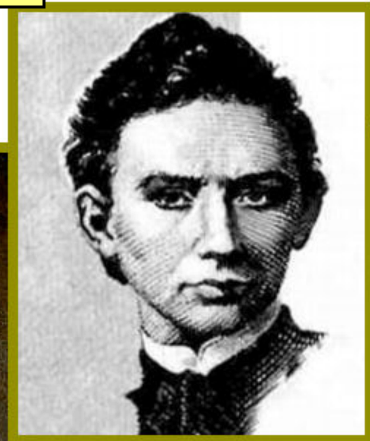
8.286 Class 10
October 12, 2022

INTRODUCTION TO NON-EUCLIDEAN SPACES, PART 2

Review from the previous lecture

~1750-1850

Gauss

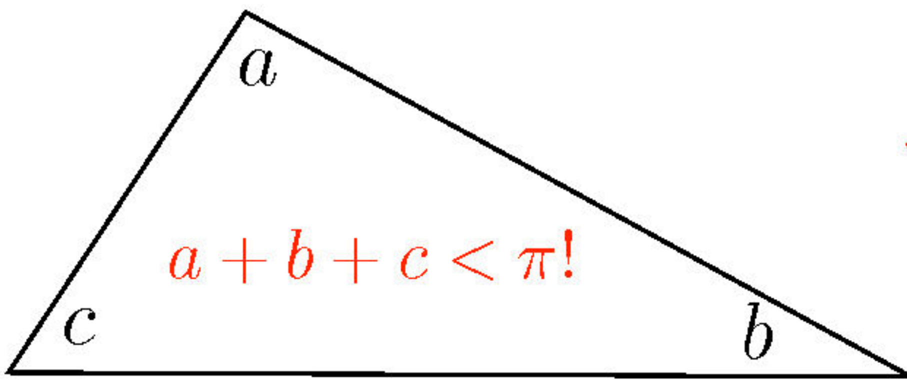


Bolyai

Lobachevski

- infinite
- constant negative curvature

~~5th Postulate~~



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Review from the previous lecture

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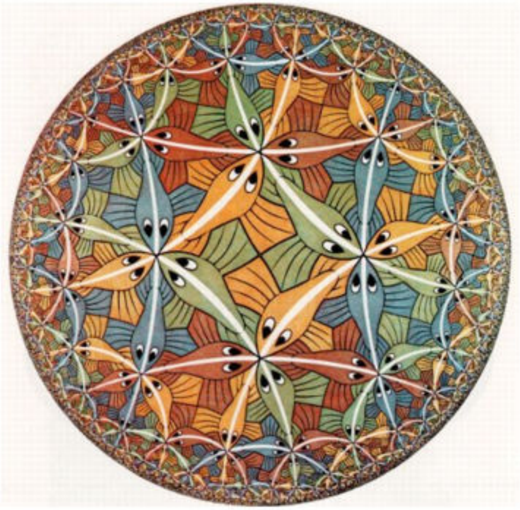
GBL geometry with Klein

1. constant negative curvature
2. infinite
3. ~~5th postulate~~



(x_1, y_1)

(x_2, y_2)

$$x^2 + y^2 < 1$$

$$d(1,2) = a \cosh^{-1}\left[\frac{1 - x_1 x_2 - y_1 y_2}{\sqrt{1 - x_1^2 - y_1^2}\sqrt{1 - x_2^2 - y_2^2}}\right]$$

Note: no global embedding in 3D Euclidean space possible

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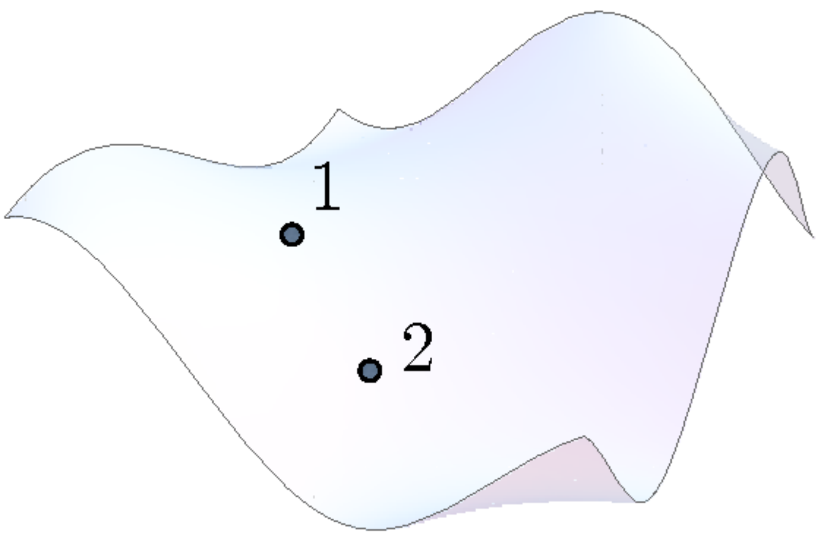
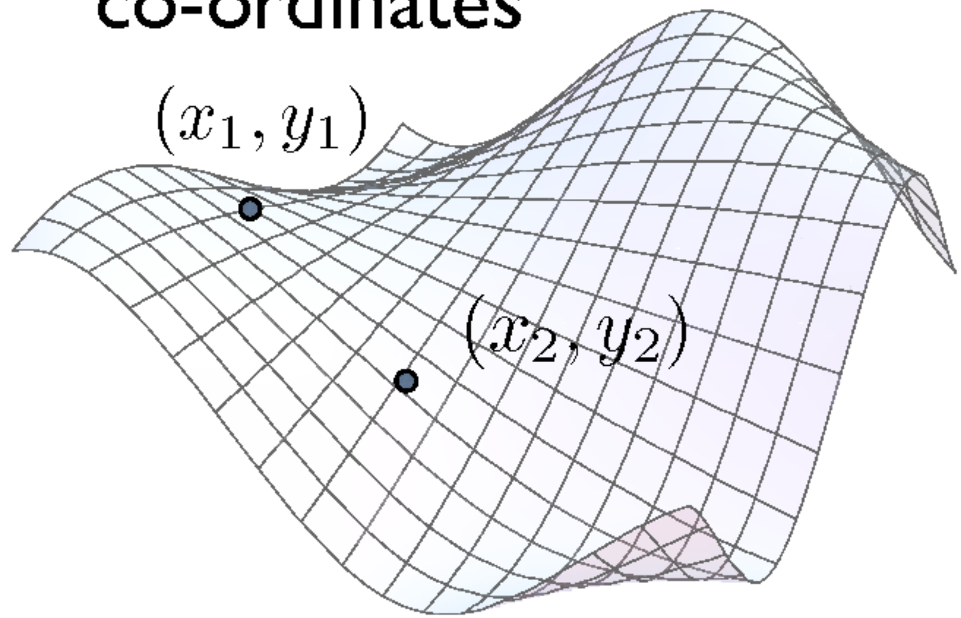
Geometry (after Klein)



