8.286 Class 12
October 19, 2022

INTRODUCTION TO NON-EUCLIDEAN SPACES, PART 4

Review from the previous lecture

Summary: Metrics of Interest

Minkowski Metric: (Special relativity)

$$\begin{aligned} ds^2 &= -c^2 dt^2 + dx^2 + dy^2 + dz^2 \\ &= -c^2 dt^2 + dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2) \ . \end{aligned}$$

Robertson–Walker Metric:

$$ds^2 = -c^2 \, dt^2 + a^2(t) \left\{ \frac{dr^2}{1 - kr^2} + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right) \right\} \ .$$

Meaning: If $ds^2 > 0$, ds is distance in freely falling frame in which events are simultaneous. If $ds^2 < 0$, $ds^2 = -c^2 d\tau^2$, where $d\tau$ is time interval in freely falling frame in which events occur at same point. If $ds^2 = 0$, events are lightlike separated.

Review from the previous lecture

Geodesics in General Relativity

A geodesic is a path connecting two points in spacetime, with the property that the length of the curve is stationary with respect to small changes in the path. It can be a maximum, minimum, or saddle point.

In a curved spacetime, a geodesic is the closest thing to a straight line that exists.

In general relativity, if no forces act on a particle other than gravity, the particle travels on a geodesic.

Geodesics in Two Spatial Dimensions

Metric:

$$ds^2 = g_{xx} dx^2 + g_{xy} dx \, dy + g_{yx} dy \, dx + g_{yy} dy^2 \ .$$

Let $x^1 \equiv x$, $x^2 \equiv y$, so x^i is either, as $i = 1$ or 2.

$$\begin{aligned} ds^2 &= \sum_{i=1}^{2} \sum_{j=1}^{2} g_{ij}(x^\ell) \, dx^i \, dx^j \\ &= g_{ij}(x^\ell) \, dx^i \, dx^j \ . \end{aligned}$$

Einstein summation convention: repeated indices within one term are summed over coordinate indices (1 and 2), unless otherwise specified.

The sum is always over one upper index and one lower, but we will not discuss why some indices are written as upper and some as lower.

$g_{ij}\left(x^\ell\right)$ indicates that g_{ij} is a function of all the components of x^ℓ. I.e., when x^ℓ occurs as an argument of a function, it is shorthand for (x^1, x^2). By contrast, dx^i denotes the i'th component of dx, meaning dx^1 if $i = 1$, or dx^2 if $i = 2$.

The Length of Path

Consider a path from A to B.

Path description: $x^i(\lambda)$, where λ is parameter running from 0 to λ_f.

$$x^i(0) = x^i_A , \qquad x^i(\lambda_f) = x^i_B \ .$$

Between λ and $\lambda + \mathrm{d}\lambda$,

$$\mathrm{d}x^i = \frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\mathrm{d}\lambda \ ,$$

so

$$\mathrm{d}s^2 = g_{ij}(x^\ell)\,\mathrm{d}x^i\,\mathrm{d}x^j = g_{ij}\left(x^\ell(\lambda)\right)\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\mathrm{d}\lambda^2 \ ,$$

and then

$$\mathrm{d}s = \sqrt{g_{ij}\left(x^\ell(\lambda)\right)\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}}\,\mathrm{d}\lambda \ ,$$

and

$$S[x^i(\lambda)] = \int_0^{\lambda_f} \sqrt{g_{ij}\left(x^\ell(\lambda)\right)\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}}\,\mathrm{d}\lambda \ .$$

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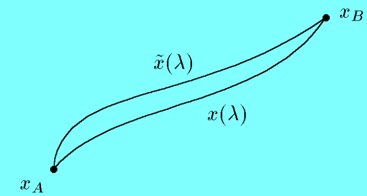
Varying the Path



$$\tilde{x}^i(\lambda) = x^i(\lambda) + \alpha w^i(\lambda) \ ,$$

where

$$w^i(0) = 0 \ , \qquad w^i(\lambda_f) = 0 \ .$$

Geodesic condition:

$$\left.\frac{\mathrm{d}\,S\left[\tilde{x}^i(\lambda)\right]}{\mathrm{d}\alpha}\right|_{\alpha=0} = 0 \qquad \text{for all } w^i(\lambda) \ .$$

