8.286 Class 14
October 26, 2022

BLACK-BODY RADIATION AND THE EARLY HISTORY OF THE UNIVERSE

$E = mc^2$

- THE most famous equation in physics. But I was not able to find any actual surveys.
- **Meaning: Mass and energy are equivalent.** They are just two different ways of expressing exactly the same thing. The total energy of any system is equal to the total mass of the system — sometimes called the relativistic mass — times c^2, the square of the speed of light.
- One can imagine measuring the mass/energy of an object in either kilograms, joules, or kilowatt-hours, with

$$1\text{ kg} = 8.9876 \times 10^{16}\text{ joule} = 2.497 \times 10^{10}\text{ kW-hr.}$$

$E = mc^2$ and the World Power Supply

- The total amount of power produced in the world, on average, is about 1.89×10^{10} kW, according to the International Energy Agency (2020).
- This amounts to about 2.5 kW per person.
- If a 15 gallon tank of gasoline could be converted *entirely* into usable energy, it would power the world for $2\frac{1}{2}$ days.
- However, it is not so easy! Even with nuclear power, when a uranium-235 nucleus undergoes fission, only about 0.09% of its mass is converted to energy.

$E = mc^2$ and Particle Masses

- Nuclear and particle physicists tend to measure the mass of elementary particles in energy units, usually using either MeV (10^6 eV) or GeV (10^9 eV) as the unit of energy, where

$$1\,\text{eV} = 1\text{ electron volt} = 1.6022 \times 10^{-19}\text{ J},$$

and then

$$1\,\text{GeV} = 1.7827 \times 10^{-27}\text{ kg} \cdot c^2\ .$$

The mass of a proton is 0.938 GeV/c^2, and the mass of an electron is 0.511 MeV/c^2.

Energy and Momentum in Special Relativity

- We have talked about the kinematic consequences of special relativity (time dilation, Lorentz contraction, and the relativity of simultaneity), but now we need to bring in the dynamical consequences, involving energy and momentum.
- In special relativity, the definitions of energy and momentum are different from those in Newtonian mechanics.
- Why? Because special relativity is based on the principle that the laws of physics in any inertial reference frame are the same, and furthermore, in order for the speed of light be the same in any inertial reference frame, these frames cannot be related to each other as in Newtonian physics. They must instead be related by Lorentz transformations, which take into account the kinematic effects mentioned above.

- Two important laws of physics are the conservation of energy and momentum.
- If energy and momentum kept their Newtonian definitions, then, if they were conserved in one frame, they would not be conserved in other frames.
- The requirement that the conservation equations hold in all frames requires the standard special relativity definitions.

Energy, Momentum, and the Energy-Momentum Four-Vector

- The energy-momentum four-vector is defined by starting with the momentum three-vector $(p^1, p^2, p^3) \equiv (p^x, p^y, p^z)$, and appending a fourth component

$$p^0 = \frac{E}{c} \; ,$$

so the four-vector can be written as

$$p^\mu = \left(\frac{E}{c}, \vec{p}\right) \; .$$

As with the three-vector momentum, the energy-momentum four-vector can be defined for a system of particles as the sum of the vectors for the individual particles.

- The 4-vector p^μ transforms, when we change frames of reference, according to the Lorentz transformation, exactly like the 4-vector $x^\mu = (ct, \vec{x})$.
- Furthermore, the total energy-momentum 4-vector is conserved — in any inertial frame of reference.

Relation of Energy and Momentum to Rest Mass and Velocity

The mass of a particle in its own rest frame is called its rest mass, which we denote by m_0. At velocity $\vec{v}$

$$\vec{p} = \gamma m_0 \vec{v} ,$$
$$E = \gamma m_0 c^2 = \sqrt{(m_0 c^2)^2 + |\vec{p}|^2 c^2} ,$$

where as usual γ is defined by

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} .$$

Lorentz Invariance of p^2

$$\vec{p} = \gamma m_0 \vec{v} ,$$
$$E = \gamma m_0 c^2 = \sqrt{(m_0 c^2)^2 + |\vec{p}|^2 c^2} ,$$

