8.286 Class 13
October 24, 2022

INTRODUCTION TO NON-EUCLIDEAN SPACES, PART 5

Review from the previous lecture

Geodesics in General Relativity

A geodesic is a path connecting two points in spacetime, with the property that the length of the curve is stationary with respect to small changes in the path. It can be a maximum, minimum, or saddle point.

In a curved spacetime, a geodesic is the closest thing to a straight line that exists.

In general relativity, if no forces act on a particle other than gravity, the particle travels on a geodesic.

Review from the previous lecture

Geodesics in Two Spatial Dimensions

Metric:

$$ds^2 = g_{xx}dx^2 + g_{xy}dx\,dy + g_{yx}dy\,dx + g_{yy}dy^2 \ .$$

Let $x^1 \equiv x$, $x^2 \equiv y$, so x^i is either, as $i = 1$ or 2.

$$ds^2 = \sum_{i=1}^{2}\sum_{j=1}^{2} g_{ij}(x^\ell)\,dx^i\,dx^j$$

$$= g_{ij}(x^\ell)\,dx^i\,dx^j \ .$$

Einstein summation convention: repeated indices within one term are summed over coordinate indices (1 and 2), unless otherwise specified.

The sum is always over one upper index and one lower, but we will not discuss why some indices are written as upper and some as lower.

$g_{ij}\left(x^\ell\right)$ indicates that g_{ij} is a function of all the components of x^ℓ. I.e., when x^ℓ occurs as an argument of a function, it is shorthand for (x^1, x^2). By contrast, dx^i denotes the i'th component of dx, meaning dx^1 if $i = 1$, or dx^2 if $i = 2$.

Review from the previous lecture

The Length of Path

Consider a path from A to B.

Path description: $x^i(\lambda)$, where λ is parameter running from 0 to λ_f.

$$x^i(0) = x^i_A, \qquad x^i(\lambda_f) = x^i_B \ .$$

Between λ and $\lambda + d\lambda$,

$$dx^i = \frac{dx^i}{d\lambda}d\lambda \ ,$$

so

$$ds^2 = g_{ij}(x^\ell)\,dx^i\,dx^j = g_{ij}\left(x^\ell(\lambda)\right)\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda}\,d\lambda^2 \ ,$$

and then

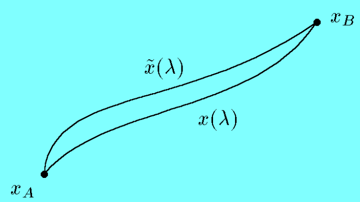
$$ds = \sqrt{g_{ij}\left(x^\ell(\lambda)\right)\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda}}\,d\lambda \ ,$$

and

$$S[x^i(\lambda)] = \int_0^{\lambda_f} \sqrt{g_{ij}\left(x^\ell(\lambda)\right)\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda}}\,d\lambda \ .$$

Review from the previous lecture

Varying the Path



$$\tilde{x}^i(\lambda) = x^i(\lambda) + \alpha w^i(\lambda) \ ,$$

where

$$w^i(0) = 0 \ , \qquad w^i(\lambda_f) = 0 \ .$$

Geodesic condition:

$$\left.\frac{\mathrm{d}\,S\left[\tilde{x}^i(\lambda)\right]}{\mathrm{d}\alpha}\right|_{\alpha=0} = 0 \qquad \text{for all } w^i(\lambda) \ .$$

Review from the previous lecture

$$\tilde{x}^i(\lambda) = x^i(\lambda) + \alpha w^i(\lambda) \ .$$

$$S\left[\tilde{x}^i(\lambda)\right] = \int_0^{\lambda_f} \sqrt{g_{ij}\left(\tilde{x}^\ell(\lambda)\right)\frac{\mathrm{d}\tilde{x}^i}{\mathrm{d}\lambda}\frac{\mathrm{d}\tilde{x}^j}{\mathrm{d}\lambda}}\,\mathrm{d}\lambda \ .$$

