MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

8.323: Relativistic Quantum Field Theory I

The Dirac Field

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Eq. (55) was corrected on 5/6/08

Transformations of the Dirac Field

Let us assume that we are trying to construct a free field $\psi_a(x)$ for electrons which, like the free field $\phi(x)$ for scalar particles, is linear in creation and annihilation operators. Then we expect a nonzero value for

$$\langle 0 \,|\psi_a(x)|\, 1 \text{ electron}, \vec{p} = 0\rangle \ .$$

Under rotations the state(s) on the right transform under the spin-$\frac{1}{2}$ representation, so $\psi_a(x)$ must contain this representation, or else the matrix element vanishes (i.e. the only spin that can be added to spin-$\frac{1}{2}$ to get spin-0 is spin-$\frac{1}{2}$).

But $\psi_a(x)$ must transform under some representation of the Lorentz group, generated by

$$\begin{aligned} \vec{J}_+ &= \frac{1}{2}(\vec{J} + i\vec{K}) \qquad & \vec{J} &= \vec{J}_+ + \vec{J}_- \\ \vec{J}_- &= \frac{1}{2}(\vec{J} - i\vec{K}) \qquad & \vec{K} &= i\left(\vec{J}_- - \vec{J}_+\right) \ . \end{aligned} \tag{41}$$

The simplest allowed representations are $(j_+, j_-) = \left(\frac{1}{2}, 0\right)$ or $\left(0, \frac{1}{2}\right)$. We will use both.

Mixing of $\left(\frac{1}{2},0\right)$ and $\left(0,\frac{1}{2}\right)$

Note that under a parity transformation,

$$\vec{K} \to -\vec{K}$$
$$\vec{J} \to \vec{J} \ ,$$

so $\vec{J}_+ = \frac{1}{2}(\vec{J} + i\vec{K})$, $\vec{J}_- = \frac{1}{2}(\vec{J} - i\vec{K})$, implies that

$$\vec{J}_+ \longleftrightarrow \vec{J}_- \ .$$

In addition, recall that the 4-vector rep is $\left(\frac{1}{2},\frac{1}{2}\right)$. So if χ transforms under, say, the $\left(\frac{1}{2},0\right)$ rep, then $\partial_\mu\chi$ must belong to the $\left(\frac{1}{2},\frac{1}{2}\right) \times \left(\frac{1}{2},0\right)$ rep, which contains the $\left(0,\frac{1}{2}\right)$ rep but not the $\left(\frac{1}{2},0\right)$. Thus, $\partial_\mu\chi$ must interchange these two reps.

The Lorentz Group and SL(2,$\mathbb{C}$)

From the $\left(\frac{1}{2},0\right)$ or $\left(0,\frac{1}{2}\right)$ representations, one can see the relation between the proper orthochronous Lorentz group $L_+^\uparrow$ and SL(2,$\mathbb{C}$).

For the $\left(\frac{1}{2},0\right)$ rep,

$$\vec{J}_+ = \frac{1}{2}\vec{\sigma} \ , \qquad \vec{J}_- = 0 \quad \Longrightarrow \quad \vec{J} = \frac{1}{2}\vec{\sigma} \ , \qquad \vec{K} = -\frac{i}{2}\vec{\sigma} \ ,$$

where the σ_i are the (traceless) Pauli spin matrices. Exponentiation of the σ_i with imaginary coefficients produces all unitary determinant one 2×2 matrices, or SU(2). Since $e^{-2\pi i J_z} = -1$, there are 2 matrices in SU(2) for every rotation group element, so the rotation group is SU(2)/$\mathbb{Z}_2$. The Lorentz group includes the exponentiation of the σ_i matrices with arbitrary complex coefficients, so unitarity is lost. Exponentiation generates SL(2,$\mathbb{C}$), the group of complex 2×2 matrices with determinant 1. One still has $e^{-2\pi i J_z} = -1$ and the resulting 2:1 relationship, so

$$\boxed{L_+^\uparrow = \mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2 \ .} \qquad (42)$$

The Dirac Field

Let $\psi_L(x)$ be a 2-component field, represented as a 1×2 column vector, transforming according to the $\left(\frac{1}{2}, 0\right)$ rep.

