Equation of motion:

\[ (\Box + m^2)\phi(x) = j(x) . \]  

Initial condition:

\[ \phi(x) = \phi_{in}(x) . \]  

Eqs. (2.1) and (2.2) \( \implies \) unique solution for Heisenberg operator \( \phi(x) \).

Solution:

\[ \phi(x) = \phi_{in}(x) + i \int d^4y D_R(x - y)j(y) , \]  

where \( D_R(x - y) \) is the retarded propagator:

\[ (\Box_x + m^2)D_R(x - y) = -i\delta^{(4)}(x - y) \]  

where \( D_R(x - y) = 0 \) if \( x^0 < y^0 \) (retarded).
We know that
\[ D_R(x - y) = \theta(x^0 - y^0) \langle 0 | [\phi_{in}(x), \phi_{in}(y)] | 0 \rangle \]
\[ = \theta(x^0 - y^0) \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \left[ e^{-ip \cdot (x-y)} - e^{ip \cdot (x-y)} \right] \]
\[ p^0 = E_p = \sqrt{p^2 + m^2} \]  
(2.5)

Note that \( D_R(x - y) \) is defined by the free wave equation. It can be written in terms of \( [\phi_{in}(x), \phi_{in}(y)] \) as above, or in terms of \( [\phi_{out}(x), \phi_{out}(y)] \), but not in terms of \( [\phi(x), \phi(y)] \).

\( \theta(x^0 - y^0) \) in \( D_R \) is hard to deal with, but for \( x^0 \equiv t > t_2 \) we can set \( \theta(x^0 - y^0) = 1 \). Then
\[ \phi(x) = \phi_{in}(x) + i \int d^4 y \, j(y) \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \left[ e^{-ip \cdot (x-y)} - e^{ip \cdot (x-y)} \right] \]  
(2.6)

Repeating,
\[ \phi(x) = \phi_{in}(x) + i \int d^4 y \, j(y) \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \left[ e^{-ip \cdot (x-y)} - e^{ip \cdot (x-y)} \right] \]  
(2.6)

Define
\[ \tilde{j}(p) \equiv \int d^4 y \, e^{ip \cdot y} j(y) \]  
(2.7)

so
\[ \phi(x) = \phi_{in}(x) + i \int d^3p \, \frac{1}{2E_p} \left[ \tilde{j}(p)e^{-ip \cdot x} - \tilde{j}(-p)e^{ip \cdot x} \right] \]
\[ = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ a_{in}(\vec{p}) + \frac{i}{\sqrt{2E_p}} \tilde{j}(p) \right\} e^{-ip \cdot x} \]
\[ + \left\{ a_{in}^\dagger(\vec{p}) - \frac{i}{\sqrt{2E_p}} \tilde{j}(-p) \right\} e^{ip \cdot x} \]
\[ = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left[ a_{out}(\vec{p})e^{-ip \cdot x} + \text{h.c.} \right] \]  
(2.8)
So

\[
\begin{align*}
    a_{\text{out}}(\vec{p}) &= a_{\text{in}}(\vec{p}) + \frac{i}{\sqrt{2E_p}} \tilde{j}(p) \\
    a_{\text{out}}^\dagger(\vec{p}) &= a_{\text{in}}^\dagger(\vec{p}) - \frac{i}{\sqrt{2E_p}} \tilde{j}(-p),
\end{align*}
\]  

(2.9)

where

\[
\tilde{j}(-p) = \tilde{j}^*(p),
\]  

(2.10)

since \( j(x) \) is real, and

\[
p^0 = \sqrt{\vec{p}^2 + m^2}.
\]  

(2.11)

Thus, only the mass shell component \( (p^0 = \sqrt{\vec{p}^2 + m^2}) \) of \( j(p) \) results in particle creation. This is just the classical phenomenon of resonance occurring in the quantum field theory setting.