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$$\tilde{x}^i(\lambda) = x^i(\lambda) + \alpha w^i(\lambda) \ .$$

$$S\left[\tilde{x}^i(\lambda)\right] = \int_0^{\lambda_f} \sqrt{g_{ij}\left(\tilde{x}^\ell(\lambda)\right)\frac{\mathrm{d}\tilde{x}^i}{\mathrm{d}\lambda}\frac{\mathrm{d}\tilde{x}^j}{\mathrm{d}\lambda}}\,\mathrm{d}\lambda \ .$$

Define

$$A(\lambda,\alpha) = g_{ij}\left(\tilde{x}^\ell(\lambda)\right)\frac{\mathrm{d}\tilde{x}^i}{\mathrm{d}\lambda}\frac{\mathrm{d}\tilde{x}^j}{\mathrm{d}\lambda} \ ,$$

so we can write

$$S\left[\tilde{x}^i(\lambda)\right] = \int_0^{\lambda_f} \sqrt{A(\lambda,\alpha)}\,\mathrm{d}\lambda \ .$$

Using chain rule,

$$\frac{\mathrm{d}f\left(x(\alpha),y(\alpha)\right)}{\mathrm{d}\alpha} = \frac{\partial f(x,y)}{\partial x}\frac{\mathrm{d}x(\alpha)}{\mathrm{d}\alpha} + \frac{\partial f(x,y)}{\partial y}\frac{\mathrm{d}y(\alpha)}{\mathrm{d}\alpha} = \frac{\partial f(x^\ell)}{\partial x^i}\frac{\mathrm{d}x^i}{\mathrm{d}\alpha} \ ,$$

$$\left.\frac{\mathrm{d}}{\mathrm{d}\alpha}g_{ij}\left(\tilde{x}^\ell(\lambda)\right)\right|_{\alpha=0} = \left[\frac{\partial g_{ij}}{\partial \tilde{x}^k}\frac{\partial \tilde{x}^k}{\partial \alpha}\right]_{\alpha=0} = \frac{\partial g_{ij}}{\partial x^k}\left(x^\ell(\lambda)\right)\left.\frac{\partial \tilde{x}^k}{\partial \alpha}\right|_{\alpha=0} = \frac{\partial g_{ij}}{\partial x^k}\left(x^\ell(\lambda)\right)w^k \ ,$$

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$$\tilde{x}^i(\lambda) = x^i(\lambda) + \alpha w^i(\lambda) \ .$$

$$A(\lambda,\alpha) = g_{ij}\left(\tilde{x}^\ell(\lambda)\right)\frac{\mathrm{d}\tilde{x}^i}{\mathrm{d}\lambda}\frac{\mathrm{d}\tilde{x}^j}{\mathrm{d}\lambda} \ .$$

Using chain rule, $\dfrac{\mathrm{d}f\left(x(\alpha),y(\alpha)\right)}{\mathrm{d}\alpha} = \dfrac{\partial f(x,y)}{\partial x}\dfrac{\mathrm{d}x(\alpha)}{\mathrm{d}\alpha} + \dfrac{\partial f(x,y)}{\partial y}\dfrac{\mathrm{d}y(\alpha)}{\mathrm{d}\alpha}$,

$$\left.\frac{\mathrm{d}}{\mathrm{d}\alpha}g_{ij}\left(\tilde{x}^\ell(\lambda)\right)\right|_{\alpha=0} = \left[\frac{\partial g_{ij}}{\partial \tilde{x}^k}\frac{\partial \tilde{x}^k}{\partial \alpha}\right]_{\alpha=0} = \frac{\partial g_{ij}}{\partial x^k}\left(x^\ell(\lambda)\right)\left.\frac{\partial \tilde{x}^k}{\partial \alpha}\right|_{\alpha=0} = \frac{\partial g_{ij}}{\partial x^k}\left(x^\ell(\lambda)\right)w^k \ .$$

Furthermore,

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\left(\frac{\mathrm{d}\tilde{x}^i}{\mathrm{d}\lambda}\right) = \frac{\mathrm{d}}{\mathrm{d}\alpha}\left[\frac{\mathrm{d}x^i(\lambda)}{\mathrm{d}\lambda} + \alpha\frac{\mathrm{d}w^i(\lambda)}{\mathrm{d}\lambda}\right] = \frac{\mathrm{d}w^i(\lambda)}{\mathrm{d}\lambda} \ .$$