co-ordinates



distance function

$$d[(x_1, y_1), (x_2, y_2)]$$

Review from the previous lecture

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Intrinsic Geometry

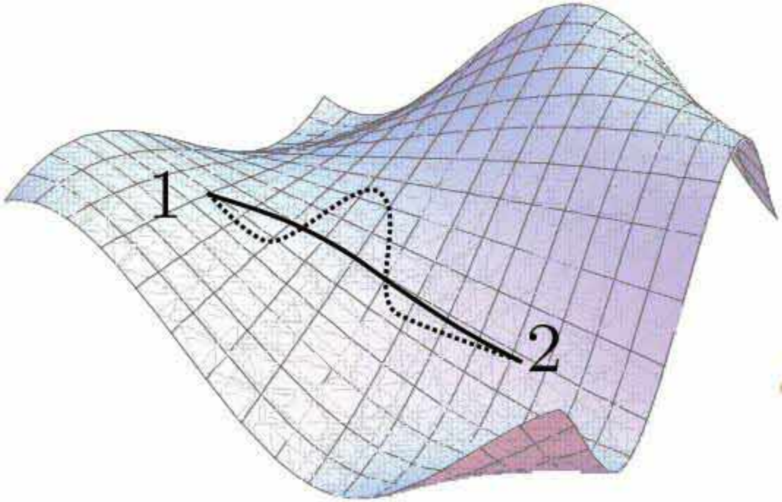

Image of ant from: http://www.termiteterry.com/

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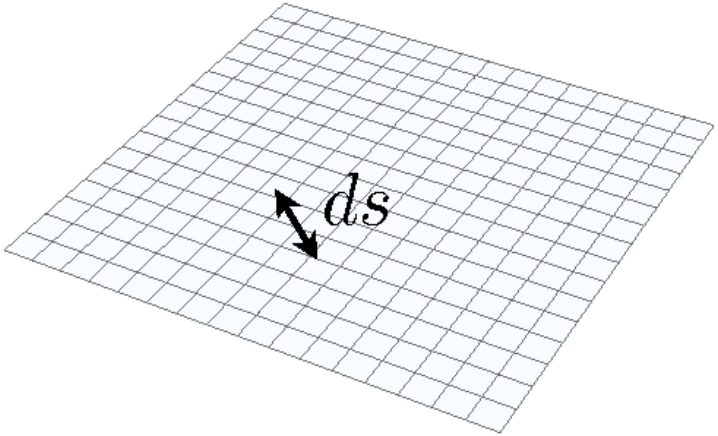
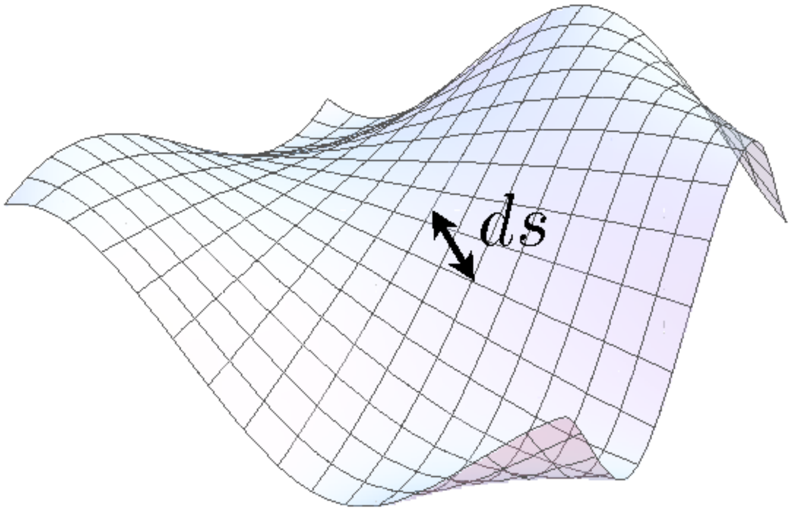
tiny distances