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**Unitary Transformation Between In and Out**

It is useful to construct a unitary transformation that relates in and out quantities. Remembering that \( D_R(x-y) = \theta(x^0 - y^0) \langle 0 | [\phi_{\text{in}}(x), \phi_{\text{in}}(y)] | 0 \rangle \), recall also that \( [\phi_{\text{in}}(x), \phi_{\text{in}}(y)] \) is a c-number, so \( \langle 0 | [\phi_{\text{in}}(x), \phi_{\text{in}}(y)] | 0 \rangle = [\phi_{\text{in}}(x), \phi_{\text{in}}(y)] \). So for \( x^0 \equiv t > t_2 \),

\[
\phi(x) = \phi_{\text{out}}(x) = \phi_{\text{in}}(x) + i \int d^4y \ [\phi_{\text{in}}(x), \phi_{\text{in}}(y)] j(y).
\]  

(2.12)

If we define

\[
B \equiv \int d^4y j(y) \phi_{\text{in}}(y),
\]  

(2.13)

then

\[
\phi_{\text{out}}(x) = \phi_{\text{in}}(x) + i [\phi_{\text{in}}(x), B].
\]  

(2.14)

But \( [\phi_{\text{in}}(x), B] \) is also a c-number, so we can write

\[
\phi_{\text{out}}(x) = e^{-iB} \phi_{\text{in}}(x) e^{iB}.
\]  

(2.15)
Since
\[ \varphi_{\text{out}}(x) = e^{-iB} \varphi_{\text{in}}(x) e^{iB}, \tag{2.15} \]
we know from the uniqueness of the Fourier expansion that
\[ a_{\text{out}}(\vec{p}) = e^{-iB} a_{\text{in}}(\vec{p}) e^{iB}. \tag{2.16} \]

We can also verify that this equation is true by using
\[ a_{\text{out}}(\vec{p}) = a_{\text{in}}(\vec{p}) + \frac{i}{\sqrt{2E_p}} \vec{j}(p) \tag{2.9a} \]
with
\[ \left[ a_{\text{in}}(\vec{p}), a_{\text{in}}^\dagger(\vec{q}) \right] = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}). \tag{2.17} \]

Define
\[ S \equiv e^{iB} \text{(the FAMOUS } S\text{-Matrix)} \tag{2.18} \]

Mapping of states:
\[ a_{\text{out}}(\vec{p}) \ket{0_{\text{out}}} = 0 \]
\[ S^{-1} a_{\text{in}}(\vec{p}) S \ket{0_{\text{out}}} = 0 \tag{2.19} \]
\[ \implies a_{\text{in}}(\vec{p}) S \ket{0_{\text{out}}} = 0 \]

This implies, up to a phase, the \( S \ket{0_{\text{out}}} = \ket{0_{\text{in}}} \). We can redefine the phase of \( \ket{0_{\text{out}}} \) (or \( \ket{0_{\text{in}}} \)) so that
\[ S \ket{0_{\text{out}}} = \ket{0_{\text{in}}}. \tag{2.20} \]
On one particle states,

\[ S |\vec{p}_{\text{out}}\rangle = S a_{\text{out}}^\dagger(\vec{p}) |0_{\text{out}}\rangle = S a_{\text{out}}^\dagger(\vec{p}) S^{-1} S |0_{\text{out}}\rangle = |\vec{p}_{\text{in}}\rangle \]  

(2.21)

In general, we could show that

\[ S |\vec{p}_1 \ldots \vec{p}_N_{\text{out}}\rangle = |\vec{p}_1 \ldots \vec{p}_N_{\text{in}}\rangle . \]

(2.22)

We know that

\[ S = e^{iB} = e^{i \int d^4 y j(y) \phi_{\text{in}}(y)} . \]

(2.23)

It is useful to write \( S \) so that all the annihilation operators are on the right. Let

\[ iB = i \int d^4 y j(y) \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left[ a_{\text{in}}(\vec{p}) e^{-ip \cdot y} + a_{\text{in}}^\dagger(\vec{p}) e^{ip \cdot y} \right] = G + F , \]