Define

$$A(\lambda,\alpha) = g_{ij}\left(\tilde{x}^\ell(\lambda)\right)\frac{\mathrm{d}\tilde{x}^i}{\mathrm{d}\lambda}\frac{\mathrm{d}\tilde{x}^j}{\mathrm{d}\lambda} \ ,$$

so we can write

$$S\left[\tilde{x}^i(\lambda)\right] = \int_0^{\lambda_f} \sqrt{A(\lambda,\alpha)}\,\mathrm{d}\lambda \ .$$

Using chain rule,

$$\frac{\mathrm{d}f\big(x(\alpha),y(\alpha)\big)}{\mathrm{d}\alpha} = \frac{\partial f(x,y)}{\partial x}\frac{\mathrm{d}x(\alpha)}{\mathrm{d}\alpha} + \frac{\partial f(x,y)}{\partial y}\frac{\mathrm{d}y(\alpha)}{\mathrm{d}\alpha} = \frac{\partial f(x^\ell)}{\partial x^i}\frac{\mathrm{d}x^i}{\mathrm{d}\alpha} \ ,$$

$$\left.\frac{\mathrm{d}}{\mathrm{d}\alpha}g_{ij}\left(\tilde{x}^\ell(\lambda)\right)\right|_{\alpha=0} = \left[\frac{\partial g_{ij}}{\partial \tilde{x}^k}\frac{\partial \tilde{x}^k}{\partial \alpha}\right]_{\alpha=0} = \frac{\partial g_{ij}}{\partial x^k}\left(x^\ell(\lambda)\right)\left.\frac{\partial \tilde{x}^k}{\partial \alpha}\right|_{\alpha=0} = \frac{\partial g_{ij}}{\partial x^k}\left(x^\ell(\lambda)\right)w^k \ ,$$

Review from the previous lecture

$$\tilde{x}^i(\lambda) = x^i(\lambda) + \alpha w^i(\lambda) \ .$$

$$A(\lambda,\alpha) = g_{ij}\left(\tilde{x}^\ell(\lambda)\right)\frac{\mathrm{d}\tilde{x}^i}{\mathrm{d}\lambda}\frac{\mathrm{d}\tilde{x}^j}{\mathrm{d}\lambda} \ .$$

Using chain rule, $\dfrac{\mathrm{d}f\big(x(\alpha),y(\alpha)\big)}{\mathrm{d}\alpha} = \dfrac{\partial f(x,y)}{\partial x}\dfrac{\mathrm{d}x(\alpha)}{\mathrm{d}\alpha} + \dfrac{\partial f(x,y)}{\partial y}\dfrac{\mathrm{d}y(\alpha)}{\mathrm{d}\alpha}$,

$$\left.\frac{\mathrm{d}}{\mathrm{d}\alpha}g_{ij}\left(\tilde{x}^\ell(\lambda)\right)\right|_{\alpha=0} = \left[\frac{\partial g_{ij}}{\partial \tilde{x}^k}\frac{\partial \tilde{x}^k}{\partial \alpha}\right]_{\alpha=0} = \frac{\partial g_{ij}}{\partial x^k}\left(x^\ell(\lambda)\right)\left.\frac{\partial \tilde{x}^k}{\partial \alpha}\right|_{\alpha=0} = \frac{\partial g_{ij}}{\partial x^k}\left(x^\ell(\lambda)\right)w^k \ .$$

Furthermore,

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\left(\frac{\mathrm{d}\tilde{x}^i}{\mathrm{d}\lambda}\right) = \frac{\mathrm{d}}{\mathrm{d}\alpha}\left[\frac{\mathrm{d}x^i(\lambda)}{\mathrm{d}\lambda} + \alpha\frac{\mathrm{d}w^i(\lambda)}{\mathrm{d}\lambda}\right] = \frac{\mathrm{d}w^i(\lambda)}{\mathrm{d}\lambda} \ .$$