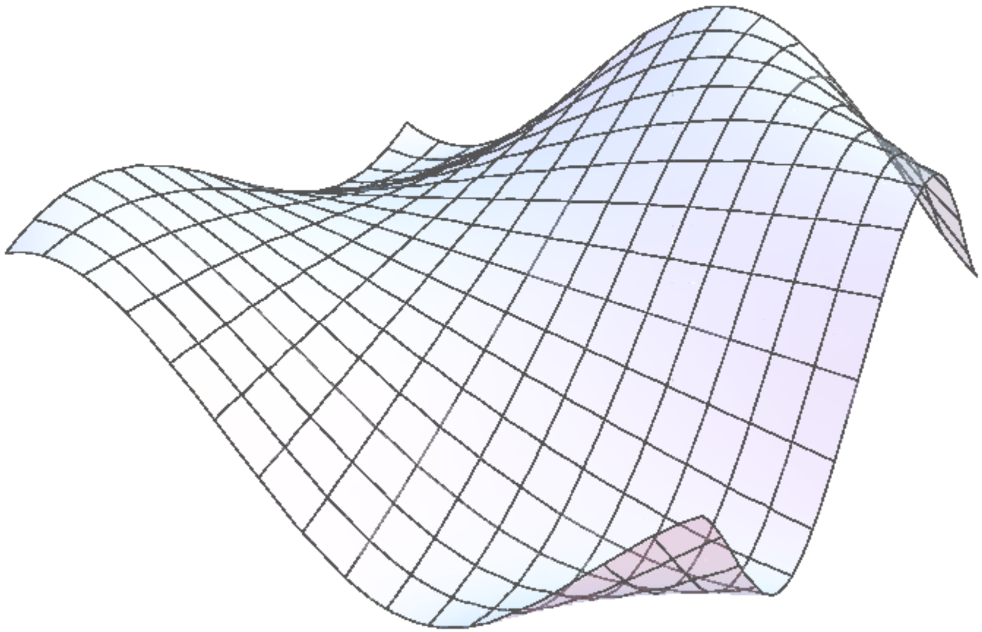


$$ds^2 = dx^2 + dy^2$$

$$ds^2 = g_{xx}(x,y)dx^2 + 2g_{xy}(x,y)dxdy + g_{yy}(x,y)dy^2$$

quadratic form

Image:www.easternct.edu/career/webresources.htm



$$ds^2 = g_{xx}(x,y)dx^2 + 2g_{xy}(x,y)dxdy + g_{yy}(x,y)dy^2$$

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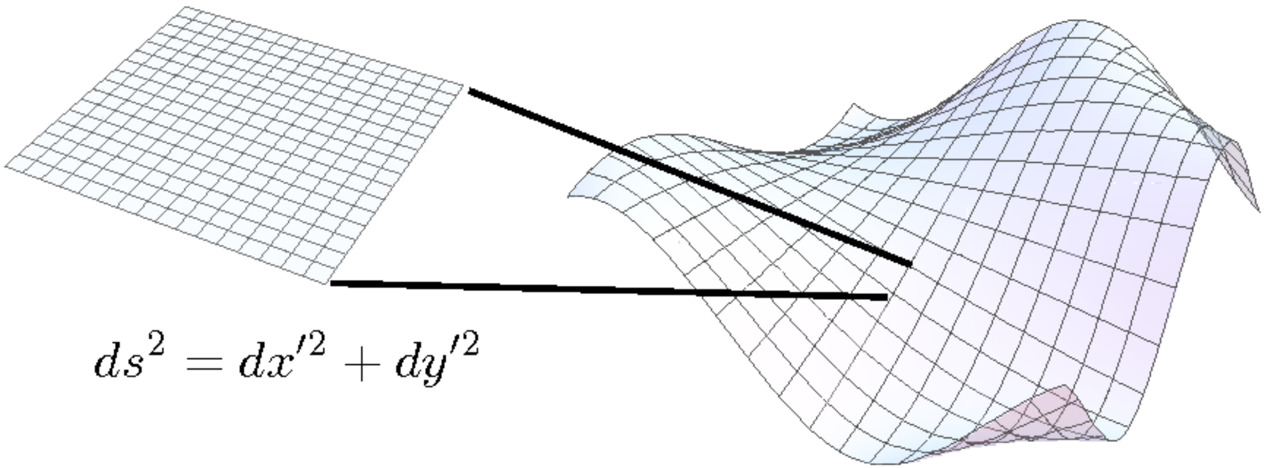
locally Euclidean



$$ds^2 = dx'^2 + dy'^2$$

$$ds^2 = g_{xx}(x,y)dx^2 + 2g_{xy}(x,y)dxdy + g_{yy}(x,y)dy^2$$

$$g_{xx}g_{yy} - g_{xy}^2 > 0$$

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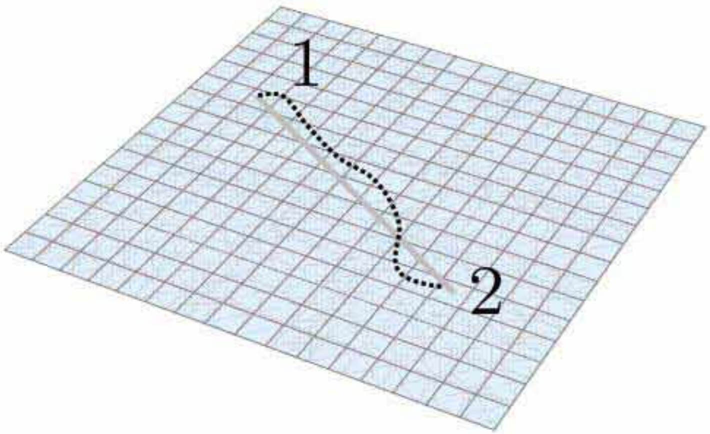
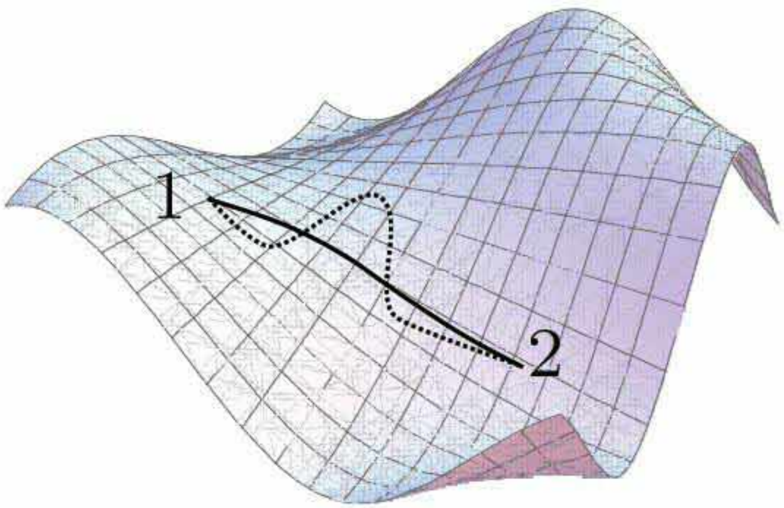
geodesics