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$$S\left[\tilde{x}^i(\lambda)\right] = \int_0^{\lambda_f} \sqrt{A(\lambda,\alpha)}\,\mathrm{d}\lambda\ ,$$

where

$$A(\lambda,\alpha) = g_{ij}\left(\tilde{x}^\ell(\lambda)\right)\frac{\mathrm{d}\tilde{x}^i}{\mathrm{d}\lambda}\frac{\mathrm{d}\tilde{x}^j}{\mathrm{d}\lambda}\ ,$$

with

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}g_{ij}\left(\tilde{x}^\ell(\lambda)\right)\Big|_{\alpha=0} = \frac{\partial g_{ij}}{\partial x^k}\left(x^\ell(\lambda)\right)w^k\ , \qquad \frac{\mathrm{d}}{\mathrm{d}\alpha}\left(\frac{\mathrm{d}\tilde{x}^i}{\mathrm{d}\lambda}\right) = \frac{\mathrm{d}w^i(\lambda)}{\mathrm{d}\lambda}\ .$$

Then

$$\frac{\mathrm{d}S\left[\tilde{x}^\ell(\lambda)\right]}{\mathrm{d}\alpha}\Bigg|_{\alpha=0} = \frac{1}{2}\int_0^{\lambda_f}\frac{1}{\sqrt{A(\lambda,0)}}\left\{\frac{\partial g_{ij}}{\partial x^k}w^k\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda} + \right.$$
$$\left. + g_{ij}\frac{\mathrm{d}w^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda} + g_{ij}\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}w^j}{\mathrm{d}\lambda}\right\}\mathrm{d}\lambda\ ,$$

where the metric g_{ij} is to be evaluated at $x^\ell(\lambda)$.

$$\frac{\mathrm{d}S\left[\tilde{x}^\ell(\lambda)\right]}{\mathrm{d}\alpha}\Bigg|_{\alpha=0} = \frac{1}{2}\int_0^{\lambda_f}\frac{1}{\sqrt{A(\lambda,0)}}\left\{\frac{\partial g_{ij}}{\partial x^k}w^k\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda} + \right.$$
$$\left. + g_{ij}\frac{\mathrm{d}w^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda} + g_{ij}\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}w^j}{\mathrm{d}\lambda}\right\}\mathrm{d}\lambda\ .$$

Manipulating "dummy" indices: in third term, replace $i \to j$ and $j \to i$, and recall that $g_{ij} = g_{ji}$. Then 2nd & 3rd term are equal:

$$\frac{\mathrm{d}S\left[\tilde{x}^\ell(\lambda)\right]}{\mathrm{d}\alpha}\Bigg|_{\alpha=0} = \frac{1}{2}\int_0^{\lambda_f}\frac{1}{\sqrt{A(\lambda,0)}}\left\{\frac{\partial g_{ij}}{\partial x^k}w^k\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda} + 2g_{ij}\frac{\mathrm{d}w^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\right\}\mathrm{d}\lambda\ .$$

Repeating,

$$\frac{\mathrm{d}S\left[\tilde{x}^\ell(\lambda)\right]}{\mathrm{d}\alpha}\Bigg|_{\alpha=0} = \frac{1}{2}\int_0^{\lambda_f}\frac{1}{\sqrt{A(\lambda,0)}}\left\{\frac{\partial g_{ij}}{\partial x^k}w^k\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda} + 2g_{ij}\frac{\mathrm{d}w^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\right\}\mathrm{d}\lambda\ .$$

Integration by Parts: Integral depends on both w^k and $\mathrm{d}w^i/\mathrm{d}\lambda$. Can eliminate $\mathrm{d}w^i/\mathrm{d}\lambda$ by integrating by parts:

$$\int_0^{\lambda_f}\left[\frac{1}{\sqrt{A}}g_{ij}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\right]\frac{\mathrm{d}w^i}{\mathrm{d}\lambda}\,\mathrm{d}\lambda = \int_0^{\lambda_f}\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\frac{1}{\sqrt{A}}g_{ij}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}w^i\right]\mathrm{d}\lambda$$
$$- \int_0^{\lambda_f}\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\frac{1}{\sqrt{A}}g_{ij}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\right]w^i\,\mathrm{d}\lambda\ .$$