Like the Lorentz-invariant interval that we discussed as $\mathrm{d}s^2 = -c^2 \mathrm{d}t^2 + \mathrm{d}\vec{x}^2$, the energy-momentum four-vector has a Lorentz-invariant square:

$$p^2 \equiv |\vec{p}|^2 - \left(p^0\right)^2 = |\vec{p}|^2 - \frac{E^2}{c^2} = -\,(m_0 c)^2 .$$

For a particle at rest,

$$E = m_0 c^2 .$$

Energy Exchange in a Simple Chemical Reaction

Consider the reaction

$$p + e^- \longrightarrow H + \gamma .$$

Assuming that the proton and electron begin at rest, and ignoring the very small kinetic energy of the hydrogen atom when it recoils from the emitted photon, conservation of energy implies that

$$m_H = m_p + m_e - E_\gamma / c^2 .$$

The energy given off when the proton and electron bind is called the **binding energy** of the hydrogen atom. It is 13.6 eV.

Relativistic Mass

- Since $E = mc^2$, we can define the *relativistic mass* of any particle or system as simply

$$m_{\text{rel}} \equiv \frac{E}{c^2} .$$

- Some authors avoid using the concept of relativistic mass, reserving the word "mass" to mean rest mass m_0. Relativistic mass is certainly a redundant concept, since anything that can be described in terms of m_{rel} can also be described in terms of E.
- For cosmology the concept of relativistic mass will be helpful, since relativistic mass is the source of gravity. By calling E/c^2 a mass, we are indicating our recognition that it is the source of gravity.

The Source of Gravity in General Relativity

This is beyond the level of what we need, but for those who are interested, I mention that the Einstein field equations imply that the source of gravitational fields is the *energy-momentum tensor* $T^{\mu\nu}$, where μ and ν are 4-vector indices that take on values from 0 to 3.

$T^{00} = u =$ energy density,

$T^{0i} = T^{i0}$ is $\frac{1}{c}$ times the flow of energy in the i'th direction (i=1,2,3) and is also c times the density of the i'th component of momentum,

$T^{ij} = T^{ji}$ is the flow in the j'th direction of the i'th component of momentum. T^{ij} is often diagonal, with $T^{ij} = p\,\delta^{ij}$, where p is the pressure.

For a homogeneous, isotropic universe model, only u and p will serve as sources for gravity.

Mass of Radiation

- Electromagnetic radiation has energy. The energy density is given by

$$u = \frac{1}{2}\left[\epsilon_0 \left|\vec{E}\right|^2 + \frac{1}{\mu_0}\left|\vec{B}\right|^2\right] .$$

We won't need this equation, but we need to know that electromagnetic radiation **has** an energy density u.

- Energy density implies a (relativistic) mass density

$$\rho = u/c^2 .$$

(*Relativistic mass* is defined to be the energy divided by c^2.)

Energy and Momentum of Photons

Photons have zero rest mass.

In general,

$$p^2 = |\vec{p}\,|^2 - \frac{E^2}{c^2} = -\left(m_0 c\right)^2 ,$$

but for photons, $m_0 = 0$, so

$$|\vec{p}\,|^2 - \frac{E^2}{c^2} = 0 , \quad \text{or} \quad E = c|\vec{p}\,| .$$

Radiation in an Expanding Universe

- From the end of inflation (maybe about 10^{-35} second, to be discussed later) until stars form, the number of photons is almost exactly conserved.
- Therefore,

$$n_\gamma \propto \frac{1}{a^3(t)} .$$

- But the frequency of each photon redshifts:

$$\nu \propto \frac{1}{a(t)} .$$

$$n_\gamma \propto \frac{1}{a^3(t)} , \qquad \nu \propto \frac{1}{a(t)} .$$

☆ But according to quantum mechanics, the energy of each photon is

$$E = h\nu ,$$

so the energy of each photon is proportional to $1/a(t)$.