Let $\psi_R(x)$ be a 2-component field, represented as a 1×2 column vector, transforming according to the $\left(0, \frac{1}{2}\right)$ rep.

The Dirac field $\psi_a(x)$ is 4-component field, represented as a 1×4 column vector, constructed by placing $\psi_L(x)$ on top of $\psi_R(x)$:

$$\psi_a(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix} = \begin{pmatrix} \psi_L(x) \\ \psi_R(x) \end{pmatrix} . \tag{43}$$

The Dirac Matrices

Dirac discovered that one can construct the $\left(\frac{1}{2}, 0\right) + \left(0, \frac{1}{2}\right)$ representation of the Lorentz group by starting with four 4×4 "Dirac" matrices γ^μ, chosen to satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} , \tag{44}$$

where the curly brackets denote the anticommutator, and the right-hand side is multiplied by an implicit 4×4 identity matrix. Given Eq. (44), the matrices

$$S^{\mu\nu} = \frac{i}{4} [\gamma^\mu , \gamma^\nu] \tag{45}$$

automatically have the Lorentz group commutation relations,

$$[S^{\mu\nu} , S^{\rho\sigma}] = 4i \, \{g^{\rho\nu} \, S^{\mu\sigma}\}_{\substack{\mu\nu \\ \rho\sigma}} ,$$

where the pairs of subscripts denote antisymmetrizations as defined by Eq. (21), 4/13/06.

Choice of Dirac Matrix Conventions

Following Peskin and Schroeder, we will use

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} , \tag{46}$$

where the entries in γ^0 represent 2×2 blocks of zeros or the 2×2 identity matrix, and the σ^i represent the Pauli spin matrices,

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{47}$$

This is called the Weyl or the chiral representation. Another popular choice is to diagonalize γ^0.

Properties of this Choice of Dirac Matrices

With some calculation, one finds that

$$J^k = \frac{1}{2}\epsilon^{k\ell m} S^{\ell m} = \frac{1}{2}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} , \quad K^k = S^{0k} = -\frac{i}{2}\begin{pmatrix} \sigma^k & 0 \\ 0 & -\sigma^k \end{pmatrix} . \tag{48}$$

This gives

$$J_+^k = \frac{1}{2}(J^k + iK^k) = \frac{1}{2}\begin{pmatrix} \sigma^k & 0 \\ 0 & 0 \end{pmatrix} , \quad J_-^k = \frac{1}{2}(J^k - iK^k) = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & \sigma^k \end{pmatrix} . \tag{49}$$

One-Particle States and the Poincaré Group

A one-particle state must remain a one-particle state under a Poincaré transformation, so the space of one-particle states forms a representation of the Poincaré group.

$P^2 \equiv P_\mu P^\mu$ is a Casimir operator of the Poincaré group, so a single particle has a unique value of P^2, $P^2 = m^2$, where m is called the mass of the particle.

Particles seem to also have a definite spin. Is this a Casimir operator?

Answer: Yes. The operator is

$$W^2 \equiv W_\mu W^\mu \ , \quad \text{where} \ \ W_\mu = \frac{1}{2}\epsilon_{\mu\nu\lambda\sigma} J^{\nu\lambda} P^\sigma \ . \tag{50}$$

W_μ is called the Pauli-Lubanski pseudovector. In the rest frame of P^μ one has $W^2 = -m^2 \left|\vec{J}\right|^2$. One knows that $\left[W^2\, , \, J^{\mu\nu}\right] = 0$, since W_μ transforms as a vector under Lorentz transformations. It is also translation-invariant, since

$$\left[J^{\nu\lambda}\, , \, P^\rho\right] = i\left(g^{\rho\lambda}\, P^\nu - g^{\rho\nu}\, P^\lambda\right) \ ,$$

and the sum with $\epsilon_{\mu\nu\lambda\sigma}$ will vanish when two indices are contracted with P.

One-Particle Representations of the Poincaré Group

Consider first $P^2 > 0$, i.e., massive particles.

Diagonalize P^μ and consider a particle in its rest frame, $P^\mu = (m, 0, 0, 0)$.

