MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

8.323: Relativistic Quantum Field Theory I

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QUANTUM MECHANICS AND PATH INTEGRALS

The goal of this section is to derive the path integral formulation of quantum mechanics.

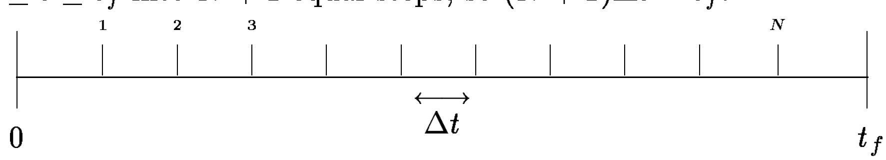
Consider first a free particle, moving in one dimension:

$$H = \frac{p^2}{2m} \quad . \tag{5.1}$$

The evolution of a state is described by applying the operator $U(t_f) \equiv e^{-iHt_f/\hbar}$. Let

$$U_{fi} \equiv \left\langle x_f \left| e^{-iHt_f/\hbar} \right| x_i \right\rangle \quad . \tag{5.2}$$

To develop a path integral expression for this matrix element, begin by dividing the interval $0 \le t \le t_f$ into $N+1$ equal steps, so $(N+1)\Delta t = t_f$:



Now express the evolution operator $e^{-iHt_f/\hbar}$ as the product of an evolution operator for each interval Δt:

$$e^{-iHt_f/\hbar} = \left(e^{-iH\Delta t/\hbar}\right)^{N+1} \quad . \tag{5.3}$$

Then insert a complete set of intermediate states between each factor, using

$$1 = \int_{-\infty}^{\infty} dx \, |x\rangle\langle x| \quad . \tag{5.4}$$

Calling the variables of integration $x_1, x_2, \ldots, x_N$,

$$\begin{aligned} U_{fi} = \int_{-\infty}^{\infty} & dx_1 \ldots dx_N \, \left\langle x_f \left| e^{-iH\Delta t/\hbar} \right| x_N \right\rangle \\ & \times \left\langle x_N \left| e^{-iH\Delta t/\hbar} \right| x_{N-1} \right\rangle \ldots \left\langle x_1 \left| e^{-iH\Delta t/\hbar} \right| x_i \right\rangle \quad . \end{aligned} \tag{5.5}$$

The matrix elements in the above expression can be evaluated exactly by using a momentum space representation:

$$\begin{aligned} \left\langle y \left| e^{-iH\Delta t/\hbar} \right| x \right\rangle &= \int_{-\infty}^{\infty} dp \, \langle y|p\rangle \exp\left\{-i\frac{p^2}{2m\hbar}\Delta t\right\} \langle p|x\rangle \\ &= \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dp \, \exp\left\{ip(y-x)/\hbar\right\} \exp\left\{-i\frac{p^2}{2m\hbar}\Delta t\right\} \end{aligned} \quad . \tag{5.6}$$

Completing the square,

$$\left\langle y \left| e^{-iH\Delta t/\hbar} \right| x \right\rangle = \frac{1}{2\pi\hbar} \exp\left\{ \frac{i}{\hbar} \frac{m(y-x)^2}{2\Delta t} \right\} \times \int_{-\infty}^{\infty} dp \exp\left\{ -\frac{i\Delta t}{2m\hbar} \left[p - \frac{m(y-x)}{\Delta t} \right]^2 \right\} \quad . \tag{5.7}$$

The integrand oscillates wildly at high p, and the integral is conditionally convergent. It can be rendered absolutely convergent by assigning an infinitesimal negative imaginary part to Δt. Then apply the formula

$$\int_{-\infty}^{\infty} dp\, e^{-ap^2} = \sqrt{\frac{\pi}{a}} \qquad (\mathrm{Re}\, a > 0) \tag{5.8}$$

to give

$$\left\langle y \left| e^{-iH\Delta t/\hbar} \right| x \right\rangle = \sqrt{\frac{m}{2\pi i\hbar\Delta t}} \exp\left\{ \frac{i}{\hbar} \frac{m}{2} \left(\frac{y-x}{\Delta t} \right)^2 \Delta t \right\} \quad . \tag{5.9}$$