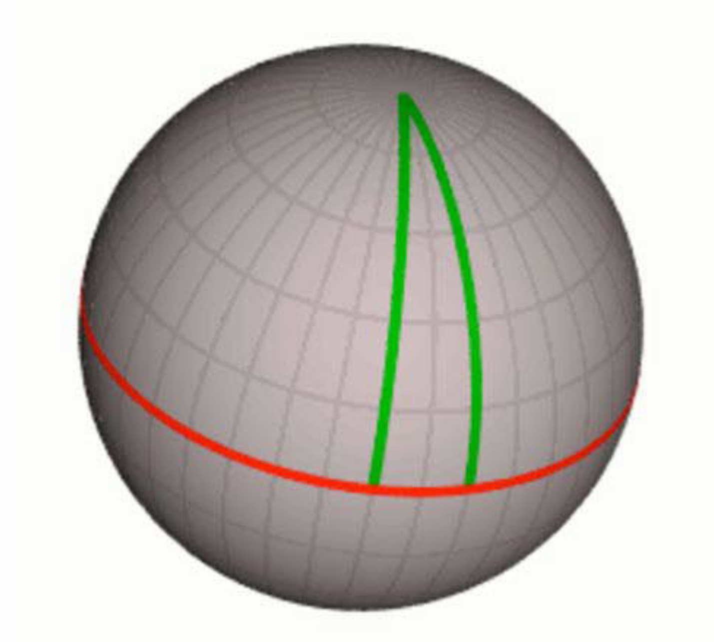


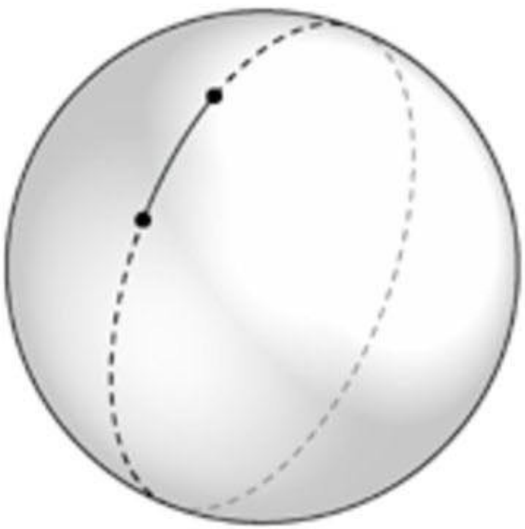


a geodesic is a curve along which the distance between two given points is extremised.

note: important!

sphere: geodesics





longitudes: yes
latitudes: no

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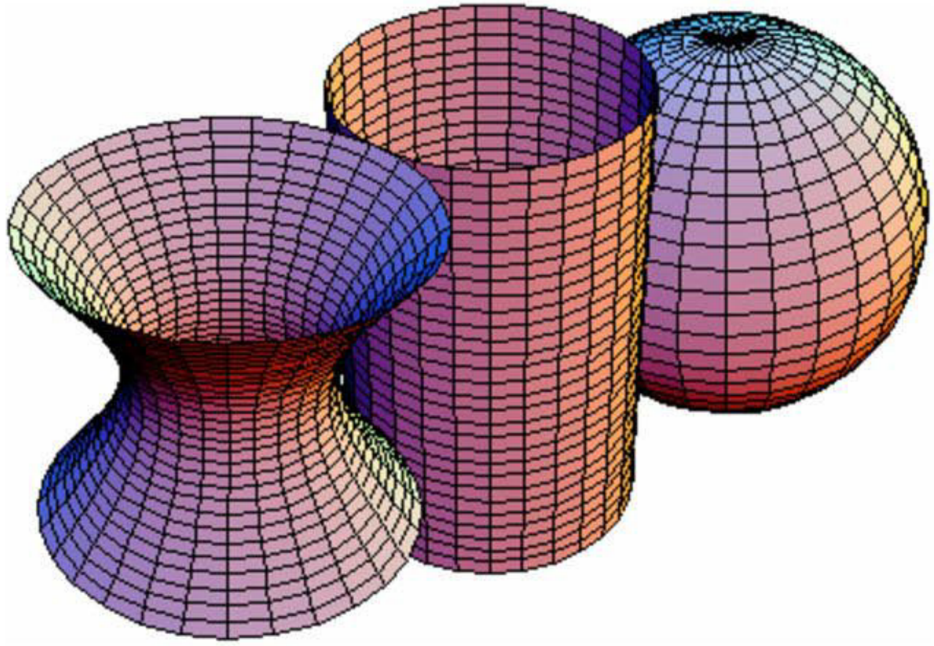
curved ?