The subgroup of the Lorentz group that leaves P^μ invariant is called the little group.

In this case, the little group is the rotation group.

But we know about the rotation group!

The particle must have spin j, an integer or half-integer, with $2j+1$ spin states described by $s \equiv J^z$, with $s = -j, -j+1, \ldots\ , j$. (The eigenvalue of J^z is often called m, but m = mass.)

$$\boxed{\left\langle \vec{p} = 0, s' \left| U\big(R(\hat{n}, \theta)\big) \right| \vec{p} = 0, s \right\rangle = \left[e^{-i\theta\hat{n}\cdot\vec{J}}\right]_{s's} \equiv D_{s's}\big(R(\hat{n}, \theta)\big) \ .} \tag{51}$$

States with Nonzero $\vec{P}$

By Wigner's theorem, we know that we should expect to construct a unitary representation of the Poincare group in the Hilbert space. (Anti-unitary is excluded here, since the group is continuous.)

We can therefore describe states with $\vec{p} \neq 0$ by boosting the $\vec{p} = 0$ states. So we can define

$$\boxed{|\vec{p}, s\rangle \equiv U(B_{\vec{p}})\,|\vec{p} = 0, s\rangle \quad ,} \tag{52}$$

where $B_{\vec{p}}$ is a boost that transforms the rest vector to the specified momentum $\vec{p}$. $B_{\vec{p}}$ is not uniquely defined by this criterion.

For Eq. (52) to be unitary, we must be using the covariant normalization:

$$\langle \vec{p}\,', s'\,|\vec{p}, s\,\rangle = 2E_{\vec{p}}\,(2\pi)^3\,\delta_{s's}\,\delta^{(3)}(\vec{p}\,' - \vec{p}\,)\;.$$

Canonical vs. Helicity Basis

There are two standard ways to choose $B_{\vec{p}}$, and therefore to define a basis for the Hilbert space of 1-particle states:

Canonical: boost in direction of $\vec{p}$:

$$\boxed{B^{(c)}_{\vec{p}} = B\left(\hat{p}, \xi(|\vec{p}\,|)\right) = e^{-i\xi\hat{p}\cdot\vec{K}} \quad ,} \tag{53}$$

where $\xi(|\vec{p}\,|)$ is the rapidity associated with $\vec{p}$, $\tanh\xi = |\vec{p}\,|/E$.

Helicity: boost in positive z-direction, then rotate in the z-$\hat{p}$ plane (along the shorter of the two options):

$$\boxed{B^{(h)}_{\vec{p}} = R(\hat{p})\,B_z\left(\xi(|\vec{p}\,|)\right)\;.} \tag{54}$$

This process preserves the helicity, the component of the spin in the direction of the momentum. Thus, $s =$ helicity.

Lorentz Transformations of Basis States

Translations are no problem, since these are eigenstates of momentum.

To apply a Lorentz transformation, use

$$\begin{aligned} U(\Lambda)\,|\vec{p},s\rangle &= U(\Lambda)\,U(B_{\vec{p}}\,)\;|\vec{p}=0,s\rangle \\ &= U(B_{\Lambda\vec{p}}\,)\,U^{-1}(B_{\Lambda\vec{p}}\,)\,U(\Lambda)\,U(B_{\vec{p}}\,)\;|\vec{p}=0,s\rangle \;\;. \end{aligned}$$

Now define the *Wigner rotation*

$$\boxed{R_W(\Lambda,\vec{p}) \equiv B^{-1}_{\Lambda\vec{p}}\,\Lambda\,B_{\vec{p}}\;\;.} \qquad (55)$$

Note that R_W maps a rest vector to a rest vector, so it is a rotation. But we know about rotations!

