8.286 Class 15
October 31, 2022

BLACK-BODY RADIATION AND THE EARLY HISTORY OF THE UNIVERSE, PART 2

PROBLEM 3: THE CRUNCH OF A CLOSED, MATTER-DOMINATED UNIVERSE *(25 points)*

This is Problem 5.7 (Problem 6.5 in the first edition) from Barbara Ryden's ***Introduction to Cosmology****, with some paraphrasing to make it consistent with the language used in lecture.*

Consider a closed universe containing only nonrelativistic matter. This is the closed universe discussed in Lecture Notes 4, and it is also the "Big Crunch" model discussed in Ryden's section Section 5.4.1 (Section 6.1 in the first edition). At some time during the contracting phase (i.e., when $\theta > \pi$), an astronomer named Elbbuh Niwde discovers that nearby galaxies have blueshifts ($-1 \le z < 0$) proportional to their distance. He then measures the present values of the Hubble expansion rate, H_0, and the mass density parameter, Ω_0. He finds, of course, that $H_0 < 0$ (because he is in the contracting phase) and $\Omega_0 > 1$ (because the universe is closed). **In terms of H_0 and Ω_0, how long a time will elapse between Dr. Niwde's observation at $t = t_0$ and the final Big Crunch at $t = t_{\text{Crunch}} = 2\pi\alpha/c$?** Assuming that Dr. Niwde is able to observe all objects within his horizon, **what is the most blueshifted (i.e., most negative) value of z that Dr. Niwde is able to see? What is the lookback time to an object with this blueshift?** (By lookback time, one means the difference between the time of observation t_0 and the time at which the light was emitted.)

Review from the previous lecture

Radiation In An Expanding Universe

☆ As the universe expands,

$$n_\gamma \propto \frac{1}{a^3(t)}\ , \quad \nu \propto \frac{1}{a(t)}\ , \quad E_\gamma = h\nu \quad \Longrightarrow \quad u_\gamma \propto \rho_\gamma \propto \frac{1}{a^4(t)}\ ,$$

where n_γ = number density of photons, ν = frequency of any one photon, E_γ = energy of any one photon, and u_γ and ρ_γ are the energy density and mass density of photons, respectively.

Review from the previous lecture

The Radiation Dominated Era

Radiation energy density today (including photons and neutrinos):

$$u_r = 7.01 \times 10^{-14}\ \text{J/m}^3\ , \qquad \rho_r = u_r/c^2 = 7.80 \times 10^{-34}\ \text{g/cm}^3\ .$$

Total mass density today, ρ_0, is equal, to within uncertainties of a fraction of a percent, to the critical density,

$$\rho_c = \frac{3H_0^2}{8\pi G} = 1.88\, h_0^2\ \times 10^{-29}\ \text{g/cm}^3\ ,$$

where

$$H_0 = 100\, h_0\, \text{km-s}^{-1}\text{-Mpc}^{-1}\ , \qquad h_0 \approx 0.67\ ,$$

which gives the present value of Ω_r as $\boxed{\Omega_r \approx 9.2 \times 10^{-5}}$.

Review from the previous lecture

Since $\rho_r \propto 1/a^4(t)$, while $\rho_m \propto 1/a^3(t)$.

ρ_m = mass density of nonrelativistic matter: baryonic matter plus dark matter. It follows that

$$\rho_r/\rho_m \propto 1/a(t)\ .$$

Today $\rho_m \approx 0.30\rho_c$.

Carrying out the calculations, we found that matter-radiation equality occurred at

$$z_{\rm eq} \approx 3200\ ,$$

and

$$t_{\rm eq} = 5.5 \times 10^{-6}\, t_0 = 5.5 \times 10^{-6} \times 13.8 \text{ Gyr} \approx 75{,}000 \text{ years.}$$

Ryden (p. 96) gives 50,000 years, which is more accurate. We have not yet included a cosmological constant, and we treated the universe as completely matter-dominated even just after $t_{\rm eq}$.

