8.286 Lecture 6
September 26, 2022

THE DYNAMICS OF NEWTONIAN COSMOLOGY, PART 2

Can a Uniform Infinite Distribution of Mass Be Stable?

- Newton (1692): Yes.
- Gauss's Law of Gravity (Lagrange, 1773, Gauss, 1835): No, but not noticed by anybody.

$$\vec{g} = -\frac{GM}{r^2}\hat{r} \quad \Longrightarrow \quad \boxed{\oint \vec{g} \cdot d\vec{a} = -4\pi G M_{\text{enclosed}}}$$

- Poisson's Equation (1829): No, but not noticed by anyone.

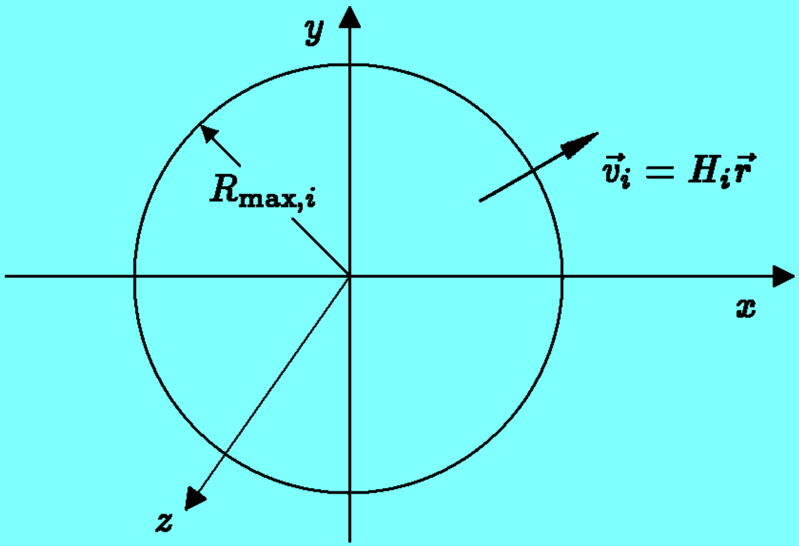
$$\boxed{\nabla^2 \phi = 4\pi G \rho \ ,} \quad \text{where } \vec{g} = -\vec{\nabla}\phi \ ,$$

where ρ is the mass density.

- ☆ Einstein (1917): No.

  Einstein discovered that a uniform infinite distribution of mass is unstable in Newtonian physics, and also in the original form of General Relativity. (Einstein was, however, still convinced that the universe must be static. He found that he could add a new term to the equations that describe how matter creates a gravitational field (the Einstein field equations), which he called the *cosmological term*, which described a repulsive force which could be adjusted to just balance the attractive force, allowing a static universe. Einstein endorsed this model until Hubble discovered in 1929 that the universe was expanding.

Mathematical Model



$$t_i \equiv \text{time of initial picture}$$

$$R_{\max,i} \equiv \text{initial maximum radius}$$

$$\rho_i \equiv \text{initial mass density}$$

$$\vec{v}_i = H_i\vec{r}\ .$$

Miraculous Scaling Relations

$$\ddot{r} = -\frac{4\pi}{3}\frac{G r_i^3 \rho_i}{r^2} \ , \quad r(r_i, t_i) = r_i \ , \quad \dot{r}(r_i, t_i) = H_i r_i \ .$$

☆ Suppose we define

$$u(r_i, t) \equiv \frac{r(r_i, t)}{r_i} \ .$$

Then

$$\ddot{u} = \frac{\ddot{r}}{r_i} = -\frac{4\pi}{3}\frac{G\rho_i}{u^2} \ .$$

There is no r_i-dependence. This "miracle" depended on gravity being a $1/r^2$ force.

Review from previous lecture

$$\ddot{r} = -\frac{4\pi}{3}\frac{G r_i^3 \rho_i}{r^2}\ , \quad r(r_i, t_i) = r_i\ , \quad \dot{r}(r_i, t_i) = H_i r_i\ .$$

$$u(r_i, t) \equiv \frac{r(r_i, t)}{r_i} \quad \Longrightarrow \quad \ddot{u} = -\frac{4\pi}{3}\frac{G\rho_i}{u^2}\ .$$

What about the initial conditions for $u(r_i, t)$?

$$u(r_i, t_i) = \frac{r(r_i, t_i)}{r_i} = 1\ , \quad \dot{u}(r_i, t_i) = \frac{\dot{r}(r_i, t_i)}{r_i} = H_i\ .$$

Since the differential equation and the intial conditions determine $u(r_i, t)$, it does not depend on r_i. We can rename it

$$u(r_i, t) \equiv a(t)\ , \qquad \text{so} \qquad r(r_i, t) = a(t)\, r_i\ .$$

This describes uniform expansion by a scale factor $a(t)$.

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Time Dependence of $\rho(t)$

We know how the mass density depends on time, because we assumed that $M(r_i)$ — the total mass contained inside a shell of particles whose initial radius was r_i — does not change with time. The radius of the shell at time t is $a(t)r_i$. The mass density is just the mass divided by the volume,

$$\rho(t) = \frac{M(r_i)}{\frac{4\pi}{3}a^3(t)r_i^3} = \frac{\frac{4\pi}{3}r_i^3\rho_i}{\frac{4\pi}{3}a^3(t)r_i^3} = \frac{\rho_i}{a^3(t)} \ .$$

So

$$\ddot{u} = -\frac{4\pi}{3}\frac{G\rho_i}{u^2} \quad \Longrightarrow \quad \ddot{a} = -\frac{4\pi}{3}\frac{G\rho_i}{a^2} \ .$$

$$\Longrightarrow \quad \ddot{a} = -\frac{4\pi}{3}G\rho(t)\,a(t) \ . \qquad \text{Friedmann equation.}$$

Nothing Depends on $R_{\mathrm{max},i}$

- An observer living in this model universe would see uniform expansion all around herself, and would only be aware of the boundary at R_{max} if she was close enough to the boundary to see it.
- Thus, we can take the limit $R_{\mathrm{max},i} \to \infty$ without doing anything, since nothing of interest depends on $R_{\mathrm{max},i}$.