$$U(\Lambda)\,|\vec{p},s\rangle = \sum_{s'} U(B_{\Lambda\vec{p}}\,)\;\big|\vec{p}=0,s'\big\rangle\,\big\langle\vec{p}=0,s'\,\big|U\big(R_W(\Lambda,\vec{p})\big)\big|\,\vec{p}=0,s\big\rangle \;\;.$$

$$U(\Lambda)\,|\vec{p},s\rangle = \sum_{s'} U(B_{\Lambda\vec{p}}\,)\;\big|\vec{p}=0,s'\big\rangle\,\big\langle\vec{p}=0,s'\,\big|U\big(R_W(\Lambda,\vec{p})\big)\big|\,\vec{p}=0,s\big\rangle \;\;.$$

$$\boxed{U(\Lambda)\,|\vec{p},s\rangle = \sum_{s'} \big|\Lambda\vec{p},s'\big\rangle\,D_{s's}\big(R_W(\Lambda,\vec{p})\big)\;\;.} \qquad (56)$$

Basis States for 1-Particle Hilbert Space

(Review, Massive Particles)

Basis States in Rest Frame:

$$\langle \vec{p} = 0, s' \left| U\big(R(\hat{n}, \theta)\big) \right| \vec{p} = 0, s \rangle = \left[e^{-i\theta \hat{n}\cdot\vec{J}} \right]_{s's} \equiv D_{s's}\big(R(\hat{n},\theta)\big) \ . \qquad (51)$$

Basis States in Arbitrary Frame:

$$|\vec{p}, s\rangle \equiv U(B_{\vec{p}})\, |\vec{p} = 0, s\rangle \quad , \qquad (52)$$

where $B_{\vec{p}}$ is a boost (canonical or helicity) that boosts a rest vector to $\vec{p}$.

Lorentz Transformations of Basis States

$$U(\Lambda)\, |\vec{p}, s\rangle = \sum_{s'} \left| \Lambda\vec{p}, s' \right\rangle D_{s's}\big(R_W(\Lambda, \vec{p})\big) \ , \qquad (56)$$

where

$$R_W(\Lambda, \vec{p}) \equiv B^{-1}_{\Lambda\vec{p}}\, \Lambda\, B_{\vec{p}} \ . \qquad (55)$$

is called the Wigner rotation.

Recall: $B_{\vec{p}}$ is the standard boost, in either the canonical or helicity basis.

Multiparticle States

Free particle Fock space basis vectors:

$$|\vec{p}_1 s_1, \ldots, \vec{p}_N s_N\rangle \ .$$

If two kets $|\psi_1\rangle$ and $|\psi_2\rangle$ are identical except for the ordering of the particles, they represent the same physical state. For scalar particles, $|\psi_1\rangle = |\psi_2\rangle$. We **will** learn (soon!) that for spin-$\frac{1}{2}$ particles, $|\psi_1\rangle = \pm|\psi_2\rangle$, depending on whether the permutation is even $(+)$ or odd (-).

Lorentz Transformations: each particle transforms independently.

$$\begin{aligned} U(\Lambda)\,|\vec{p}_1 s_1, \ldots, \vec{p}_N s_N\rangle = \\ \sum_{\{s'_j\}} |\Lambda\vec{p}_1 s'_1, \ldots, \Lambda\vec{p}_N s'_N\rangle \\ \times D_{s'_1 s_1}\big(R_W(\Lambda,\vec{p}_1)\big)\ldots D_{s'_N s_N}\big(R_W(\Lambda,\vec{p}_N)\big) \ . \end{aligned} \tag{57}$$

Transformation of Creation and Annihilation Operators

$$\begin{aligned} &U(\Lambda) a^\dagger_{s_N}(\vec{p}_N)\sqrt{2E_{\vec{p}_N}}\ |\vec{p}_1 s_1, \ldots, \vec{p}_{N-1} s_{N-1}\rangle = \\ &\quad \sum_{\{s'_j\}} a^\dagger_{s'_N}(\Lambda\vec{p}_N)\sqrt{2E_{\Lambda\vec{p}_N}}\ \big|\Lambda\vec{p}_1 s'_1, \ldots, \Lambda\vec{p}_{N-1} s'_{N-1}\big\rangle \\ &\qquad \times D_{s'_1 s_1}\big(R_W(\Lambda,\vec{p}_1)\big)\ldots D_{s'_N s_N}\big(R_W(\Lambda,\vec{p}_N)\big) \\ &\quad = \sum_{s'_N} a^\dagger_{s'_N}(\Lambda\vec{p}_N)\sqrt{2E_{\Lambda\vec{p}_N}}\, U(\Lambda)\,|\vec{p}_1 s_1, \ldots, \vec{p}_{N-1} s_{N-1}\rangle\, D_{s'_N s_N}\big(R_W(\Lambda,\vec{p}_N)\big) \ , \end{aligned}$$

