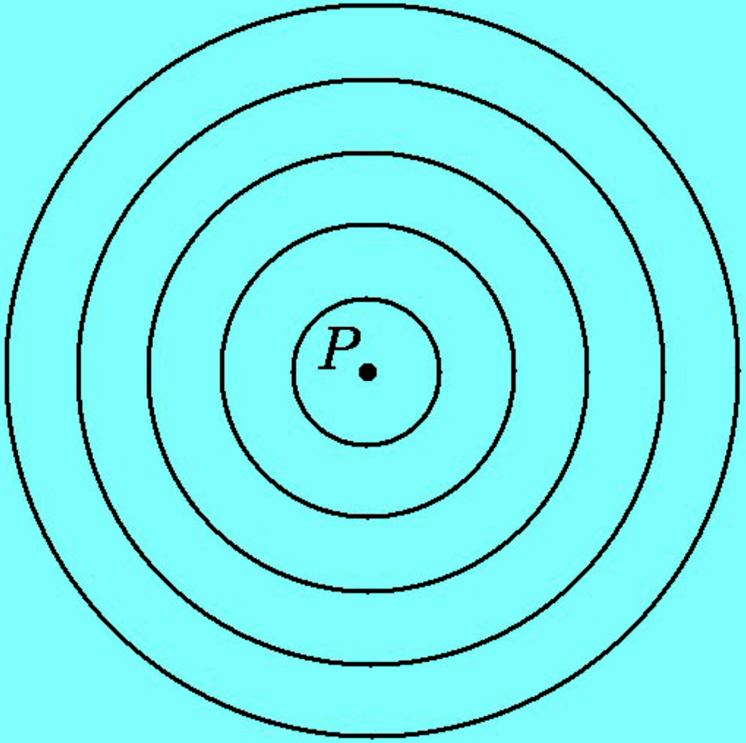
and

Attempting to Integrate the Gravitational Force

- Problem: the integral is not *absolutely* convergent, but only *conditionally* convergent.