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A Conservation Law

☆ The equation for $\ddot{a}$ has the same form as an equation for the motion of a particle with a time-independent potential energy function. So, there is a conservation law:

$$\ddot{a} = -\frac{4\pi}{3}\frac{G\rho_i}{a^2} \quad \Longrightarrow \quad \dot{a}\left\{\ddot{a} + \frac{4\pi}{3}\frac{G\rho_i}{a^2}\right\} = 0 \quad \Longrightarrow \quad \frac{dE}{dt} = 0 ,$$

where

$$E = \frac{1}{2}\dot{a}^2 - \frac{4\pi}{3}\frac{G\rho_i}{a} .$$

Summary: Equations

Want: $r(r_i, t) \equiv$ radius at t of shell initially at r_i

Find: $r(r_i, t) = a(t) r_i$, where

$$\text{Friedmann Equations} \begin{cases} \ddot{a} = -\dfrac{4\pi}{3} G\rho(t) a \\[2ex] H^2 = \left(\dfrac{\dot{a}}{a}\right)^2 = \dfrac{8\pi}{3} G\rho - \dfrac{kc^2}{a^2} \quad \text{(Friedmann Eq.)} \end{cases}$$

and

$$\rho(t) \propto \frac{1}{a^3(t)} \;, \text{ or } \rho(t) = \left[\frac{a(t_1)}{a(t)}\right]^3 \rho(t_1) \text{ for any } t_1 .$$

☆ Note that t_i no longer plays any role. It does not appear on this slide!

The Return of the 'Notch'

- Definition: $r(r_i, t) = a(t) r_i$.
- In the previous derivation, r_i was the initial radius of some particle, measured in meters. But when we finished, r_i was being used only as a coordinate to label shells, where the shell corresponding to $r_i = 1$ had a radius of one meter only at time t_i.
- But t_i no longer appears, and will not be mentioned again! So, the connection between the numerical value of r_i and the length of a meter has disappeared from the formalism.
- Bottom line: r_i is the radial coordinate in a comoving coordinate system, measured in units that have no particular meaning. I will refer to the units of r_i as "notches," but you should be aware that the term is not standard.

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Conventions for the Notch

Us: For us, the notch is an arbitrary unit that we use to mark off intervals on the comoving coordinate system. We are free to use a different definition every time we use the notch.

Ryden: $a(t_0) = 1$ (where $t_0 =$ now). (In our language, Ryden's convention is $a(t_0) = 1$ m/notch.)

Many Other Books: if $k \neq 0$, then $k = \pm 1$.

In our language, this means $k = \pm 1/\text{notch}^2$. To see the units of k, recall that the Friedmann equation is

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{a^2} \ .$$

We will use $[x]$ to mean the units of x, and we will use T and L to denote the units of time and length, respectively. The units of the left-hand side are $1/T^2$, with the units of a canceling. So

$$[k] = \frac{1}{T^2}\left[\frac{a}{c}\right]^2 = \frac{1}{T^2}\left[\frac{L/\text{notch}}{L/T}\right]^2 = \frac{1}{\text{notch}^2} \ .$$

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Types of Solutions

$$\dot{a}^2 = \frac{8\pi G}{3} \frac{\rho(t_1)a^3(t_1)}{a(t)} - kc^2 \quad \text{(for any } t_1\text{)} \ .$$

For intuition, remember that $k \propto -E$, where E is a measure of the energy of the system.

Types of Solutions:

1) $k < 0$ $(E > 0)$: unbound system. $\dot{a}^2 > (-kc^2) > 0$, so the universe expands forever. **Open Universe.**

2) $k > 0$ $(E < 0)$: bound system. $\dot{a}^2 \geq 0 \quad \Longrightarrow$

$$a_{\max} = \frac{8\pi G}{3} \frac{\rho(t_1)a^3(t_1)}{kc^2} \ .$$

Universe reaches maximum size and then contracts to a Big Crunch. **Closed Universe.**

3) $k = 0$ ($E = 0$): critical mass density.

$$H^2 = \frac{8\pi G}{3}\rho - \underbrace{\frac{kc^2}{a^2}}_{=0} \quad \Longrightarrow \quad \boxed{\rho \equiv \rho_c = \frac{3H^2}{8\pi G}\ .}$$

Flat Universe.

Summary: $\rho > \rho_c \Longleftrightarrow$ closed, $\rho < \rho_c \Longleftrightarrow$ open, $\rho = \rho_c \Longleftrightarrow$ flat.

Numerical value: For $H = 68$ km-s^{-1}-Mpc^{-1} (Planck 2015 plus other experiments),

$$\rho_c = 8.7 \times 10^{-27}\ \text{kg/m}^3 = 8.7 \times 10^{-30}\ \text{g/cm}^3$$

$$\approx 5 \text{ proton masses per m}^3.$$

Definition: $\Omega \equiv \dfrac{\rho}{\rho_c}$.