Review from the previous lecture

$$S\left[\tilde{x}^i(\lambda)\right] = \int_0^{\lambda_f} \sqrt{A(\lambda,\alpha)}\,\mathrm{d}\lambda \ ,$$

where

$$A(\lambda,\alpha) = g_{ij}\left(\tilde{x}^\ell(\lambda)\right)\frac{\mathrm{d}\tilde{x}^i}{\mathrm{d}\lambda}\frac{\mathrm{d}\tilde{x}^j}{\mathrm{d}\lambda} \ ,$$

with

$$\left.\frac{\mathrm{d}}{\mathrm{d}\alpha}g_{ij}\left(\tilde{x}^\ell(\lambda)\right)\right|_{\alpha=0} = \frac{\partial g_{ij}}{\partial x^k}\left(x^\ell(\lambda)\right)w^k \ , \qquad \frac{\mathrm{d}}{\mathrm{d}\alpha}\left(\frac{\mathrm{d}\tilde{x}^i}{\mathrm{d}\lambda}\right) = \frac{\mathrm{d}w^i(\lambda)}{\mathrm{d}\lambda} \ .$$

Then

$$\left.\frac{\mathrm{d}S\left[\tilde{x}^\ell(\lambda)\right]}{\mathrm{d}\alpha}\right|_{\alpha=0} = \frac{1}{2}\int_0^{\lambda_f}\frac{1}{\sqrt{A(\lambda,0)}}\left\{\frac{\partial g_{ij}}{\partial x^k}w^k\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda} + \right.$$

$$\left. + g_{ij}\frac{\mathrm{d}w^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda} + g_{ij}\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}w^j}{\mathrm{d}\lambda}\right\}\mathrm{d}\lambda \ ,$$

where the metric g_{ij} is to be evaluated at $x^\ell(\lambda)$.

Review from the previous lecture

$$\left.\frac{\mathrm{d}S\left[\tilde{x}^\ell(\lambda)\right]}{\mathrm{d}\alpha}\right|_{\alpha=0} = \frac{1}{2}\int_0^{\lambda_f} \frac{1}{\sqrt{A(\lambda,0)}}\left\{\frac{\partial g_{ij}}{\partial x^k}w^k\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda} + +g_{ij}\frac{\mathrm{d}w^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda} + g_{ij}\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}w^j}{\mathrm{d}\lambda}\right\}\mathrm{d}\lambda\,.$$

Manipulating "dummy" indices: in third term, replace $i \to j$ and $j \to i$, and recall that $g_{ij} = g_{ji}$. Then 2nd & 3rd term are equal:

$$\left.\frac{\mathrm{d}S\left[\tilde{x}^\ell(\lambda)\right]}{\mathrm{d}\alpha}\right|_{\alpha=0} = \frac{1}{2}\int_0^{\lambda_f} \frac{1}{\sqrt{A(\lambda,0)}}\left\{\frac{\partial g_{ij}}{\partial x^k}w^k\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda} + 2g_{ij}\frac{\mathrm{d}w^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\right\}\mathrm{d}\lambda\,.$$

Review from the previous lecture

Repeating,

$$\left.\frac{\mathrm{d}S\left[\tilde{x}^\ell(\lambda)\right]}{\mathrm{d}\alpha}\right|_{\alpha=0} = \frac{1}{2}\int_0^{\lambda_f} \frac{1}{\sqrt{A(\lambda,0)}}\left\{\frac{\partial g_{ij}}{\partial x^k}w^k\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda} + 2g_{ij}\frac{\mathrm{d}w^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\right\}\mathrm{d}\lambda\,.$$

Integration by Parts: Integral depends on both w^k and $\mathrm{d}w^i/\mathrm{d}\lambda$. Can eliminate $\mathrm{d}w^i/\mathrm{d}\lambda$ by integrating by parts:

$$\int_0^{\lambda_f}\left[\frac{1}{\sqrt{A}}g_{ij}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\right]\frac{\mathrm{d}w^i}{\mathrm{d}\lambda}\,\mathrm{d}\lambda = \int_0^{\lambda_f}\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\frac{1}{\sqrt{A}}g_{ij}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}w^i\right]\mathrm{d}\lambda - \int_0^{\lambda_f}\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\frac{1}{\sqrt{A}}g_{ij}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\right]w^i\,\mathrm{d}\lambda\,.$$

But

$$\int_0^{\lambda_f}\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\frac{1}{\sqrt{A}}g_{ij}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}w^i\right]\mathrm{d}\lambda = \left.\left[\frac{1}{\sqrt{A}}g_{ij}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}w^i\right]\right|_{\lambda=0}^{\lambda=\lambda_f} = 0\,,$$

since $w^i(\lambda)$ vanishes at $\lambda = 0$ and $\lambda = \lambda_f$.