Review from the previous lecture

Dynamics of the Radiation-Dominated Era

$$\rho \propto \frac{1}{a^3} \implies \dot\rho = -3\frac{\dot a}{a}\rho\ , \quad \rho(t) \propto \frac{1}{a^4(t)} \implies \dot\rho = -4\frac{\dot a}{a}\rho\ .$$

$\dot\rho$ and pressure p: (Problem 1, Problem Set 7)

$$dU = -p\,dV \quad \implies \quad \frac{dU}{dt} = -p\frac{dV}{dt}$$
$$= \frac{d}{dt}\left(a^3\rho c^2\right) = -p\frac{d}{dt}(a^3)$$
$$\implies \quad \dot\rho = -3\frac{\dot a}{a}\left(\rho + \frac{p}{c^2}\right)\ .$$

Review from the previous lecture

Friedmann equations:

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{a^2}$$
$$\ddot a = -\frac{4\pi}{3}G\rho a\ ,$$
$$\dot\rho = -3\frac{\dot a}{a}\rho$$

(matter-dominated universe)

Any two of the above equations implies the third. So they become inconsistent if

$$\dot\rho = -3\frac{\dot a}{a}\left(\rho + \frac{p}{c^2}\right)\ .$$

Review from the previous lecture

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{a^2}\ , \qquad \ddot a = -\frac{4\pi}{3}G\rho a\ , \qquad \dot\rho = -3\frac{\dot a}{a}\rho\ .$$

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$$\dot\rho = -3\frac{\dot a}{a}\left(\rho + \frac{p}{c^2}\right)\ .$$

So, if we believe the equation for $\dot\rho$, we must modify one of the two Friedmann equations. First order equation represents conservation of energy: pressure does not belong! (Pressures can change suddenly, as when dynamite explodes, so it does not make sense to have pressure in a conservation equation.) So modify the 2nd order equation, deriving it from the first order equation and the $\dot\rho$ equation:

$$\ddot a = -\frac{4\pi}{3}G\left(\rho + \frac{3p}{c^2}\right)a\ .$$

Summary: Complete Friedmann Equations and Energy Conservation

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{a^2}$$

$$\ddot a = -\frac{4\pi}{3}G\left(\rho + \frac{3p}{c^2}\right)a$$

$$\dot\rho = -3\frac{\dot a}{a}\left(\rho + \frac{p}{c^2}\right) .$$

The items in red are new.

Dynamics of a Flat Radiation-dominated Universe

$$H^2 = \frac{8\pi G}{3}\rho \;,\; \rho \propto 1/a^4 \quad \Longrightarrow \quad \left(\frac{\dot a}{a}\right)^2 = \frac{\text{const}}{a^4} .$$

Then

$$a\,da = \sqrt{\text{const}}\,dt \quad \Longrightarrow \quad \frac{1}{2}a^2 = \sqrt{\text{const}}\,t + \text{const}' .$$

So, setting our clocks so that $\text{const}' = 0$,

$$a(t) \propto \sqrt{t} \qquad \text{(flat radiation-dominated)} .$$

$$H(t) = \frac{\dot a}{a} = \frac{1}{2t} \qquad \text{(flat radiation-dominated)} .$$

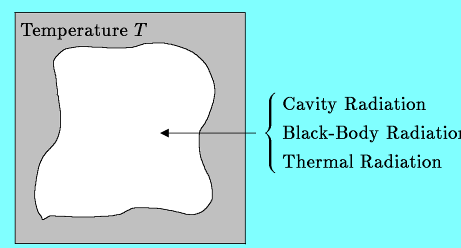
$$\ell_{p,\text{horizon}}(t) = a(t)\int_0^t \frac{c}{a(t')}dt'$$

$$= 2ct \qquad \text{(flat radiation-dominated)} .$$

$$H^2 = \frac{8\pi G}{3}\rho \quad \Longrightarrow \quad \rho = \frac{3}{32\pi G t^2} .$$

Black Body Radiation

- If a cavity is carved out of any material, and the walls are kept at a uniform temperature T, then the cavity will fill with radiation.