Attempting to Integrate the Gravitational Force

- Problem: the integral is not *absolutely* convergent, but only *conditionally* convergent.
- Discussion of absolute and conditional convergence (on black-board).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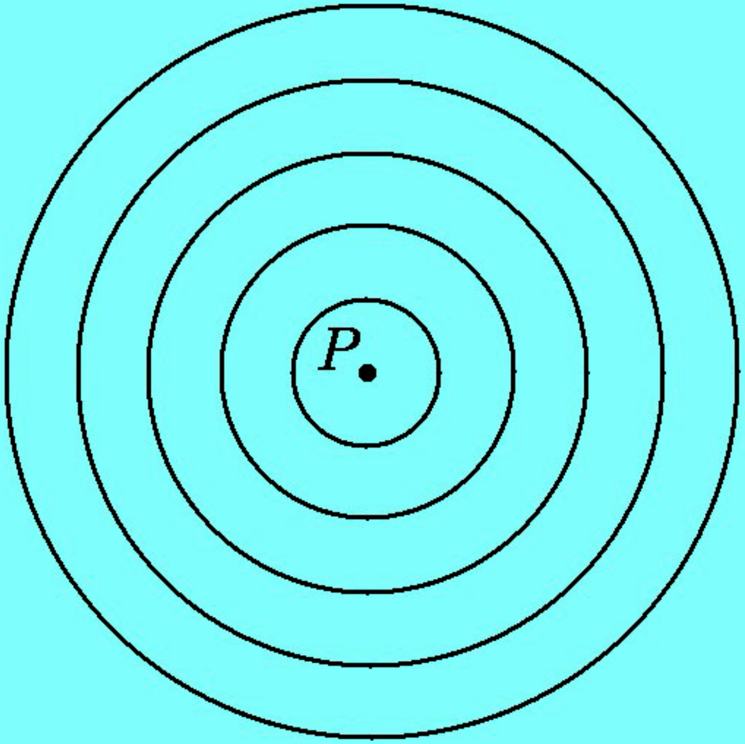
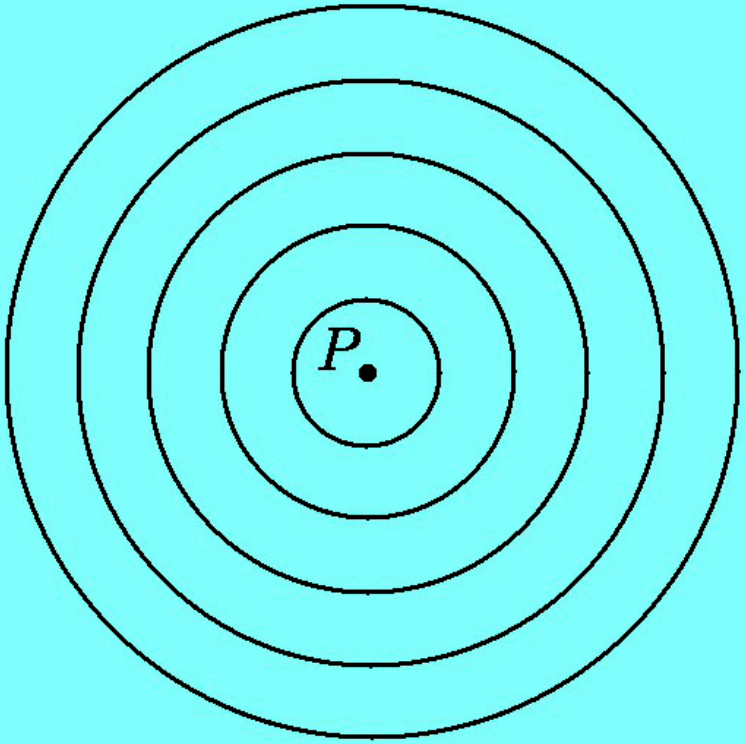
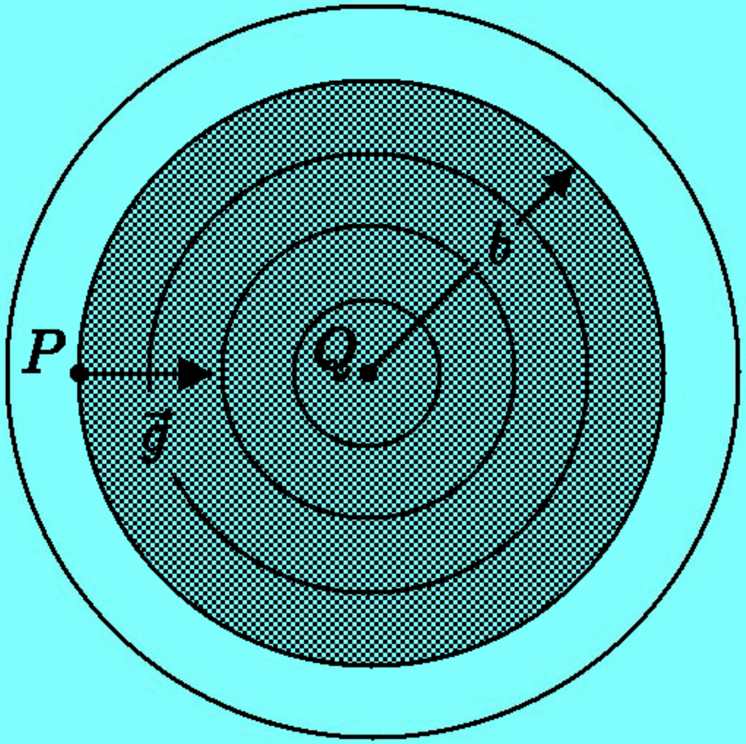
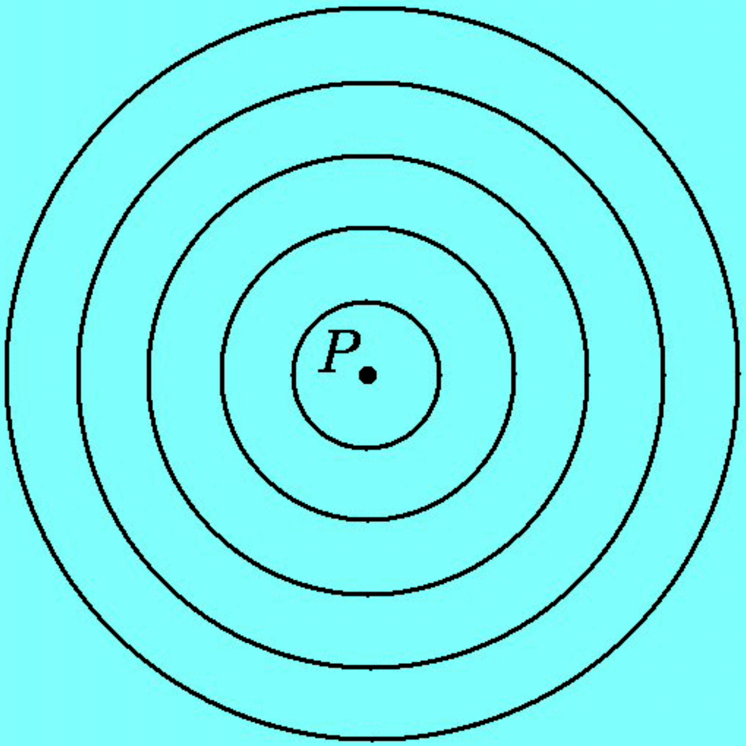