Review from the previous lecture

$$\left.\frac{\mathrm{d}S\left[\tilde{x}^\ell(\lambda)\right]}{\mathrm{d}\alpha}\right|_{\alpha=0} = \frac{1}{2}\int_0^{\lambda_f} \frac{1}{\sqrt{A(\lambda,0)}}\left\{\frac{\partial g_{ij}}{\partial x^k}w^k\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda} + 2g_{ij}\frac{\mathrm{d}w^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\right\}\mathrm{d}\lambda\,.$$

$$\left.\frac{\mathrm{d}S}{\mathrm{d}\alpha}\right|_{\alpha=0} = \frac{1}{2}\int_0^{\lambda_f}\left\{\frac{1}{\sqrt{A}}\frac{\partial g_{ij}}{\partial x^k}\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}w^k - 2\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\frac{1}{\sqrt{A}}g_{ij}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\right]w^i\right\}\mathrm{d}\lambda\,.$$

$$\left.\frac{\mathrm{d}S}{\mathrm{d}\alpha}\right|_{\alpha=0} = \frac{1}{2}\int_0^{\lambda_f}\left\{\frac{1}{\sqrt{A}}\frac{\partial g_{ij}}{\partial x^k}\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}w^k - 2\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\frac{1}{\sqrt{A}}g_{ij}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\right]w^i\right\}\mathrm{d}\lambda\,.$$

Complication: one term is proportional to w^k, and the other is proportional to w^i. But with more index juggling, we can fix that. In 1st term replace $i \to j, j \to k, k \to i$:

$$\left.\frac{\mathrm{d}S}{\mathrm{d}\alpha}\right|_{\alpha=0} = \int_0^{\lambda_f}\left\{\frac{1}{2\sqrt{A}}\frac{\partial g_{jk}}{\partial x^i}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\frac{\mathrm{d}x^k}{\mathrm{d}\lambda} - \frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\frac{1}{\sqrt{A}}g_{ij}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\right]\right\}w^i(\lambda)\,\mathrm{d}\lambda\,.$$

To vanish ***for all*** $\boldsymbol{w^i(\lambda)}$ which vanish at $\lambda = 0$ and $\lambda = \lambda_f$, the quantity in curly brackets must vanish. If not, then suppose that $\{\,\}_i > 0$ for some $i = i_0$ and for some $\lambda = \lambda_0$. By continuity, $\{\,\}_{i_0} > 0$ in some neighborhood of λ_0. Choose $w^{i_0}(\lambda)$ to be positive in this neighborhood, and zero everywhere else, with $w^j(\lambda) \equiv 0$ for $j \neq i_0$, and one has a contradiction.

So

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\frac{1}{\sqrt{A}}g_{ij}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\right] = \frac{1}{2\sqrt{A}}\frac{\partial g_{jk}}{\partial x^i}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\frac{\mathrm{d}x^k}{\mathrm{d}\lambda}\quad.$$

Repeating,

$$\frac{d}{d\lambda}\left[\frac{1}{\sqrt{A}} g_{ij} \frac{dx^j}{d\lambda}\right] = \frac{1}{2\sqrt{A}} \frac{\partial g_{jk}}{\partial x^i} \frac{dx^j}{d\lambda} \frac{dx^k}{d\lambda} \; .$$

This is ***complicated,*** since A is complicated.

Simplify by choice of parameterization: This result is valid for any parameterization. We don't need that! We can choose λ to be the path length. Since

$$ds = \sqrt{g_{ij}\left(x^\ell(\lambda)\right)\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda}}\, d\lambda = \sqrt{A}\, d\lambda \; ,$$

we see that $d\lambda = ds$ implies

$$A = 1 \qquad (\text{for } \lambda = \text{path length}).$$

Then

$$\frac{d}{ds}\left[g_{ij} \frac{dx^j}{ds}\right] = \frac{1}{2} \frac{\partial g_{jk}}{\partial x^i} \frac{dx^j}{ds} \frac{dx^k}{ds} \; .$$

Alternative Form of Geodesic Equation

Most books write the geodesic equation differently, as

$$\frac{d^2 x^i}{ds^2} = -\Gamma^i_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds} \; ,$$

where

$$\Gamma^i_{jk} = \frac{1}{2} g^{i\ell}\left(\partial_j g_{\ell k} + \partial_k g_{\ell j} - \partial_\ell g_{jk}\right)$$

and $g^{i\ell}$ is the matrix inverse of g_{ij}. The quantity Γ^i_{jk} is called the affine connection.