- If no radiation can get through the wall, then the energy density and spectrum of the radiation is determined by T alone — the material of the wall is irrelevant.
- The radiation is known as cavity radiation, black-body radiation, or thermal radiation.
- It can be thought of simply as radiation at temperature T.

Why Is It Called Black-Body?

- ☆ A black body at temperature T in empty space emits radiation with exactly this intensity and spectrum.
- ☆ Definitions:
  - A *black* object absorbs all light that hits it; none is reflected or transmitted.
  - Reflection vs. emission: *reflection* is immediate. If the body absorbs radiation and emits it later, that is *emission*.
- ☆ Equilibrium: if a black body were placed in the cavity, it would reach an equilibrium in which no further energy would be exchanged. The body would be at the same temperature T as the box and the cavity radiation.
- ☆ Since the black body absorbs all the radiation that hits it, it must emit exactly this much radiation.

- ☆ Furthermore, in every frequency interval the block must emit exactly as much radiation as it absorbs.

  Otherwise, we could imagine surrounding the body by a filter that transmits only in this frequency interval, and otherwise reflects. If the emission in this interval did not match the aborption, the body would then become hotter or colder than T, which violates a basic property of thermal equilibrium — once it is reached, the temperature will remain uniform, unless energy is exchanged with some external mechanism.
- ☆ Since the black body reflects nothing, all of the emitted radiation is thermal radiation, which will continue even if the body is taken out of the cavity.
- ☆ Thus, a black body at temperature T will emit with exactly the same intensity and spectrum as the radiation in the cavity.

Vague Description of the Black-Body Radiation Calculation

- ☆ We will leave the full derivation of black-body radiation to some stat mech class.
- ☆ But here we will summarize the basic ideas.
- ☆ **Prelude: The "equipartition theorem" of classical stat mech:** each degree of freedom of a system at temperature T acquires a mean thermal energy of $\frac{1}{2}kT$, where k = Boltzmann constant = 8.617×10^{-5} eV/K. For example, a gas of spinless particles has 3 degrees of freedom per atom: the x, y, and z components of velocity. In thermal equilbrium, the thermal energy is $\frac{3}{2}kT$ per particle. A harmonic oscillator has 2 degrees of freedom: its kinetic and potential energies.

- ☆ **Equipartition and the electromagnetic radiation:**
  - Imagine describing the electromagnetic field inside a rectangular box. With reflecting boundary conditions, the field can be described by standing waves, each with an integral number of half wavelengths in each of the 3 directions.
  - There are 2 polarizations (right and left circular polarization, or x and y linear polarization — these are two different bases for the same space of solutions; any polarization can be written as a superposition of left and right circular polarization, OR x and y polarization; either way, it counts as TWO polarizations). Each standing wave, with a specified polarization, is called a *mode*. Each mode is 2 degrees of freedom, like a harmonic oscillator.
  - **Jeans Catastrophe:** The number of modes is **infinite**, since there is no shortest wavelength. If classical physics applied, the electromagnetic field could never reach thermal equilibrium. Instead, it would continue to absorb energy, exciting shorter and shorter wavelength modes. It would be an infinite heat sink, absorbing all thermal energy.

☆ **Quantum Theory to the Rescue:**

- Classically, each mode can be excited by any amount.
- Quantum mechanically, however, a harmonic oscillator with frequency ν can only acquire energy in lumps of size $h\nu$. For the E&M field, each excitation of energy $h\nu$ is a *photon*.
- For modes for which $h\nu \ll kT$, the classical physics works, and each mode acquires energy kT. (Note: Lecture Notes 6 incorrectly states that for $h\nu \ll kT$, each mode acquires energy $\frac{1}{2}kT$ — it's really kT, with the 2 degrees of freedom of a harmonic oscillator.)
- For modes with $h\nu \gg kT$, the typical energy available ($\sim kT$) is much smaller than the minimum possible excitation ($h\nu$). These modes are excited only very rarely. The Jeans catastrophe is avoided, and the total energy density is finite.

