MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

8.323: Relativistic Quantum Field Theory I

Particle Creation by a Classical Source

(Part II, and incomplete)

March 18, 2008

— Alan Guth

Solution to Differential Equation

Equation of motion:

$$(\Box + m^2)\phi(x) = j(x) \; . \tag{2.1}$$

Initial condition:

$$\phi(x) = \phi_{\text{in}}(x) \; . \tag{2.2}$$

Eqs. (2.1) and (2.2) $\Longrightarrow$ unique solution for Heisenberg operator $\phi(x)$.

Solution:

$$\phi(x) = \phi_{\text{in}}(x) + i \int \mathrm{d}^4 y \, D_R(x-y) j(y) \; , \tag{2.3}$$

where $D_R(x-y)$ is the retarded propagator:

$$\begin{gathered} (\Box_x + m^2) D_R(x-y) = -i\delta^{(4)}(x-y) \\ \text{where } D_R(x-y) = 0 \text{ if } x^0 < y^0 \text{ (retarded)} \; . \end{gathered} \tag{2.4}$$

We know that

$$D_R(x-y) = \theta(x^0 - y^0)\,\langle 0\,|[\phi_{\rm in}(x)\,,\,\phi_{\rm in}(y)]|\,0\rangle$$
$$= \theta(x^0 - y^0)\int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2E_p}\left[e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}\right]_{p^0=E_p=\sqrt{\vec{p}^{\,2}+m^2}} \quad (2.5)$$

Note that $D_R(x-y)$ is defined by the **free** wave equation. It can be written in terms of $[\phi_{\rm in}(x)\,,\,\phi_{\rm in}(y)]$ as above, or in terms of $[\phi_{\rm out}(x)\,,\,\phi_{\rm out}(y)]$, but **not** in terms of $[\phi(x)\,,\,\phi(y)]$.

$\theta(x^0-y^0)$ in D_R is hard to deal with, but for $x^0 \equiv t > t_2$ we can set $\theta(x^0-y^0) = 1$. Then

$$\phi(x) = \phi_{\rm in}(x) + i\int d^4y\, j(y)\int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2E_p}\left[e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}\right] \;. \quad (2.6)$$

Repeating,

$$\phi(x) = \phi_{\rm in}(x) + i\int d^4y\, j(y)\int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2E_p}\left[e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}\right] \;. \quad (2.6)$$

Define

$$\tilde{\jmath}(p) \equiv \int d^4y\, e^{ip\cdot y}\, j(y)\;, \quad (2.7)$$

so

$$\begin{aligned}\phi(x) &= \phi_{\rm in}(x) + i\int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2E_p}\left[\tilde{\jmath}(p)e^{-ip\cdot x} - \tilde{\jmath}(-p)e^{ip\cdot x}\right]\\ &= \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{\sqrt{2E_p}}\left\{\left[a_{\rm in}(\vec{p}\,) + \frac{i}{\sqrt{2E_p}}\tilde{\jmath}(p)\right]e^{-ip\cdot x}\right.\\ &\qquad\left. + \left[a^\dagger_{\rm in}(\vec{p}\,) - \frac{i}{\sqrt{2E_p}}\tilde{\jmath}(-p)\right]e^{ip\cdot x}\right\}\\ &= \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{\sqrt{2E_p}}\left[a_{\rm out}(\vec{p}\,)e^{-ipx} + \text{h.c.}\right]\;.\end{aligned} \quad (2.8)$$

So

$$\boxed{\begin{aligned} a_{\text{out}}(\vec{p}) &= a_{\text{in}}(\vec{p}) + \frac{i}{\sqrt{2E_p}}\,\tilde{\jmath}(p) \\ a^\dagger_{\text{out}}(\vec{p}) &= a^\dagger_{\text{in}}(\vec{p}) - \frac{i}{\sqrt{2E_p}}\,\tilde{\jmath}(-p)\ , \end{aligned}} \tag{2.9}$$

where

$$\tilde{\jmath}(-p) = \tilde{\jmath}^*(p)\ , \tag{2.10}$$

since $j(x)$ is real, and

$$p^0 = \sqrt{\vec{p}^{\,2} + m^2}\ . \tag{2.11}$$

Thus, only the mass shell component ($p^0 = \sqrt{\vec{p}^{\,2} + m^2}$) of $\tilde{\jmath}(p)$ results in particle creation. This is just the classical phenomenon of resonance occurring in the quantum field theory setting.

Unitary Transformation Between In and Out

It is useful to construct a unitary transformation that relates in and out quantities.