If you are interested, see the lecture notes. If you are not interested, you can skip this.

BLACK HOLES (Fun!)

The Schwarzschild Metric:

For any spherically symmetric distribution of mass, outside the mass the metric is given by the Schwarzschild metric,

$$ds^2 = -c^2 d\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \; ,$$

where M is the total mass, G is Newton's gravitational constant, c is the speed of light, and θ and ϕ have the usual polar-angle ranges.

Schwarzschild Horizon

$$ds^2 = -c^2 d\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \; .$$

The metric is singular at

$$r = R_S \equiv \frac{2GM}{c^2} \; ,$$

where the coefficient of $c^2 dt^2$ vanishes, and the coefficient of dr^2 is infinite.

Surprisingly, this singularity is not real — it is a coordinate artifact. There are other coordinate systems where the metric is smooth at R_S.

But R_S is a ***horizon:*** If you fall past the horizon, there is no return, even if you are photon.

Schwarzschild Radius of the Sun

$$R_{S,\odot} = \frac{2GM}{c^2}$$
$$= \frac{2 \times 6.673 \times 10^{-11}\ \text{m}^3\text{-kg}^{-1}\text{-s}^{-2} \times 1.989 \times 10^{30}\ \text{kg}}{(2.998 \times 10^8\ \text{m-s}^{-1})^2}$$
$$= 2.95\ \text{km}\ .$$

- ☆ If the Sun were compressed to this radius, it would become a black hole. Since the Sun is much larger than R_S, and the Schwarzschild metric is only valid outside the matter, there is no Schwarzschild horizon in the Sun.
- ☆ At the center of our galaxy is a supermassive black hole, with $M = 4.1 \times 10^6\, M_\odot$. This gives $R_S = 1.2 \times 10^{10}$ meters $\approx 1/4$ of radius of orbit of Mercury ≈ 17 times radius of Sun.

Radial Geodesics in the Schwarzschild Metric

$$ds^2 = -c^2 d\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\ .$$

Consider a particle released from rest at $r = r_0$.

r is a "radial coordinate," but not the radius, since it is not the distance from some center. If r is varied by dr, the distance traveled is not dr, but $dr/\sqrt{1 - 2GM/rc^2}$. r can be called the "circumferential radius," since the term $r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$ in the metric implies that the circumference of a circle about the origin is $2\pi r$.

By symmetry, the particle will fall straight down, with no change in θ or ϕ. Spherical symmetry implies that all directions in θ and ϕ are equivalent, so any motion in θ–ϕ space would violate this symmetry.

Particle Trajectories in Spacetime

Particle trajectories are timelike, so we use proper time τ to parameterize them, where $ds^2 \equiv -c^2 d\tau^2$. This implies that $A = -c^2$, instead of $A = 1$, but as long as A is constant, it drops out of the geodesic equation.

By tradition, the spacetime indices in general relativity are denoted by Greek letters such as μ, ν, λ, σ, and are summed from 0 to 3, where $x^0 \equiv t$.

The geodesic equation

$$\frac{d}{ds}\left[g_{ij}\frac{dx^j}{ds}\right] = \frac{1}{2}\frac{\partial g_{jk}}{\partial x^i}\frac{dx^j}{ds}\frac{dx^k}{ds}$$

is then rewritten as

$$\frac{d}{d\tau}\left[g_{\mu\nu}\frac{dx^\nu}{d\tau}\right] = \frac{1}{2}\frac{\partial g_{\lambda\sigma}}{\partial x^\mu}\frac{dx^\lambda}{d\tau}\frac{dx^\sigma}{d\tau}\ .$$

