8.286 Class 23
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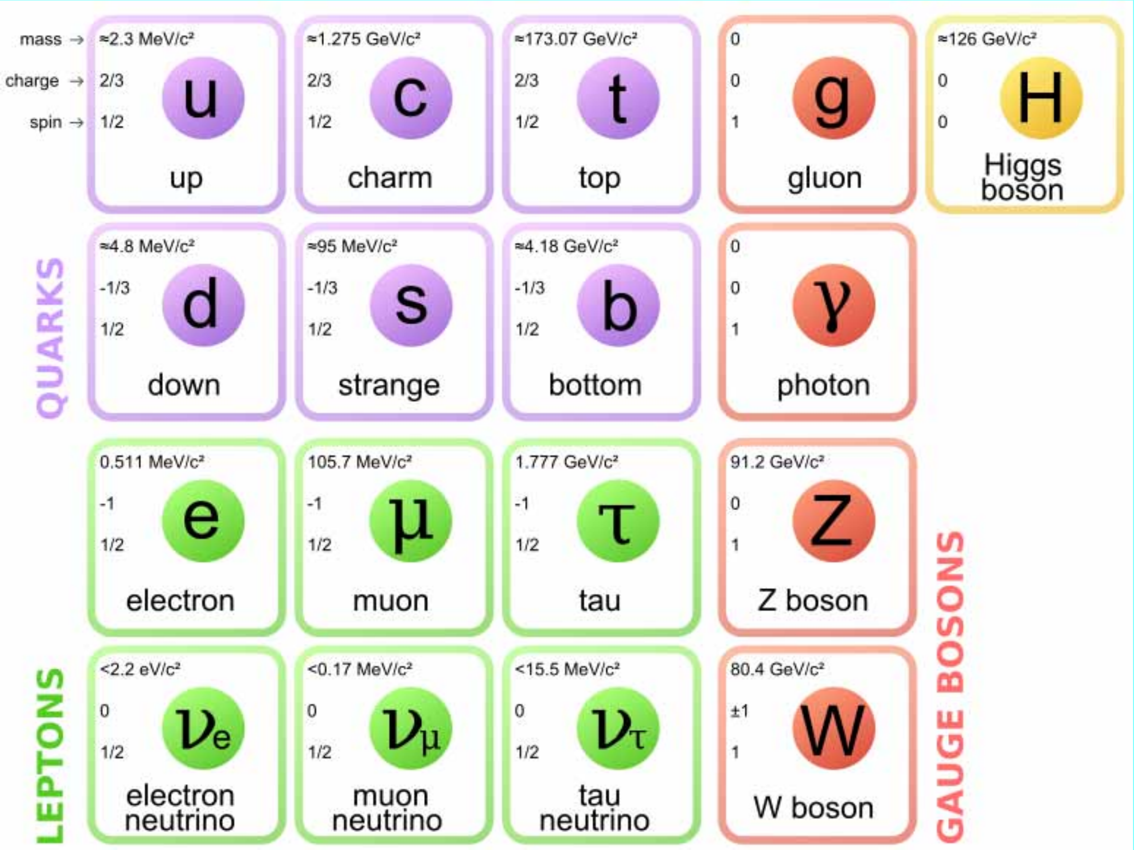
GRAND UNIFIED THEORIES
AND THE
MAGNETIC MONOPOLE
PROBLEM

The Standard Model of Particle Physics

Particle Content:



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Quarks are Colored

- A quark is specified by its *flavor* [u(p), d(own), c(harmed), s(trange), t(op), b(ottom)], its spin [up or down, along any chosen z axis], whether it is a quark or antiquark, AND ITS COLOR [three choices, often red, blue, or green].

- Quarks that differ only in color are completely indistinguishable, but the color is relevant for the Pauli exclusion principle: one can't have 3 identical quarks all in the lowest energy state, but one can have one red quark, one blue quark, and one green quark.

- Color is also relevant for the way quarks interact. The colors act like a generalized form of electric charge. Two red quarks interact with each other exactly the same way as two blue quarks, but a red quark and a blue quark interact with each other differently.

- Any isolated system of quarks must be a "color singlet". The simplest color singlets — and the only ones known to exist in nature — are 3-quark states (baryons), with equal parts of red, blue, and green, and quark-antiquark states (mesons), with equal parts of each color and its anticolor. If one tries to pull a quark from a proton, it is pulled back with a force independent of distance, as if it were attached by a string. (Called "confinement".)

Gauge Theories: Electromagnetic Example

Fields and potentials*: $\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$, $\vec{B} = \vec{\nabla} \times \vec{A}$.

Four-vector notation: $A_\mu = \left(-\frac{\phi}{c}, A_i\right)$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$,

$E_i = cF_{i,0}$, $B_i = \frac{1}{2}\epsilon_{ijk} F_{jk}$.