But

$$\int_0^{\lambda_f}\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\frac{1}{\sqrt{A}}g_{ij}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}w^i\right]\mathrm{d}\lambda = \left[\frac{1}{\sqrt{A}}g_{ij}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}w^i\right]\Bigg|_{\lambda=0}^{\lambda=\lambda_f} = 0\ ,$$

since $w^i(\lambda)$ vanishes at $\lambda = 0$ and $\lambda = \lambda_f$.

$$\frac{\mathrm{d}S\left[\tilde{x}^\ell(\lambda)\right]}{\mathrm{d}\alpha}\Bigg|_{\alpha=0} = \frac{1}{2}\int_0^{\lambda_f}\frac{1}{\sqrt{A(\lambda,0)}}\left\{\frac{\partial g_{ij}}{\partial x^k}w^k\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda} + 2g_{ij}\frac{\mathrm{d}w^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\right\}\mathrm{d}\lambda\ .$$

$$\frac{\mathrm{d}S}{\mathrm{d}\alpha}\Bigg|_{\alpha=0} = \frac{1}{2}\int_0^{\lambda_f}\left\{\frac{1}{\sqrt{A}}\frac{\partial g_{ij}}{\partial x^k}\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}w^k - 2\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\frac{1}{\sqrt{A}}g_{ij}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\right]w^i\right\}\mathrm{d}\lambda\ .$$

$$\left.\frac{\mathrm{d}S}{\mathrm{d}\alpha}\right|_{\alpha=0} = \frac{1}{2}\int_0^{\lambda_f}\left\{\frac{1}{\sqrt{A}}\frac{\partial g_{ij}}{\partial x^k}\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}w^k - 2\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\frac{1}{\sqrt{A}}g_{ij}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\right]w^i\right\}\mathrm{d}\lambda \ .$$

Complication: one term is proportional to w^k, and the other is proportional to w^i. But with more index juggling, we can fix that. In 1st term replace $i \to j, j \to k, k \to i$:

$$\left.\frac{\mathrm{d}S}{\mathrm{d}\alpha}\right|_{\alpha=0} = \int_0^{\lambda_f}\left\{\frac{1}{2\sqrt{A}}\frac{\partial g_{jk}}{\partial x^i}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\frac{\mathrm{d}x^k}{\mathrm{d}\lambda} - \frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\frac{1}{\sqrt{A}}g_{ij}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\right]\right\} w^i(\lambda)\,\mathrm{d}\lambda \ .$$

To vanish ***for all*** $\boldsymbol{w^i(\lambda)}$ which vanish at $\lambda = 0$ and $\lambda = \lambda_f$, the quantity in curly brackets must vanish. If not, then suppose that $\{\,\}_i > 0$ for some $i = i_0$ and for some $\lambda = \lambda_0$. By continuity, $\{\,\}_{i_0} > 0$ in some neighborhood of λ_0. Choose $w^{i_0}(\lambda)$ to be positive in this neighborhood, and zero everywhere else, with $w^j(\lambda) \equiv 0$ for $j \neq i_0$, and one has a contradiction.

So

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\frac{1}{\sqrt{A}}g_{ij}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\right] = \frac{1}{2\sqrt{A}}\frac{\partial g_{jk}}{\partial x^i}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\frac{\mathrm{d}x^k}{\mathrm{d}\lambda} \ .$$

Repeating,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\frac{1}{\sqrt{A}}g_{ij}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\right] = \frac{1}{2\sqrt{A}}\frac{\partial g_{jk}}{\partial x^i}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\frac{\mathrm{d}x^k}{\mathrm{d}\lambda} \ .$$

This is ***complicated,*** since A is complicated.

Simplify by choice of parameterization: This result is valid for any parameterization. We don't need that! We can choose λ to be the path length. Since

$$\mathrm{d}s = \sqrt{g_{ij}\left(x^\ell(\lambda)\right)\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}}\,\mathrm{d}\lambda = \sqrt{A}\,\mathrm{d}\lambda \ ,$$

we see that $\mathrm{d}\lambda = \mathrm{d}s$ implies

$$A = 1 \qquad (\text{for } \lambda = \text{path length}).$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[g_{ij}\frac{\mathrm{d}x^j}{\mathrm{d}s}\right] = \frac{1}{2}\frac{\partial g_{jk}}{\partial x^i}\frac{\mathrm{d}x^j}{\mathrm{d}s}\frac{\mathrm{d}x^k}{\mathrm{d}s} \ .$$