Above image:http://en.wikipedia.org/wiki/Gaussian_curvature

$$ds^2 = dx^2 + dy^2 \qquad ds^2 = g_{xx}(x,y)dx^2 + 2g_{xy}(x,y)dxdy + g_{yy}(x,y)dy^2$$

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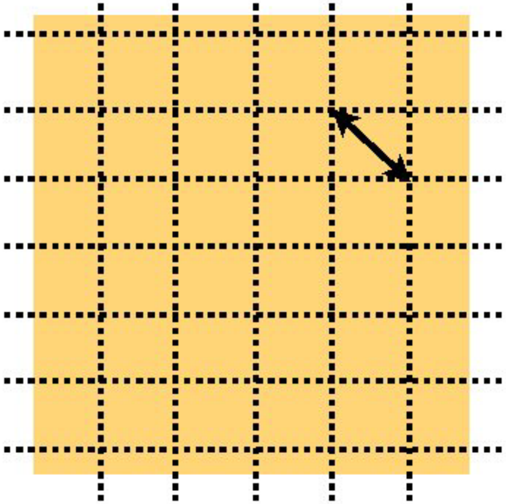
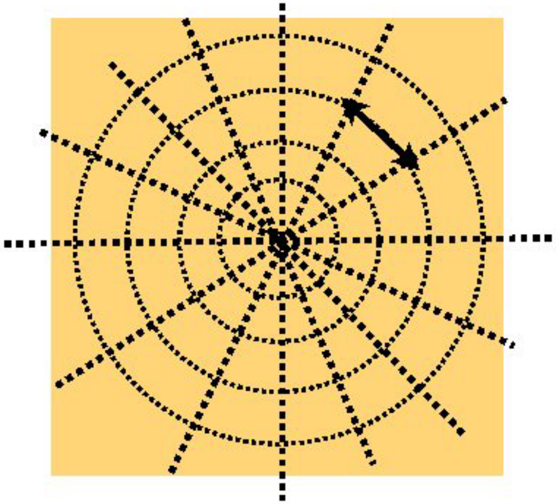
metric and co-ordinates

