MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

**8.323: Relativistic Quantum Field Theory I**

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## INFORMAL NOTES
# DIRAC DELTA FUNCTION AS A DISTRIBUTION

### Why the Dirac Delta Function is not a Function:

The Dirac delta function $\delta(x)$ is often described by considering a function that has a narrow peak at $x = 0$, with unit total area under the peak. In the limit as the peak becomes infinitely narrow, keeping fixed the area under the peak, the function is sometimes said to approach a Dirac delta function. One example of such a limit is

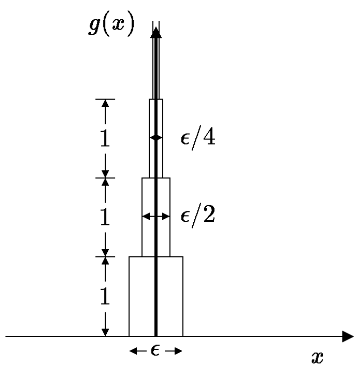
$$g(x) \equiv \lim_{\sigma \to 0} g_\sigma(x) \ , \tag{4.1}$$

where

$$g_\sigma(x) \equiv \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-\frac{1}{2}x^2/\sigma^2} \ . \tag{4.2}$$

The area under $g_\sigma(x)$ is 1, for any value of $\sigma > 0$, and $g_\sigma(x)$ approaches 0 as $\sigma \to 0$ for any $x$ other than $x = 0$.

However, it was pointed out long ago that the delta function cannot be rigorously defined this way. The function $g(x)$ is equal to zero for any $x \neq 0$, and is infinite at $x = 0$; it can be shown that any such function integrates to zero. To see this, define the integral as the area under the curve, and consider the construction:




In this picture the vertical axis is entirely encased in rectangles, each of which has height 1. The width of the rectangles vary, with the lowest rectangle having width $\epsilon$, for some $\epsilon > 0$, and each successive rectangle has half the width of the rectangle below. Note that the outline of the boxes is everywhere above the curve $g(x)$, so



the area under $g(x)$ must be less than the total area of the boxes. But the total area of the boxes is given by a geometric series,

$$A = \epsilon + \frac{1}{2}\epsilon + \frac{1}{4}\epsilon + \ldots \leq 2\epsilon \ . \tag{4.3}$$

Since $\epsilon$ can be chosen as small as one likes, the area under the limit function $g(x)$ must be zero.

This result does not contradict the statement that the area under $g_\sigma(x)$ is 1 for any $\sigma > 0$. Rather, this is a case where the limit of an integral is not the same as the integral of the limit of the integrand. The integral has the value 1 for every $\sigma > 0$, so the limit of the integral as $\sigma \to 0$ is 1. However, if one takes the limit of the integrand first, and then integrates, the answer is zero.

### Dirac Delta Function as a Distribution:

A Dirac delta function is defined to have the property that

$$\boxed{\int_{-\infty}^{\infty} \mathrm{d}x\, \varphi(x)\, \delta(x-a) \equiv \varphi(a) \ .} \tag{4.4}$$

But we have just seen that there is no function $\delta(x)$ which has this property, as long as integration is defined by the area under a curve. However, there is no problem defining a distribution that behaves this way. Using the notation we defined earlier for distributions, we define

$$\boxed{T_{\delta(x-a)}[\varphi] \equiv \varphi(a) \ .} \tag{4.5}$$

Here $T_{\delta(x-a)}$ is the name that we will use for the distribution that acts like integration over the delta function $\delta(x-a)$, and $\varphi(x)$ is the test function. This is certainly a linear functional that is defined on all Schwartz functions, and therefore is a tempered distribution. We interpret Eqs. (4.4) and (4.5) as meaning the same thing, where the second form emphasizes the definition as a distribution, and the first form emphasizes that distributions can be viewed as a generalized kind of function, with a notation that makes them look like functions. That is, we define

$$\boxed{\int_{-\infty}^{\infty} \mathrm{d}x\, \varphi(x)\, \delta(x-a) \equiv T_{\delta(x-a)}[\varphi] \equiv \varphi(a) \ .} \tag{4.6}$$

To put it differently, we must remember that an integral over a delta function, such as Eq. (4.4), is not defined as a standard integral— instead it is symbolic integral, which is defined as a distribution which maps the function that multiplies the delta function to its value at the point where the argument of the delta function vanishes.