Gauge transformations:

$$\phi'(t, \vec{x}) = \phi(t, \vec{x}) - \frac{\partial \Lambda(t, \vec{x})}{\partial t} , \quad \vec{A}'(t, \vec{x}) = \vec{A}(t, \vec{x}) + \vec{\nabla}\Lambda(t, \vec{x}) ,$$

or in four-vector notation,

$$A'_\mu(x) = A_\mu(x) + \frac{\partial \Lambda}{\partial x^\mu} , \text{ where } x^\mu \equiv (ct, \vec{x}).$$

$\vec{E}$ and $\vec{B}$ are gauge-invariant (i.e., are unchanged by a gauge transformation):

$$\vec{B}' = \vec{\nabla} \times \vec{A}' = \vec{\nabla} \times (\vec{A} + \vec{\nabla}\Lambda) = \vec{\nabla} \times \vec{A} = \vec{B} ,$$

*Using the conventions of D.J. Griffiths, *Introduction to Electrodynamics*, Fourth Edition.

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$$\vec{E}' = -\vec{\nabla}\phi' - \frac{\partial \vec{A}'}{\partial t} = -\vec{\nabla}\left(\phi - \frac{\partial \Lambda}{\partial t}\right) - \frac{\partial}{\partial t}\left(\vec{A} + \vec{\nabla}\Lambda\right)$$

$$= -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} = \vec{E} \ ,$$

where we used $\vec{\nabla} \times \vec{\nabla}\Lambda \equiv 0$ and $\vec{\nabla}\left(\frac{\partial \Lambda}{\partial t}\right) = \frac{\partial}{\partial t}\vec{\nabla}\Lambda$. So A_μ and A'_μ both satisfy the equations of motion, and describe the SAME physical situation.

Gauge transformations can be combined, forming a group:

$$\Lambda_3(x) = \Lambda_1(x) + \Lambda_2(x) \ .$$

Gauge symmetries are also called local symmetries, since the gauge function $\Lambda(x)$ is an arbitrary function of position and time.

Electromagnetism as a $U(1)$ Gauge Theory

$\Lambda(x)$ is an element of the real numbers.

But if we included an electron field $\psi(x)$, it would transform as

$$\psi'(x) = e^{ie_0 \Lambda(x)} \psi(x) \ ,$$

where e_0 is the charge of a proton and $e = 2.71828\ldots$. So we might think of $u(x) \equiv e^{ie_0 \Lambda(x)}$ as describing the gauge transformation. u contains LESS information than Λ, since it defines Λ only mod $2\pi/e_0$.

But u is enough to define the gauge transformation, since

$$\frac{\partial \Lambda}{\partial x^\mu} = \frac{1}{ie_0}\, e^{-ie_0 \Lambda(x)}\, \frac{\partial}{\partial x^\mu}\, e^{ie_0 \Lambda(x)} \ .$$

u is an element of the group $U(1)$, the group of complex phases $u = e^{i\chi}$, where χ is real. So E&M is a $U(1)$ gauge theory.

Gauge Groups of the Standard Model

$U(1)$ is abelian (commutative), but Yang and Mills showed in 1954 how to construct a nonabelian gauge theory. The standard model contains the following gauge symmetries:

***SU*(3)**: This is the group of 3×3 complex matrices that are

S $\equiv$ Special: they have determinant 1.

U $\equiv$ Unitary: they obey $u^\dagger u = 1$, which means that when they multiply a 1×3 column vector v, they preserve the norm $|v| \equiv \sqrt{v_i^* v_i}$.

***SU*(2)**: The group of 2×2 complex matrices that are special (S) and unitary (U). As you may have learned in quantum mechanics, $SU(2)$ is almost the same as the rotation group in 3D, with a 2:1 group-preserving mapping between $SU(2)$ and the rotation group.

***U*(1)**: The group of complex phases. The $U(1)$ of the standard model is not the $U(1)$ of E&M; instead $U(1)_{\mathrm{E\&M}}$ is a linear combination of the $U(1)$ of the standard model and a rotation about one fixed direction in $SU(2)$.

Combining the groups: the gauge symmetry group of the standard model is usually described as $SU(3)\times SU(2)\times U(1)$. An element of this group is an ordered triplet (u_3, u_2, u_1), where $u_3 \in SU(3)$, $u_2 \in SU(2)$, and $u_1 \in U(1)$, so $SU(3)\times SU(2)\times U(1)$ is really no different from thinking of the 3 symmetries separately.

$SU(3)$ describes the strong interactions, and $SU(2) \times U(1)$ together describe the electromagnetic and weak interactions in a unified way, called the *electroweak interactions*.

$SU(3)$ acts on the quark fields by rotating the 3 "colors" into each other. Thus the strong interactions of the quarks are entirely due to their "colors", which act like charges.