☆ Finally,

$$n_\gamma \propto \frac{1}{a^3(t)} , \quad E_\gamma \propto \frac{1}{a(t)} \quad \Longrightarrow \quad \rho_\gamma = \frac{u_\gamma}{c^2} \propto \frac{1}{a^4(t)} .$$

The Radiation Dominated Era

Radiation energy density today (including photons and neutrinos):

$$u_r = 7.01 \times 10^{-14}\ \mathrm{J/m^3} , \qquad \rho_r = u_r/c^2 = 7.80 \times 10^{-34}\ \mathrm{g/cm^3} .$$

Total mass density today, ρ_0, is equal to within uncertainties to the critical density,

$$\rho_c = \frac{3H_0^2}{8\pi G} = 1.88\, h_0^2 \times 10^{-29}\ \mathrm{g/cm^3} ,$$

where

$$H_0 = 100\, h_0\ \mathrm{km\text{-}s^{-1}\text{-}Mpc^{-1}} , \qquad h_0 \approx 0.67 ,$$

which gives the present value of Ω_r as $\Omega_r \approx 9.2 \times 10^{-5}$.

Since $\rho_r \propto 1/a^4(t)$, while $\rho_m \propto 1/a^3(t)$.

ρ_m = mass density of nonrelativistic matter, baryonic matter plus dark matter. It follows that

$$\rho_r/\rho_m \propto 1/a(t) .$$

Today $\rho_m \approx 0.30\rho_c$, so $\rho_r/\rho_m \approx 9.2 \times 10^{-5}/0.30 \approx 3.1 \times 10^{-4}$. Thus

$$\frac{\rho_r(t)}{\rho_m(t)} = \frac{a(t_0)}{a(t)} \times 3.1 \times 10^{-4} .$$

$t_{\rm eq}$ is defined to be the time of matter-radiation equality. Thus

$$\frac{\rho_r(t_{\rm eq})}{\rho_m(t_{\rm eq})} \equiv 1 = \frac{a(t_0)}{a(t_{\rm eq})} \times 3.1 \times 10^{-4} .$$

Since $a(t_0)/a(t_{\rm eq}) = 1 + z_{\rm eq}$,

$$z_{\rm eq} = \frac{1}{3.1 \times 10^{-4}} - 1 \approx 3200 .$$

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Time of matter-radiation equality:

We are not ready to calculate this accurately, but for now we can estimate it by assuming that between $t_{\rm eq}$ and now, $a(t) \propto t^{2/3}$, as in a matter-dominated flat universe. Then

$$(t_{\rm eq}/t_0)^{2/3} = 3.1 \times 10^{-4} ,$$

so

$$t_{\rm eq} = 5.5 \times 10^{-6}\, t_0 = 5.5 \times 10^{-6} \times 13.8\ \mathrm{Gyr} \approx 75{,}000\ \mathrm{years.}$$

Ryden (p. 96) gives 50,000 years, which is more accurate.

Dynamics of the Radiation-Dominated Era

$$\rho \propto \frac{1}{a^3} \implies \dot{\rho} = -3\frac{\dot{a}}{a}\rho\,, \quad \rho(t) \propto \frac{1}{a^4(t)} \implies \dot{\rho} = -4\frac{\dot{a}}{a}\rho\,.$$

$\dot{\rho}$ and pressure p: (Problem 1, Problem Set 7)

$$dU = -p\,dV \quad \implies \quad \frac{dU}{dt} = -p\frac{dV}{dt}$$

$$= \frac{d}{dt}\left(a^3\rho c^2\right) = -p\frac{d}{dt}(a^3)$$

$$\implies \quad \dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right)\,.$$

Friedmann equations:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{a^2}$$

$$\ddot{a} = -\frac{4\pi}{3}G\rho a\,,$$

$$\dot{\rho} = -3\frac{\dot{a}}{a}\rho$$

(matter-dominated universe)

Any two of the above equations implies the third. So they become inconsistent if

$$\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right)\,.$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{a^2}\,, \qquad \ddot{a} = -\frac{4\pi}{3}G\rho a\,.$$

Any two of the above equations implies the third. So they become inconsistent if

$$\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right)\,.$$

So, if we believe the equation for $\dot{\rho}$, we must modify one of the two Friedmann equations. First order equation represents conservation of energy: pressure does not belong! (Pressures can change suddenly, as when dynamite explodes, so it does not make sense to have pressure in a conservation equation.) So modify the 2nd order equation, deriving it from the first order equation and the $\dot{\rho}$ equation:

$$\ddot{a} = -\frac{4\pi}{3}G\left(\rho + \frac{3p}{c^2}\right)a\,.$$