$$\vec{g} = 0$$

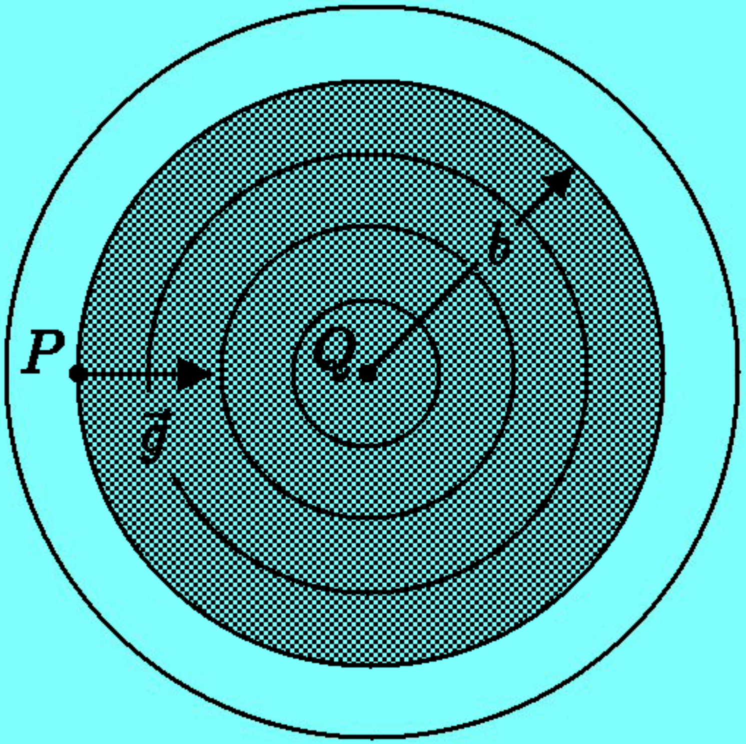




$\vec{g} = 0$



$\vec{g} = 0$



$$\vec{g} = \frac{GM}{b^2}\hat{e}_{QP}$$

But What About Symmetry?