Remembering that $D_R(x-y) = \theta(x^0 - y^0)\,\langle 0\,|[\phi_{\text{in}}(x)\,,\,\phi_{\text{in}}(y)]|\,0\rangle$, recall also that $[\phi_{\text{in}}(x)\,,\,\phi_{\text{in}}(y)]$ is a c-number, so $\langle 0\,|[\phi_{\text{in}}(x)\,,\,\phi_{\text{in}}(y)]|\,0\rangle = [\phi_{\text{in}}(x)\,,\,\phi_{\text{in}}(y)]$. So for $x^0 \equiv t > t_2$,

$$\phi(x) = \phi_{\text{out}}(x) = \phi_{\text{in}}(x) + i\int \mathrm{d}^4y\ [\phi_{\text{in}}(x)\,,\,\phi_{\text{in}}(y)]\ j(y)\ . \tag{2.12}$$

If we define

$$B \equiv \int \mathrm{d}^4y\, j(y)\,\phi_{\text{in}}(y)\ , \tag{2.13}$$

then

$$\phi_{\text{out}}(x) = \phi_{\text{in}}(x) + i\,[\phi_{\text{in}}(x)\,,\,B]\ . \tag{2.14}$$

But $[\phi_{\text{in}}(x)\,,\,B]$ is also a c-number, so we can write

$$\boxed{\phi_{\text{out}}(x) = e^{-iB}\,\phi_{\text{in}}(x)\,e^{iB}\ .} \tag{2.15}$$

Since

$$\boxed{\phi_{\text{out}}(x) = e^{-iB}\,\phi_{\text{in}}(x)\,e^{iB}\ ,} \tag{2.15}$$

we know from the uniqueness of the Fourier expansion that

$$\boxed{a_{\text{out}}(\vec{p}\,) = e^{-iB}\,a_{\text{in}}(\vec{p}\,)e^{iB}\ .} \tag{2.16}$$

We can also verify that this equation is true by using

$$a_{\text{out}}(\vec{p}\,) = a_{\text{in}}(\vec{p}\,) + \frac{i}{\sqrt{2E_p}}\,\tilde{\jmath}(p) \tag{2.9a}$$

with

$$\left[a_{\text{in}}(\vec{p}\,)\,,\,a^{\dagger}_{\text{in}}(\vec{q}\,)\right] = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q}\,)\ . \tag{2.17}$$

The S-Matrix

Define

$$\boxed{S \equiv e^{iB}\quad \text{(the **FAMOUS** } S\text{-Matrix)}} \tag{2.18}$$

Mapping of states:

$$\begin{aligned} a_{\text{out}}(\vec{p}\,)\ |0_{\text{out}}\rangle &= 0 \\ S^{-1}a_{\text{in}}(\vec{p}\,)S\ |0_{\text{out}}\rangle &= 0 \\ \Longrightarrow\quad a_{\text{in}}(\vec{p}\,)\,S\,|0_{\text{out}}\rangle &= 0 \end{aligned} \tag{2.19}$$

This implies, up to a phase, the $S\,|0_{\text{out}}\rangle = |0_{\text{in}}\rangle$. We can redefine the phase of $|0_{\text{out}}\rangle$ (or $|0_{\text{in}}\rangle$) so that

$$\boxed{S\,|0_{\text{out}}\rangle = |0_{\text{in}}\rangle\ .} \tag{2.20}$$

On one particle states,

$$\begin{aligned} S\,|\vec{p}_{\rm out}\rangle &= S\,a^\dagger_{\rm out}(\vec{p})\,|0_{\rm out}\rangle \\ &= \underbrace{S\,a^\dagger_{\rm out}(\vec{p})S^{-1}}_{a^\dagger_{\rm in}(\vec{p})}\,\underbrace{S\,|0_{\rm out}\rangle}_{|0_{\rm in}\rangle} \\ &= |\vec{p}_{\rm in}\rangle \end{aligned} \tag{2.21}$$

In general, we could show that

$$\boxed{S\,\left|\vec{p}_1\,\ldots\,\vec{p}_{N,{\rm out}}\right\rangle = \left|\vec{p}_1\,\ldots\,\vec{p}_{N,{\rm in}}\right\rangle\ .} \tag{2.22}$$

Normal Ordering of S

We know that

$$S = e^{iB} = e^{\displaystyle i\int d^4y\,j(y)\,\phi_{\rm in}(y)}\ . \tag{2.23}$$