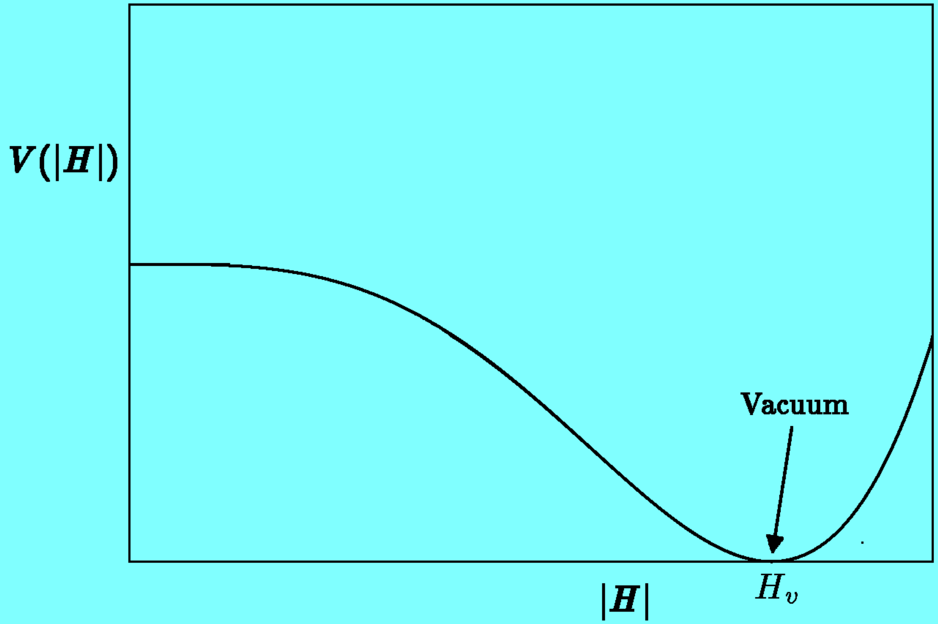
The Higgs Field and Spontaneous Symmetry Breaking

The Higgs field is a complex doublet:

$$H(x) \equiv \begin{pmatrix} h_1(x) \\ h_2(x) \end{pmatrix} \; .$$

Under $SU(2)$ transformations, $H'(x) = u_2(x)H(x)$, where $u_2(x)$ is the complex 2×2 matrix that defines the $SU(2)$ gauge transformation at x. Since the gauge symmetry implies that the potential energy density of the Higgs field $V(H)$ must be gauge-invariant, V can depend only on $|H| \equiv \sqrt{|h_1|^2 + |h_2|^2}$, which is unchanged by $SU(2)$ transformations.

Potential energy function $V(|H|)$:



The minimum is not at $|H| = 0$, but instead at $|H| = H_v$.

$|H| = 0$ is $SU(2)$ gauge-invariant, but $|H| = H_v$ is not. H randomly picks out some direction in the space of 2D complex vectors.

Spontaneous Symmetry Breaking: Whenever the ground state of a system has less symmetry than the underlying laws, it is called spontaneous symmetry breaking. Examples: crystals, ferromagnetism.

Higgs Fields Give Mass to Other Particles

When $H = 0$, all the fundamental particles of the standard model are massless. Furthermore, there is no distinction between the electron e and the electron neutrino ν_e, or between μ and ν_μ, or between τ and ν_τ. (Protons, however, would not be massless — intuitively, most of the proton mass comes from the gluon field that binds the quarks.)

For $|H| \neq 0$, H randomly picks out a direction in the space of 2D complex vectors. Since all directions are otherwise equivalent, we can assume that in the vacuum,

$$H = \begin{pmatrix} H_v \\ 0 \end{pmatrix} .$$

Components of other fields that interact with $\mathrm{Re}\,(h_1)$ then start to behave differently from fields that interact with other components of H.

Mass: mc^2 of a particle is the state of lowest energy above the ground state. In a field theory, this corresponds to a homogeneous oscillation of the field, which in turn corresponds to a particle with zero momentum.

In the free field limit, the field acts exactly like a harmonic oscillator. The first excited state has energy $h\nu = \hbar\omega$ above the ground state. So, $mc^2 = \hbar\omega$.

ω is determined by inertia and the restoring force. When $H = 0$, the standard model interactions provide no restoring forces. Any such restoring force would break gauge invariance.

When $H = \begin{pmatrix} H_v \\ 0 \end{pmatrix}$, the interactions with H creates a restoring force for some components of other fields, giving them a mass. This "Higgs mechanism" creates the distinction between electrons and neutrinos — the electrons are the particles that get a mass, and the neutrinos do not. (Neutrinos are exactly massless in the Standard Model of Particle Physics. There are various ways to modify the model to account for neutrino masses.)