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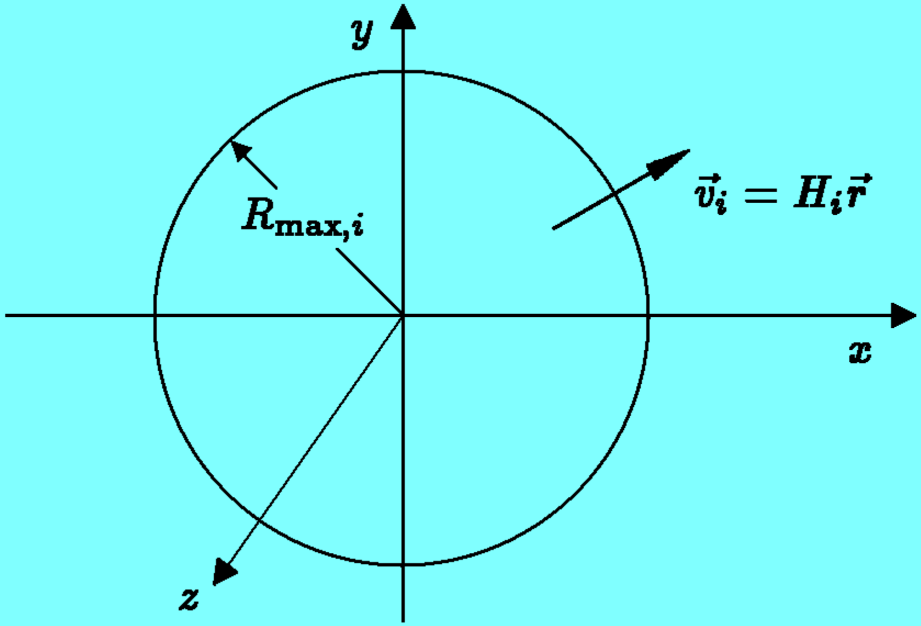
- Newton argued that there could be no acceleration, because there is no preferred direction for it to point.
- Complication: acceleration is measured relative to an inertial frame, which Newton defined as the frame of the "fixed stars". But if the universe collapses, then there are no fixed stars.
- In the absence of an inertial frame, all accelerations, like velocities, are relative.
- When all accelerations are relative, any observer can consider herself to be non-accelerating. She would then see all other objects accelerating radially toward herself. Like the velocities of Hubble expansion, this picture looks like it has a unique center, but really it is homogenous.

Mathematical Model of a Uniformly Expanding Universe

- Desired properties: homogeneity, isotropy, and Hubble's law.
- The model should be finite, to avoid the conditional convergence problems discussed last time. At the end we will take the limit as the size approaches infinity.
- Newtonian dynamics: we choose the initial conditions, and then Newton's laws of motion will determine how it will evolve.
- To impose isotropy, we model the initial state as a solid sphere, of some radius $R_{\max,i}$.
- To impose homogeneity, we take the initial mass density to be constant, ρ_i. The matter is treated as a dust, that can thin as the universe expands. By dust, we mean a system of very small closely spaced particles, that can be treated as a continuous fluid, with negligible pressure.
- We take the initial velocities according to Hubble's law, with some initial expansion rate H_i

Epochs of Cosmic History

Mathematical Model of a Uniformly Expanding Universe



$$t_i \equiv \text{time of initial picture}$$

$$R_{\max,i} \equiv \text{initial maximum radius}$$

$$\rho_i \equiv \text{initial mass density}$$

$$\vec{v}_i = H_i\vec{r}\ .$$

Description of Evolution

- As the model universe evolves, the spherical symmetry will be preserved: each gas particle will continue on a radial trajectory, since there are no forces that might pull it tangentially.

- Spherical symmetry $\Longrightarrow$ all particles that start at the same initial radius will behave the same way. So, a particle that begins at radius r_i will be found at a later time t at some radius

$$r = r(r_i, t) \ .$$

- Our goal is to figure out what determines $r(r_i, t)$

- The only relevant force is gravity. Gravity and electromagnetism are the only (known) long-range forces. The universe appears to be electrically neutral, so long-range electric forces are not present.

Reminder: the Gravitational Field of a Shell of Matter

- For points outside the shell, the gravitational force is the same as if the total mass of the shell were concentrated at the center.
- For points inside the shell, the gravitational field is **zero.**
- Newton figured this out by integration. For us, Gauss's law makes it obvious.