Black-Body Radiation: Results

Energy Density:

$$u = g\frac{\pi^2}{30}\frac{(kT)^4}{(\hbar c)^3} ,$$

where

$$\hbar = \frac{h}{2\pi} = 1.055 \times 10^{-27} \text{ erg-sec} = 6.582 \times 10^{-16} \text{ eV-sec} ,$$

and

$$g = 2 \quad \text{(for photons)} .$$

$$u = g\frac{\pi^2}{30}\frac{(kT)^4}{(\hbar c)^3} , \qquad g = 2 \quad \text{(for photons)} .$$

The factor of g is introduced so that the formula will be reusable. We will soon be talking about thermal radiation of other kinds of particles (neutrinos, e^+e^- pairs, and more!), and we'll be able to use the same formula, with different values of g. Very ecological.

For photons, $g = 2$ because the photon has two polarizations, or equivalently, two spin states.

Other Properties

Pressure: $p = \frac{1}{3}u$.

Number Density: $n = g^*\frac{\zeta(3)}{\pi^2}\frac{(kT)^3}{(\hbar c)^3}$,

where $\zeta(3)$ is the Riemann zeta function evaluated at 3,

$$\zeta(3) = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots \approx 1.202 ,$$

and

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and

$$g^* = 2 \quad \text{(for photons)} .$$

g^* is used in the equation for the number density, rather than g, again to maximize reusability. For photons, $g^* = g$, but that won't be true for all particles.

ENTROPY!!

- Entropy is often described as a measure of the "disorder" of the state of a physical system. Roughly, the entropy of a system is k times the logarithm of the number of microscopic quantum states that are consistent with its macroscropically observed state.
- Good news: **we will not really need to know what entropy means!** (We will defer to some stat mech class to make this clear.) However, we will make much use of the fact that, as long as a system remains very close to thermal equilibrium, entropy is conserved. When departures from thermal equilibrium occur, the entropy always increases (a principle called the *second law of thermodynamics*).
- In our model of the universe, a huge amount of entropy was produced at the end of the period of inflation (to be discussed later), but the subsequent expansion and cooling of the universe happens at nearly constant entropy. Once stars form, entropy production resumes.

Entropy Density of Black-Body Radiation

The entropy density s of black-body radiation is given by

$$s = g \frac{2\pi^2}{45} \frac{k^4 T^3}{(\hbar c)^3} .$$

The factor of g that appears here is the same g that occurs in the formulas for energy density and pressure. For photons, g is (still) 2.

Note that the entropy density, like the number density, is proportional to T^3. Thus the ratio

$$\frac{s}{n} = \frac{g}{g^*} 3.60157\, k .$$

For the black-body radiation of photons, entropy is just another way to count photons, with 3.6 k units of entropy per photon.

Neutrinos — A Brief History

- In 1930, Wolfgang Pauli proposed the existence of the neutrino — an unseen particle that he theorized to explain how beta decay ($n \longrightarrow p + e^-$, inside a nucleus) could be consistent with energy conservation. (Niels Bohr, by contrast, proposed that energy conservation was only valid statistically.) Pauli called it a neutron, while the particle that we know as a neutron was not discovered until 1932, by James Chadwick.
- In 1934 Enrico Fermi developed a full theory of beta decay, and gave the neutrino its current name ("little neutral one").
- The neutrino was not seen observationally until 1956 by Clyde Cowan and Frederick Reines at the Savannah River nuclear reactor.
- Cowan died in 1974 at the age of 54, and Reines was awarded the Nobel Prize for this work in 1995, at the age of 77.