It is useful to write S so that all the annihilation operators are on the right. Let

$$iB = i\int d^4y\,j(y)\int\frac{d^3p}{(2\pi)^3}\,\frac{1}{\sqrt{2E_p}}\left[a_{\rm in}(\vec{p})e^{-ip\cdot y} + a^\dagger_{\rm in}(\vec{p})e^{ip\cdot y}\right] = G + F\ , \tag{2.24}$$

where

$$F = i\int\frac{d^3p}{(2\pi)^3}\,\frac{1}{\sqrt{2E_p}}\tilde{\jmath}(p)a^\dagger_{\rm in}(\vec{p})\ ,\quad G = i\int\frac{d^3p}{(2\pi)^3}\,\frac{1}{\sqrt{2E_p}}\tilde{\jmath}(-p)a_{\rm in}(\vec{p})\ , \tag{2.25}$$

where we recall that

$$\tilde{\jmath}(p) \equiv \int d^4y\,e^{ip\cdot y}\,j(y)\ . \tag{2.7}$$

So

$$S = e^{iB} = e^{F+G} \; . \tag{2.26}$$

F and G do not commute, but $[F\,,\,G]$ is a c-number and therefore commutes with both F and G. Whenever F and G commute with $[F\,,\,G]$,

$$\boxed{e^{F+G} = e^F e^G e^{-\frac{1}{2}[F\,,\,G]} \; .} \tag{2.27}$$

Aside about $e^{F+G} = e^F e^G e^{-\frac{1}{2}[F\,,\,G]}$

To prove this identity, define

$$H_1(\lambda) \equiv e^{\lambda(F+G)} \; , \quad H_2(\lambda) = e^{\lambda F}\, e^{\lambda G}\, e^{-\frac{1}{2}\lambda^2[F,G]} \; . \tag{2.28}$$

Clearly $H_1(0) = H_2(0) = I$ (identity operator), and

$$\frac{\mathrm{d}H_1(\lambda)}{\mathrm{d}\lambda} = (F+G)\, H_1(\lambda) \; . \tag{2.29}$$

So if we can show that $H_2(\lambda)$ obeys the same differential equation as above, then it follows that $H_2(\lambda) = H_1(\lambda)$. You'll get to show this on your next problem set.

This is actually a special case of the Baker-Campbell-Hausdorff formula, which has the general form

$$e^F\, e^G = e^{F+G+\frac{1}{2}[F\,,\,G]+\ldots\text{(iterated commutators)}} \; . \tag{2.30}$$

We'll prove this, too, on a problem set soon.

Returning to the main argument:

So

$$S = e^{iB} = e^{F+G} . \tag{2.26}$$

F and G do not commute, but $[F\,,\,G]$ is a c-number and therefore commutes with both F and G. Whenever F and G commute with $[F\,,\,G]$,

$$\boxed{e^{F+G} = e^F e^G e^{-\frac{1}{2}[F\,,\,G]} .} \tag{2.27}$$

Recalling

$$F = i\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\tilde{\jmath}(p)a^\dagger_{\text{in}}(\vec{p})\,, \quad G = i\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\tilde{\jmath}(-p)a_{\text{in}}(\vec{p})\,, \tag{2.25}$$

one sees that

$$\begin{aligned}[F\,,\,G] &= -\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_q}}\,\tilde{\jmath}(p)\,\tilde{\jmath}(-q)\underbrace{[a^\dagger_{\text{in}}(\vec{p}), a_{\text{in}}(\vec{q})]}_{-(2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})} \\ &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\,|\tilde{\jmath}(p)|^2 \quad .\end{aligned} \tag{2.31}$$

So

$$\boxed{S = e^{iB} = \exp\left\{-\frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\,|\tilde{\jmath}(p)|^2\right\}\, e^F\, e^G \quad .} \tag{2.32}$$

Vacuum to Vacuum Probability

The probability that no particles are produced by the source is given by

$$P(\text{no particle production}) = \left|\langle 0_{\text{out}} | 0_{\text{in}} \rangle\right|^2 \ . \tag{2.33}$$

Logic: physical state is $|0_{\text{in}}\rangle$, independent of time (in the Heisenberg picture).

$|0_{\text{out}}\rangle$ = state with no particles for $t > t_2$.

So, $\left|\langle 0_{\text{out}} | 0_{\text{in}} \rangle\right|^2$ is the probability that the physical state of the system would be measured to have 0 particles at $t > t_2$. To express the answer in terms of the S-matrix, recall

$$|0_{\text{out}}\rangle = S^{-1} |0_{\text{in}}\rangle \quad \Longrightarrow \quad \langle 0_{\text{out}}| = \langle 0_{\text{in}}| \, S. \tag{2.34}$$

So

$$\begin{aligned} \langle 0_{\text{out}} | 0_{\text{in}} \rangle &= \langle 0_{\text{in}} | S | \, 0_{\text{in}} \rangle \\ &= \exp\left\{ -\frac{1}{2} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_p} |\tilde{\jmath}(p)|^2 \right\} \left\langle 0_{\text{in}} \left| e^F e^G \right| 0_{\text{in}} \right\rangle \ . \end{aligned} \tag{2.35}$$

Alan Guth
Massachusetts Institute of Technology
8.323, March 18, 2008