Shell Crossings?

Can shells cross? I.e., can two shells that start at different r_i ever cross each other?

The answer is no, but we don't know that when we start.

But we do know that Hubble's law implies that any two shells are initially moving apart. Therefore there must be at least some interval before any shell crossings can happen.

We will write equations that are valid assuming no shell crossings.

These equations will be valid until any possible shell crossing.

If there was a shell crossing, these equations would have to show two shells becoming arbitrarily close.

We will find, however, that the equations imply uniform expansion, so no shell crossings ever happen in this system.

Equations of Motion

- Newtonian gravity of a shell:

  Inside: $\vec{g} = 0$.

  Outside: Same as point mass at center, with same M.

- $r(r_i, t) \equiv$ radius at t of shell initially at r_i.

- Let $M(r_i) \equiv$ mass inside r_i-shell $= \frac{4\pi}{3} r_i^3 \rho_i$ at all times.

- Pressure? When a gas with pressure $p > 0$ expands, it pushes on its surroundings and loses energy. Relativistically, energy = mass (times c^2). By assuming that $M(r_i)$ is constant, we are assuming that $p \simeq 0$.

Equations:

☆ For particles at radius r,

$$\vec{g} = -\frac{GM(r_i)}{r^2}\,\hat{r}\ ,$$

where

$$M(r_i) = \frac{4\pi}{3} r_i^3 \rho_i\ .$$

Since $\vec{g}$ is the acceleration,

$$\ddot{r} = -\frac{GM(r_i)}{r^2} = -\frac{4\pi}{3}\frac{G r_i^3 \rho_i}{r^2}\ , \quad \text{where } r \equiv r(r_i, t),$$

where an overdot indicates a derivative with respect to t.

$$\ddot{r} = -\frac{GM(r_i)}{r^2} = -\frac{4\pi}{3}\frac{Gr_i^3\rho_i}{r^2} \;, \text{ where } r \equiv r(r_i, t),$$

☆ For a second order equation like this, the solution is uniquely determined if the initial value of r and $\dot{r}$ are specified:

$$r(r_i, t_i) = r_i \;,$$

and, by the Hubble law initial condition $\vec{v}_i = H_i \vec{r}_i$,

$$\dot{r}(r_i, t_i) = H_i r_i \;.$$

Miraculous Scaling Relations

$$\ddot{r} = -\frac{4\pi}{3}\frac{G r_i^3 \rho_i}{r^2} \; , \quad r(r_i, t_i) = r_i \; , \quad \dot{r}(r_i, t_i) = H_i r_i \; .$$

☆ Suppose we define

$$u(r_i, t) \equiv \frac{r(r_i, t)}{r_i} \; .$$

Then

$$\ddot{u} = \frac{\ddot{r}}{r_i} = -\frac{4\pi}{3}\frac{G\rho_i}{u^2} \; .$$

There is no r_i-dependence. This "miracle" depended on gravity being a $1/r^2$ force.

$$\ddot{r} = -\frac{4\pi}{3}\frac{G r_i^3 \rho_i}{r^2} \ , \quad r(r_i, t_i) = r_i \ , \quad \dot{r}(r_i, t_i) = H_i r_i \ .$$

$$u(r_i, t) \equiv \frac{r(r_i, t)}{r_i} \quad \Longrightarrow \quad \ddot{u} = -\frac{4\pi}{3}\frac{G\rho_i}{u^2} \ .$$

What about the initial conditions for $u(r_i, t)$?

$$u(r_i, t_i) = \frac{r(r_i, t_i)}{r_i} = 1 \ , \quad \dot{u}(r_i, t_i) = \frac{\dot{r}(r_i, t_i)}{r_i} = H_i \ .$$

Since the differential equation and the intial conditions determine $u(r_i, t)$, it does not depend on r_i. We can rename it

$$u(r_i, t) \equiv a(t) \ , \qquad \text{so} \qquad r(r_i, t) = a(t)\, r_i \ .$$

This describes uniform expansion by a scale factor $a(t)$.