**What about $\int \mathrm{d}p\, e^{-ip(x-a)}$ ?**

If we interpret this integral in the sense of Riemann or Lebesgue, it simply does not exist — it diverges. As a distribution, however, we can interpret

$$\boxed{I(x) \equiv \int_{-\infty}^{\infty} \mathrm{d}p\, e^{-ip(x-a)}} \tag{4.7}$$

as the Fourier transform of

$$f(p) = e^{ipa} \ . \tag{4.8}$$

Since $I(x)$ is the Fourier transform of the distribution $T_f[\varphi]$, it must itself be a well-defined distribution. Using the notation of distributions, the distribution corresponding to $I$ is then given by

$$T_I[\varphi] = \tilde{T}_f[\varphi] \ . \tag{4.9}$$

To evaluate the distribution $T_I[\varphi]$, we first write the distribution corresponding to function $f(p)$ in the usual way:

$$T_f[\varphi] = \int_{-\infty}^{\infty} \mathrm{d}p\, f(p)\, \varphi(p) = \int_{-\infty}^{\infty} \mathrm{d}p\, e^{ipa} \varphi(p) \ . \tag{4.10}$$

The Fourier transform of this distribution is then defined by applying the same distribution to the Fourier transform of the test function, so

$$\tilde{T}_f[\varphi] \equiv T_f[\tilde{\varphi}] = \int_{-\infty}^{\infty} \mathrm{d}p\, e^{ipa}\, \tilde{\varphi}(p) \quad . \tag{4.11}$$

But the inverse Fourier transform is given by

$$\varphi(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}p\, e^{ipx}\, \tilde{\varphi}(p) \ , \tag{4.12}$$

so by comparing the two formulas above, one sees that

$$\tilde{T}_f[\varphi] = 2\pi\varphi(a) \ . \tag{4.13}$$

But this is exactly the definition of the delta function distribution, as given in Eq. (4.6), so

$$\boxed{\tilde{T}_f[\varphi] = 2\pi\, T_{\delta(x-a)}[\varphi] \ .} \tag{4.14}$$



If we write these distributions as symbolic integrals, then the above equation becomes

$$\int_{-\infty}^{\infty} \mathrm{d}x\, I(x)\varphi(x) = 2\pi \int_{-\infty}^{\infty} \mathrm{d}x\, \delta(x-a)\varphi(x) \ , \tag{4.15}$$

where we remembered from Eq. (4.9) that $\tilde{T}_f[\varphi]$ is equal to the distribution corresponding to $I(x)$. Since Eq. (4.15) holds for an arbitrary test function $\varphi(x)$, we can equate the distributions without showing their arguments:

$$\boxed{I(x) = \int_{-\infty}^{\infty} \mathrm{d}p\, e^{-ip(x-a)} = 2\pi\delta(x-a) \ .} \tag{4.16}$$

**The Derivative of a Delta Function:**

If a Dirac delta function is a distribution, then the derivative of a Dirac delta function is, not surprisingly, the derivative of a distribution. We have not yet defined the derivative of a distribution, but it is defined in the obvious way. We first consider a distribution corresponding to a function, and ask what would be the distribution corresponding to the derivative of the function.

Starting with a well-behaved (i.e., piecewise continuous and bounded by some power of $t$) function $f(t)$, we defined the corresponding distribution by

$$T_f[\varphi] \equiv \int_{-\infty}^{\infty} \mathrm{d}t\, f(t)\varphi(t) \quad . \tag{4.17}$$

Then if we write the distribution corresponding to $\mathrm{d}f/\mathrm{d}t$, we get

$$T_{\mathrm{d}f/\mathrm{d}t}[\varphi] = \int_{-\infty}^{\infty} \mathrm{d}t\, \frac{\mathrm{d}f}{\mathrm{d}t}\, \varphi(t) \tag{4.18}$$

Since $f(t)$ is bounded for large $|t|$ by a power of $t$, and $\varphi(t)$ falls off faster than any power, we can integrate by parts without encountering a surface term:

$$T_{\mathrm{d}f/\mathrm{d}t}[\varphi] = -\int_{-\infty}^{\infty} \mathrm{d}t\, f(t)\, \frac{\mathrm{d}\varphi}{\mathrm{d}t} = -T_f\left[\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right] \ . \tag{4.19}$$

This result can then be taken as the general definition of the derivative of a distribution:

$$\boxed{T'[\varphi] \equiv -T\left[\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right] \ .} \tag{4.20}$$

Applying this result to a delta function in the notation of Eq. (4.6), it looks exactly like integration by parts:

$$\boxed{\int_{-\infty}^{\infty} \mathrm{d}t\, f(t)\frac{\mathrm{d}\delta(t-a)}{\mathrm{d}t} = -\left.\frac{\mathrm{d}f}{\mathrm{d}t}\right|_{t=a} \ .} \tag{4.21}$$