Radial Trajectory Equations

Only $dr/d\tau$ and $dt/d\tau$ are nonzero. But they are related by the metric:

$$c^2 d\tau^2 = \left(1 - \frac{2GM}{rc^2}\right) c^2\, dt^2 - \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2$$

implies that

$$c^2 = \left(1 - \frac{2GM}{rc^2}\right) c^2 \left(\frac{dt}{d\tau}\right)^2 - \left(1 - \frac{2GM}{rc^2}\right)^{-1} \left(\frac{dr}{d\tau}\right)^2\ .$$

Then, looking at the $\mu = r$ geodesic equation,

$$\frac{d}{d\tau}\left[g_{\mu\nu}\frac{dx^\nu}{d\tau}\right] = \frac{1}{2}\frac{\partial g_{\lambda\sigma}}{\partial x^\mu}\frac{dx^\lambda}{d\tau}\frac{dx^\sigma}{d\tau}$$

implies that

$$\frac{d}{d\tau}\left[g_{rr}\frac{dr}{d\tau}\right] = \frac{1}{2}\partial_r g_{rr}\left(\frac{dr}{d\tau}\right)^2 + \frac{1}{2}\partial_r g_{tt}\left(\frac{dt}{d\tau}\right)^2\ ,$$

where

$$g_{rr} = \left(1 - \frac{2GM}{rc^2}\right)^{-1}\ , \quad g_{tt} = -c^2\left(1 - \frac{2GM}{rc^2}\right)\ .$$

Repeating,

$$c^2 = \left(1 - \frac{2GM}{rc^2}\right) c^2 \left(\frac{dt}{d\tau}\right)^2 - \left(1 - \frac{2GM}{rc^2}\right)^{-1} \left(\frac{dr}{d\tau}\right)^2 .$$

$$\frac{d}{d\tau}\left[g_{rr}\frac{dr}{d\tau}\right] = \frac{1}{2}\partial_r g_{rr}\left(\frac{dr}{d\tau}\right)^2 + \frac{1}{2}\partial_r g_{tt}\left(\frac{dt}{d\tau}\right)^2 ,$$

where

$$g_{rr} = \left(1 - \frac{2GM}{rc^2}\right)^{-1} , \quad g_{tt} = -c^2\left(1 - \frac{2GM}{rc^2}\right) .$$

Expand

$$\frac{d}{d\tau}\left[g_{rr}\frac{dr}{d\tau}\right]$$

with the product rule, replace $(dt/d\tau)^2$ using the equation above, and simplify. Result:

$$\frac{d^2 r}{d\tau^2} = -\frac{GM}{r^2} ,$$

which looks just like Newton, but it is not really the same. Here τ is the proper time as measured by the infalling object, and r is not the radial distance.

Solving the Equation

$$\frac{d^2 r}{d\tau^2} = -\frac{GM}{r^2} .$$

Like Newton's equation, multiply by $dr/d\tau$, and it can then be written as

$$\frac{d}{d\tau}\left\{\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 - \frac{GM}{r}\right\} = 0 .$$

Quantity in curly brackets is conserved. Initial value (on release from rest at r_0) is $-GM/r_0$, so it always has this value. Then

$$\frac{dr}{d\tau} = -\sqrt{2GM\left(\frac{1}{r} - \frac{1}{r_0}\right)} = -\sqrt{\frac{2GM(r_0 - r)}{r r_0}} .$$

Repeating,

$$\frac{dr}{d\tau} = -\sqrt{2GM\left(\frac{1}{r} - \frac{1}{r_0}\right)} = -\sqrt{\frac{2GM(r_0 - r)}{r r_0}} .$$

Bring all r-dependent factors to one side, and bring $d\tau$ to the other side, and integrate:

$$\tau(r_f) = -\int_{r_0}^{r_f} dr \sqrt{\frac{r r_0}{2GM(r_0 - r)}}$$

$$= \sqrt{\frac{r_0}{2GM}}\left\{r_0 \tan^{-1}\left(\sqrt{\frac{r_0 - r_f}{r_f}}\right) + \sqrt{r_f(r_0 - r_f)}\right\} ,$$

where $\tan^{-1} \equiv \arctan$.