(2.24)

where

\[ F = i \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \bar{\jmath}(p) a_{\text{in}}^\dagger(\vec{p}) , \quad G = i \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \bar{\jmath}(-p) a_{\text{in}}(\vec{p}) , \]

(2.25)

where we recall that

\[ \bar{\jmath}(p) \equiv \int d^4 y e^{ip \cdot y} j(y) . \]

(2.7)
So
\[ S = e^{iB} = e^{F+G} . \]  
(2.26)

\( F \) and \( G \) do not commute, but \([F , G]\) is a c-number and therefore commutes with both \( F \) and \( G \). Whenever \( F \) and \( G \) commute with \([F , G]\),

\[ e^{F+G} = e^F e^G e^{-\frac{1}{2}[F , G]} . \]  
(2.27)

\textbf{Aside about } e^{F+G} = e^F e^G e^{-\frac{1}{2}[F , G]}

To prove this identity, define
\[ H_1(\lambda) \equiv e^{\lambda(F+G)} , \quad H_2(\lambda) = e^{\lambda F} e^{\lambda G} e^{-\frac{1}{2} \lambda^2 [F,G]} . \]  
(2.28)

Clearly \( H_1(0) = H_2(0) = I \) (identity operator), and

\[ \frac{dH_1(\lambda)}{d\lambda} = (F + G) H_1(\lambda) . \]  
(2.29)

So if we can show that \( H_2(\lambda) \) obeys the same differential equation as above, then it follows that \( H_2(\lambda) = H_1(\lambda) \). You’ll get to show this on your next problem set.

This is actually a special case of the Baker-Campbell-Hausdorff formula, which has the general form

\[ e^F e^G = e^{F+G+\frac{1}{2}[F,G]+\cdots(\text{iterated commutators})} . \]  
(2.30)

We’ll prove this, too, on a problem set soon.
Returning to the main argument:

So

\[ S = e^{iB} = e^{F+G}. \] (2.26)

\( F \) and \( G \) do not commute, but \([F, G]\) is a c-number and therefore commutes with both \( F \) and \( G \). Whenever \( F \) and \( G \) commute with \([F, G]\),

\[ e^{F+G} = e^F e^G e^{-\frac{1}{2}[F, G]}. \] (2.27)

Recalling

\[ F = i \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \tilde{j}(p) a^\dagger_{in}(\vec{p}) , \quad G = i \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \tilde{j}(-\vec{p}) a_{in}(\vec{p}), \] (2.25)

one sees that

\[ [F, G] = -\int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_q}} \tilde{j}(p) \tilde{j}(-q) [a^\dagger_{in}(\vec{p}), a_{in}(\vec{q})] \]

\[ = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} |\tilde{j}(p)|^2 . \] (2.31)

So

\[ S = e^{iB} = \exp \left\{ -\frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} |\tilde{j}(p)|^2 \right\} e^F e^G . \] (2.32)
The probability that no particles are produced by the source is given by
\[ P(\text{no particle production}) = |\langle 0_{\text{out}} | 0_{\text{in}} \rangle|^2. \]  
(2.33)

Logic: physical state is \( |0_{\text{in}}\rangle \), independent of time (in the Heisenberg picture).

\( |0_{\text{out}}\rangle = \text{state with no particles for } t > t_2. \)

So, \( |\langle 0_{\text{out}} | 0_{\text{in}} \rangle|^2 \) is the probability that the physical state of the system would be measured to have 0 particles at \( t > t_2 \). To express the answer in terms of the \( S \)-matrix, recall

\[ |0_{\text{out}}\rangle = S^{-1} |0_{\text{in}}\rangle \Rightarrow \langle 0_{\text{out}} | = \langle 0_{\text{in}} | S. \]  
(2.34)

So

\[ \langle 0_{\text{out}} | 0_{\text{in}} \rangle = \langle 0_{\text{in}} | S | 0_{\text{in}} \rangle \]

\[ = \exp \left\{ -\frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} |\tilde{f}(p)|^2 \right\} \langle 0_{\text{in}} | e^{F_G} | 0_{\text{in}} \rangle. \]  
(2.35)