This expression is then inserted back into Eq. (5.5), and then one in principle takes the limit $N \to \infty$:

$$\begin{aligned} U_{fi} = \lim_{N\to\infty} & \left(\frac{m}{2\pi i\hbar\Delta t} \right)^{(N+1)/2} \int_{-\infty}^{\infty} dx_1 \ldots dx_N \\ & \times \exp\left\{ \frac{i}{\hbar} \left[\frac{m}{2} \left(\frac{x_f - x_N}{\Delta t} \right)^2 + \frac{m}{2} \left(\frac{x_N - x_{N-1}}{\Delta t} \right)^2 + \right. \right. \\ & \left. \left. + \ldots + \frac{m}{2} \left(\frac{x_1 - x_i}{\Delta t} \right)^2 \right] \Delta t \right\} \quad . \end{aligned} \tag{5.10}$$

Although the above expression looks at first like a mess, it is actually the endpoint of the calculation. The quantity is called a path integral, or sometimes a functional integral, and it may be denoted more compactly by

$$U_{fi} = \int_{x(0)=x_i}^{x(t_f)=x_f} \mathcal{D}x(t) \exp\left\{ \frac{i}{\hbar} \int_0^{t_f} dt\, \frac{1}{2} m\dot{x}^2 \right\} \quad . \tag{5.11}$$

Note that the overall normalization of the path integral is quite complicated, but fortunately it will almost never be necessary to know it. In practical applications, whenever the normalization is needed it is determined by using the unitarity of the operator $U(t)$, or, equivalently, the unit normalization of the final state wave function.

So far we have done only the free particle, so the next step is to include a potential energy function in the Hamiltonian:

$$H = \frac{p^2}{2m} + V(x) \quad . \tag{5.12}$$

One proceeds by dividing the time interval and inserting intermediate states as before. The crucial difference comes in the evaluation of the matrix element

$$\left\langle y \left| e^{-iH\Delta t/\hbar} \right| x \right\rangle \quad , \tag{5.13}$$

which now involves a Hamiltonian with a potential. The easiest way to proceed is to use the identity

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\cdots} \quad . \tag{5.14}$$

This relation is known as the Baker–Campbell–Hausdorff identity. The terms omitted on the right-hand-side are all constructed from higher order iterated commutators of A and B, and it is somewhat difficult to prove the theorem in general. We, however, will need the theorem only to the order shown, and to this order (or any finite order) it is straightforward to demonstrate the relation by expanding both sides in a power series and then comparing. Thus

$$\exp\left\{-i\frac{p^2}{2m\hbar}\Delta t\right\} \exp\left\{-iV\Delta t/\hbar\right\} = \exp\left\{-\frac{i}{\hbar}\left(\frac{p^2}{2m}+V\right)\Delta t + \mathcal{O}(\Delta t^2)\right\} \quad . \tag{5.15}$$

The $\mathcal{O}(\Delta t^2)$ correction gives no contribution in the $N \to \infty$ ($\Delta t \to 0$) limit, so to the accuracy required we can take the evolution operator to be the operator on the left-hand side of Eq. (5.15). The potential energy factor then operates on the position-space eigenstate to the right, and can be taken outside the matrix element. The remaining matrix element is the one already evaluated, so

$$\begin{aligned}\left\langle y \left| e^{-iH\Delta t/\hbar} \right| x \right\rangle = \sqrt{\frac{m}{2\pi i\hbar\Delta t}} \exp\left\{\frac{i}{\hbar}\left[\frac{m}{2}\left(\frac{y-x}{\Delta t}\right)^2 - V(x)\right]\Delta t\right\} \\ + \text{negligible terms higher order in } \Delta t \quad .\end{aligned} \tag{5.16}$$

Putting this into an expression of the form of Eq. (5.5), taking the $N \to \infty$ limit, and using the compact notation of Eq. (5.11), one has

$$U_{fi} = \int_{x(0)=x_i}^{x(t_f)=x_f} \mathcal{D}x(t) \, \exp\left\{\frac{i}{\hbar}\int_0^{t_f} dt \left[\frac{1}{2}m\dot{x}^2 - V(x)\right]\right\} \quad . \tag{5.17}$$

Note that for any given path $x(t)$ the classical action is defined by the functional

$$\begin{aligned} S\left[x(t)\right] &= \int_0^{t_f} dt\, L(t) \qquad (L = \text{Lagrangian}) \\ &= \int_0^{t_f} dt \left\{ \frac{1}{2} m\dot{x}^2 - V(x) \right\} \quad , \end{aligned} \tag{5.18}$$

so

$$\boxed{U_{fi} = \int\limits_{x(0)=x_i}^{x(t_f)=x_f} \mathcal{D}x(t)\, e^{\frac{i}{\hbar} S[x(t)]} \quad .} \tag{5.19}$$