Conclusion: object will reach $r = 0$ in a finite proper time τ.

$$\tau(r_f) = \sqrt{\frac{r_0}{2GM}}\left\{r_0 \tan^{-1}\left(\sqrt{\frac{r_0 - r_f}{r_f}}\right) + \sqrt{r_f(r_0 - r_f)}\right\} .$$

Setting $r_f = 0$ to find the proper time when the object reaches $r = 0$,

$$\tau(0) = \sqrt{\frac{r_0}{2GM}}\left\{r_0 \tan^{-1}(\infty) + 0\right\}$$

$$= \frac{\pi}{2}\sqrt{\frac{r_0^3}{2GM}} .$$

Falling from the Schwarzschild Horizon to $r = 0$

Recall,

$$\tau(0) = \frac{\pi}{2}\sqrt{\frac{r_0^3}{2GM}} .$$

For $r_0 = R_S$,

$$\tau = \frac{\pi GM}{c^3} .$$

For $r_0 = R_S$,

$$\tau = \frac{\pi GM}{c^3} .$$

For the Sun, this gives

$$\tau = 1.55 \times 10^{-5}\ \text{s}.$$

For the black hole in the center of our galaxy,

$$\tau = 6.34\ \text{s}.$$

Note that inside the black hole,

$$ds^2 = -c^2 d\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right)c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) ,$$

but

$$\left(1 - \frac{2GM}{rc^2}\right) < 0 ,$$

which implies that t is spacelike, and r is timelike! The calculation that we just did is still correct. The singularity at $r = 0$ cannot be avoided for the same reason that we cannot prevent ourselves from reaching tomorrow!

But Coordinate Time t is Different!

$$\frac{dr}{d\tau} = -\sqrt{2GM\left(\frac{1}{r} - \frac{1}{r_0}\right)} = -\sqrt{\frac{2GM(r_0 - r)}{rr_0}} .$$

$$c^2 = \left(1 - \frac{2GM}{rc^2}\right)c^2\left(\frac{dt}{d\tau}\right)^2 - \left(1 - \frac{2GM}{rc^2}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 .$$

$$\frac{dr}{dt} = \frac{dr}{d\tau}\frac{d\tau}{dt} = \frac{dr/d\tau}{dt/d\tau}$$

$$= \frac{dr/d\tau}{\sqrt{h^{-1}(r) + c^{-2}h^{-2}(r)\left(\frac{dr}{d\tau}\right)^2}} ,$$

where $h^{-1}(r) \equiv 1/h(r)$, not the inverse function, and

$$h(r) \equiv 1 - \frac{R_S}{r} = 1 - \frac{2GM}{rc^2} .$$

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = -\sqrt{2GM\left(\frac{1}{r} - \frac{1}{r_0}\right)} = -\sqrt{\frac{2GM(r_0 - r)}{r r_0}} \,.$$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}r/\mathrm{d}\tau}{\sqrt{h^{-1}(r) + c^{-2}h^{-2}(r)\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2}} \,,$$

where

$$h(r) \equiv 1 - \frac{R_S}{r} = 1 - \frac{2GM}{rc^2} \,.$$

Look at behavior near horizon; $h^{-1}(r)$ blows up:

$$h^{-1}(r) = \frac{r}{r - R_S} \approx \frac{R_S}{r - R_S} \,.$$

Denominator of $\mathrm{d}r/\mathrm{d}t$ is dominated by 2nd term, which gives

$$\frac{\mathrm{d}r}{\mathrm{d}t} \approx -ch(r) = -c\left(\frac{r - R_S}{R_S}\right) \,.$$

Repeating,

$$\frac{\mathrm{d}r}{\mathrm{d}t} \approx -c\left(\frac{r - R_S}{R_S}\right) \,.$$

Rearranging,

$$\mathrm{d}t = -\frac{R_S}{c}\frac{\mathrm{d}r}{r - R_S} \,.$$

We can find the time needed to fall from some r_i near the horizon, to a smaller r_f which is nearer to the horizon:

$$t(r_f) \approx -\frac{R_S}{c}\int_{r_i}^{r_f} \frac{\mathrm{d}r'}{r' - R_S} \approx \boxed{\frac{R_S}{c}\ln\left(\frac{r_i - R_S}{r_f - R_S}\right) \,.}$$

Thus t diverges logarithmically as $r_f \to R_S$, so the object does not reach R_S for any finite value of t.