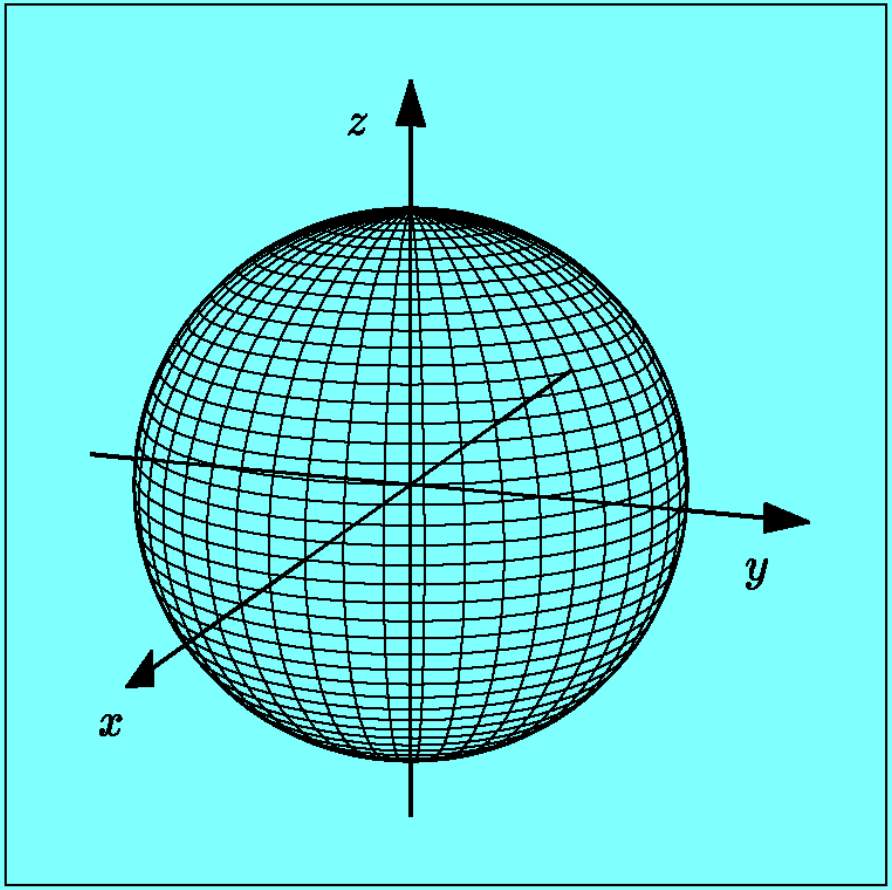


$$ds^2 = dx^2 + dy^2$$



$$ds^2 = dr^2 + r^2 d\theta^2$$

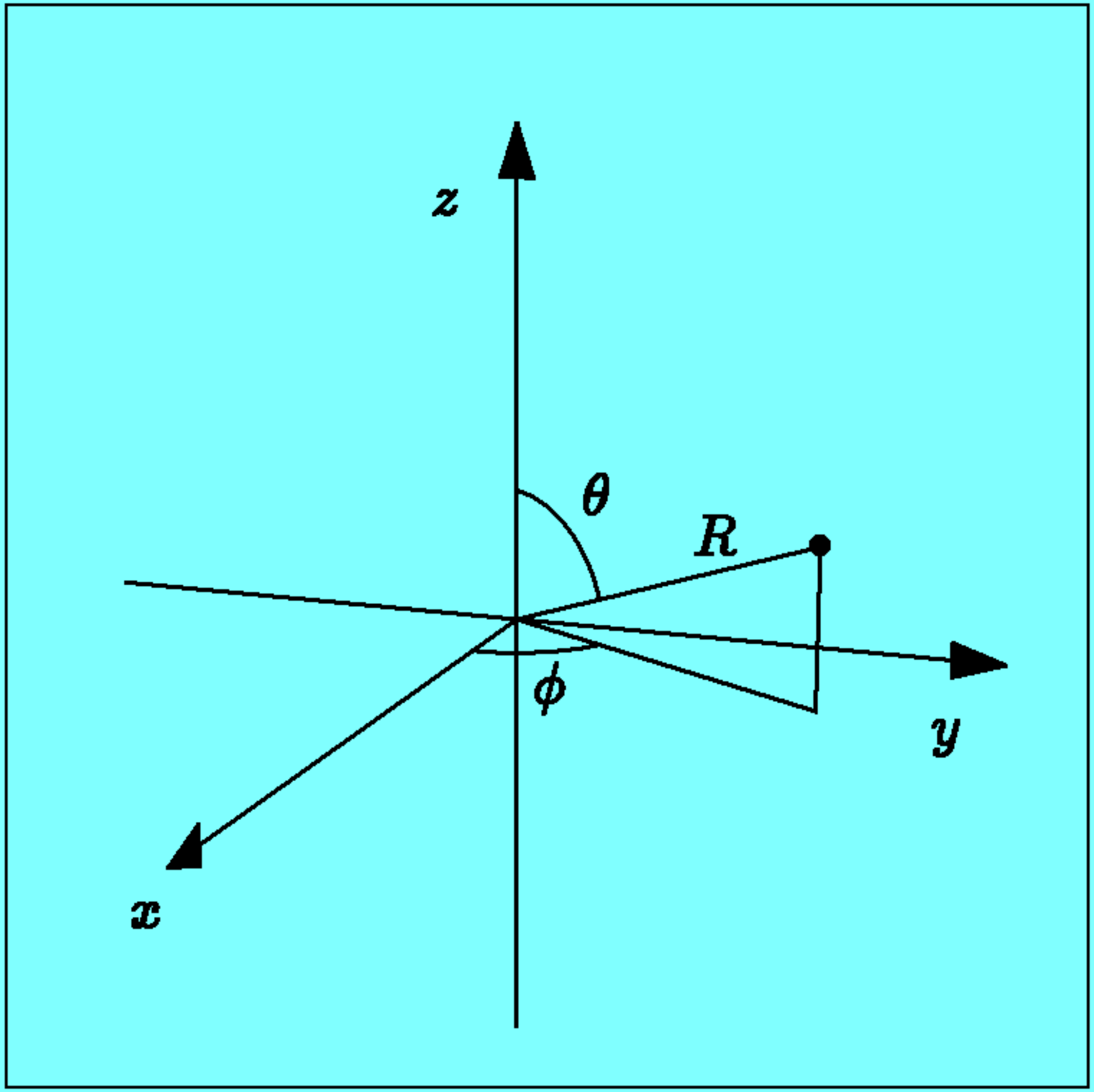
Non-Euclidean Geometry: The Surface of a Sphere



$$x^2 + y^2 + z^2 = R^2 \ .$$

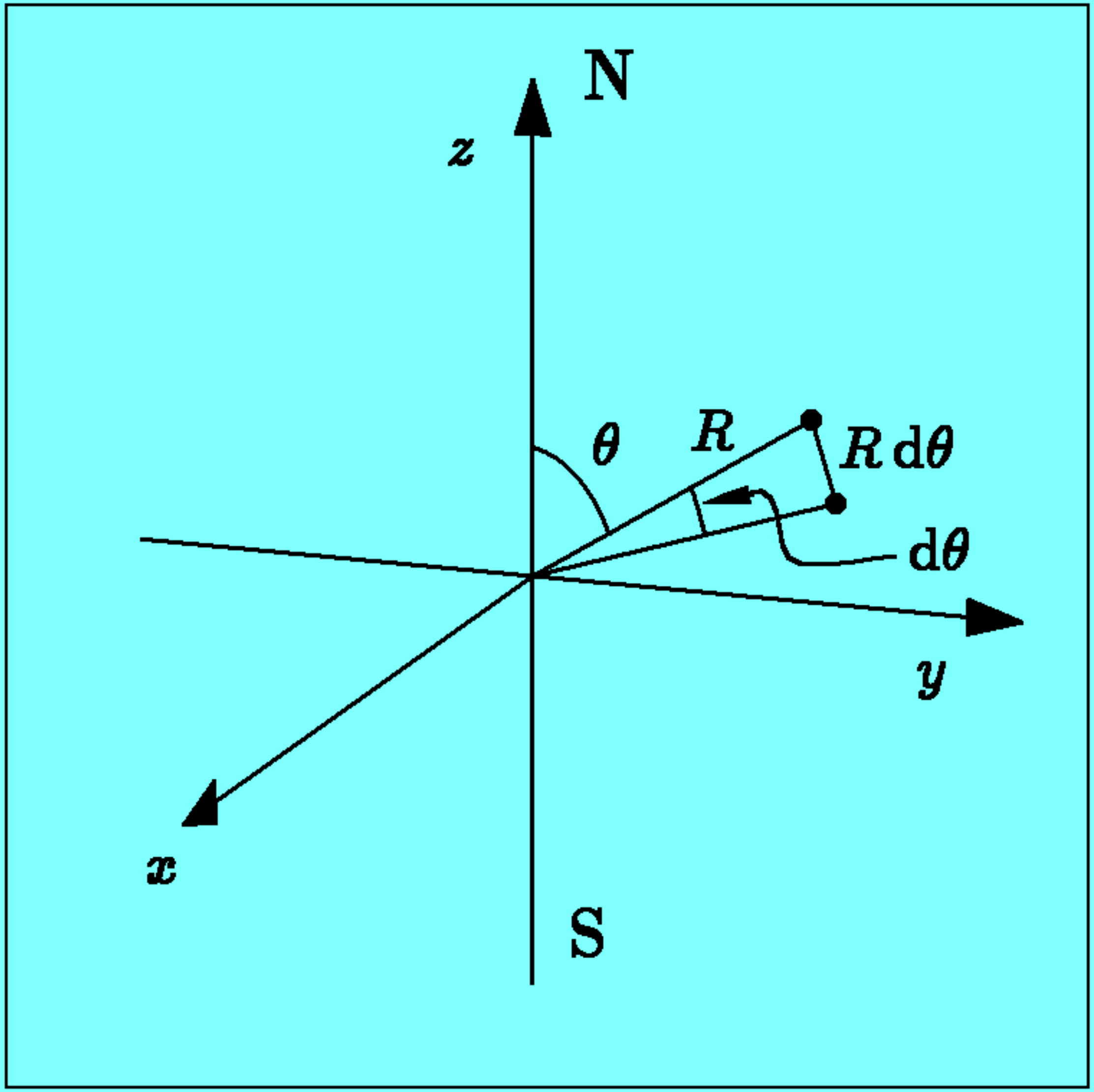
Polar Coordinates:

$$x = R \sin\theta \cos\phi$$
$$y = R \sin\theta \sin\phi$$
$$z = R \cos\theta \ ,$$



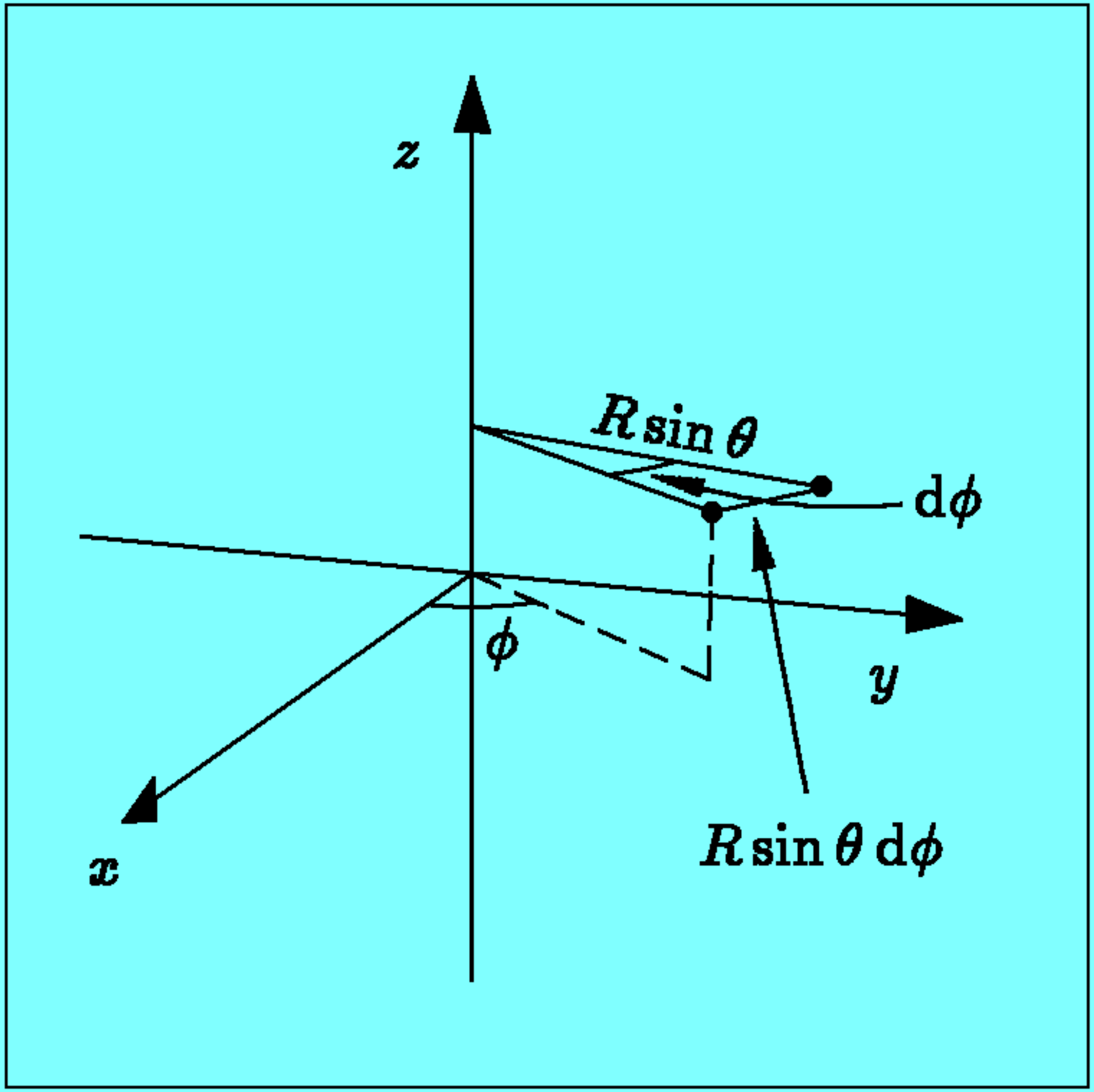
Varying θ:

$$ds = R\,d\theta$$



Varying ϕ:

$$ds = R\sin\theta\,d\phi$$



Varying θ and ϕ

Varying θ: $\quad ds = R\, d\theta$

Varying ϕ: $\quad ds = R \sin\theta\, d\phi$

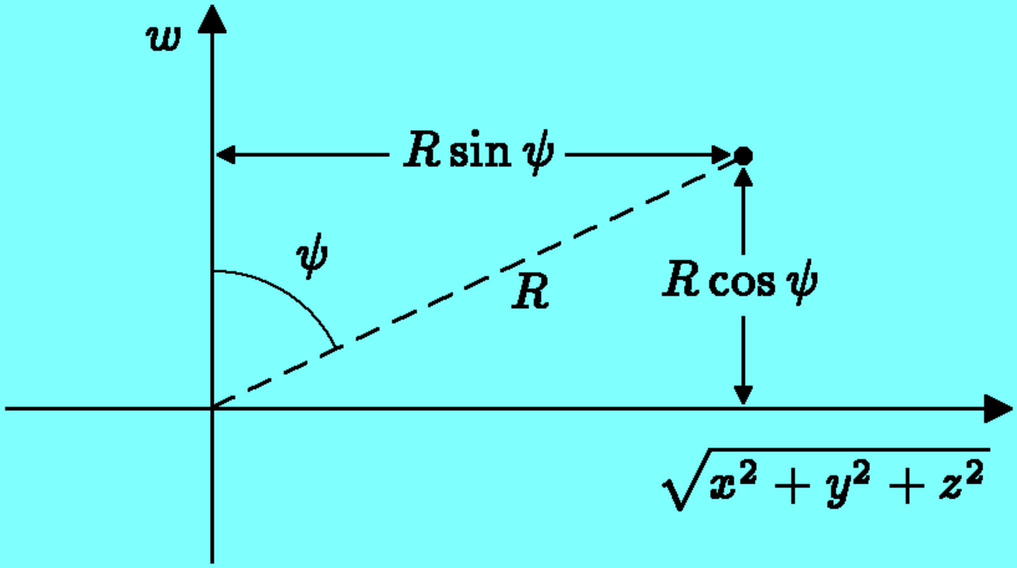
$$ds^2 = R^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2 \right)$$

A Closed Three-Dimensional Space

$$x^2 + y^2 + z^2 + w^2 = R^2$$

$$\begin{aligned} x &= R \sin\psi \sin\theta \cos\phi \\ y &= R \sin\psi \sin\theta \sin\phi \\ z &= R \sin\psi \cos\theta \\ w &= R \cos\psi \ , \end{aligned}$$

$$ds = R\, d\psi$$



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Massachusetts Institute of Technology

Metric for the Closed 3D Space

Varying ψ: $ds = R\,d\psi$

Varying θ or ϕ: $ds^2 = R^2 \sin^2\psi(d\theta^2 + \sin^2\theta\,d\phi^2)$

If the variations are orthogonal to each other, then

$$ds^2 = R^2\left[d\psi^2 + \sin^2\psi\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right]$$

Proof of Orthogonality of Variations

Let $d\vec{r}_\psi$ = displacement of point when ψ is changed to $\psi + d\psi$.

Let $d\vec{r}_\theta$ = displacement of point when θ is changed to $\theta + d\theta$.

- ☆ $d\vec{r}_\theta$ has no w-component $\Longrightarrow$ $d\vec{r}_\psi \cdot d\vec{r}_\theta = d\vec{r}_\psi^{\,(3)} \cdot d\vec{r}_\theta^{\,(3)}$, where (3) denotes the projection into the x-y-z subspace.
- ☆ $d\vec{r}_\psi^{\,(3)}$ is radial; $d\vec{r}_\theta^{\,(3)}$ is tangential

$$\Longrightarrow \quad d\vec{r}_\psi^{\,(3)} \cdot d\vec{r}_\theta^{\,(3)} = 0$$

Implications of General Relativity

- $ds^2 = R^2 \left[d\psi^2 + \sin^2\psi \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right) \right]$, where R is radius of curvature.
- According to GR, matter causes space to curve. So R, the curvature radius, should be determined by the matter.
- From the metric, or from the picture of a sphere of radius R in a 4D Euclidean embedding space, it is clear that R determines the size of the space. But $a(t)$, the scale factor, also determines the size of the space. So they must be proportional.
- But R is in meters, $a(t)$ in meters/notch. So dimensional consistency $\Longrightarrow$ $R \propto a(t)/\sqrt{k}$, since $[k] = \text{notch}^{-2}$.
- In fact,

$$R^2(t) = \frac{a^2(t)}{k} \; .$$

(I do not know any way to explain why the proportionality constant is 1, except by using the full equations of GR.)

☆ $ds^2 = R^2 \left[d\psi^2 + \sin^2\psi \left(d\theta^2 + \sin^2\theta \, d\phi^2\right)\right]$, where R is radius of curvature.

☆ In fact,

$$R^2(t) = \frac{a^2(t)}{k} \ .$$

☆ So,

$$ds^2 = \frac{a^2(t)}{k} \left[d\psi^2 + \sin^2\psi \left(d\theta^2 + \sin^2\theta \, d\phi^2\right)\right] \ .$$

☆ It is common to introduce a new radial variable $r \equiv \sin\psi / \sqrt{k}$, so $dr = \cos\psi \, d\psi/\sqrt{k} = \sqrt{1 - kr^2} \, d\psi/\sqrt{k}$. In terms of r,

$$ds^2 = a^2(t) \left\{ \frac{dr^2}{1 - kr^2} + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right) \right\} \ .$$

This is the spatial part of the Robertson-Walker metric.

Open Universes

- For $k > 0$ (closed universe),

$$ds^2 = a^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right\}$$

describes a homogeneous isotropic universe.

- For $k < 0$ (open universe),

$$ds^2 = a^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right\}$$

still describes a homogeneous isotropic universe.

- Properties are very different. The closed universe reaches its equator at $r = 1/\sqrt{k}$, which is a finite distance from the origin,

$$a(t)\int_0^{1/\sqrt{k}} \frac{dr}{\sqrt{1-kr^2}} = \frac{\pi a(t)}{2\sqrt{k}} \ .$$

The total volume is finite. For the open universe, r has no limit, and the volume is infinite.