so

$$\begin{aligned} &U(\Lambda) a^\dagger_{s_N}(\vec{p}_N)\sqrt{2E_{\vec{p}_N}}\ |\psi\rangle = \\ &\quad \sum_{s'_N} a^\dagger_{s'_N}(\Lambda\vec{p}_N)\sqrt{2E_{\Lambda\vec{p}_N}}\, D_{s'_N s_N}\big(R_W(\Lambda,\vec{p}_N)\big) U(\Lambda)\,|\psi\rangle \ . \end{aligned}$$

$$U(\Lambda) a^\dagger_{s_N}(\vec{p}_N)\sqrt{2E_{\vec{p}_N}}\,|\psi\rangle =$$

$$\sum_{s'_N} a^\dagger_{s'_N}(\Lambda\vec{p}_N)\sqrt{2E_{\Lambda\vec{p}_N}}\,D_{s'_N s_N}\big(R_W(\Lambda,\vec{p}_N)\big)U(\Lambda)\,|\psi\rangle \ .$$

So,

$$\boxed{U(\Lambda)\,a^\dagger_s(\vec{p})\,U^{-1}(\Lambda) = \sqrt{\frac{E_{\Lambda\vec{p}}}{E_{\vec{p}}}}\sum_{s'} a^\dagger_{s'}(\Lambda\vec{p})\,D_{s's}\big(R_W(\Lambda,\vec{p})\big) \ .} \qquad (58)$$

Taking the adjoint:

$$\boxed{U(\Lambda)\,a_s(\vec{p})\,U^{-1}(\Lambda) = \sqrt{\frac{E_{\Lambda\vec{p}}}{E_{\vec{p}}}}\sum_{s'} D^{-1}_{ss'}\big(R_W(\Lambda,\vec{p})\big)\,a_{s'}(\Lambda\vec{p}) \ ,} \qquad (59)$$

where I used the unitarity of $D_{s's}$:

$$D^*_{s's}\big(R_W(\Lambda,\vec{p})\big) = D^{-1}_{ss'}\big(R_W(\Lambda,\vec{p})\big) \ .$$

Massless Particles

What is the "little group" when $P^2 = 0$, so there is no rest frame?

Let $p^\mu = (\omega, 0, 0, \omega)$

Little group: generated by J^3, M^1, and M^2, where

$$\begin{aligned} J^3 &= J^{12} \\ M^1 &\equiv K^1 - J^2 = -J^{10} + J^{13} \\ M^2 &\equiv K^2 + J^1 = -J^{20} + J^{23} \ . \end{aligned} \qquad (60)$$

Algebra of little group:

$$\begin{aligned} [M^1, M^2] &= 0 \\ \big[J^3, M^1\big] &= iM^2 \\ \big[J^3, M^2\big] &= -iM^1 \ . \end{aligned} \qquad (61)$$

The little group is E(2), the Euclidean group (rotations and translations in $\mathbb{R}^2$)

Representations of E(2)

Casimir Operator: $\vec{M}^{\,2} = (M^1)^2 + (M^2)^2$

- ☆ Reps with $\vec{M}^{\,2} \neq 0$. $\vec{M}$ could then be rotated by any amount, so the rep is infinite dimensional. This would involve an infinite number of states with the same momentum, and does not correspond to anything known physically.
- ☆ Finite-dimensional (1D) representations with $\vec{M}^{\,2} = 0$. When $\vec{M}^{\,2} = 0$, J^3 becomes a Casimir operator, since $\left[J^3\,,\,M^i\right]$ is proportional to M^j. So in this case

$$M^1 = M^2 = 0$$

$$J^3 = \text{constant} \; \in \mathbb{Z}/2 \; .$$

(since $e^{4\pi i J^3} = 1$ in SL(2, $\mathbb{C}$)) So, for $m = 0$ the little group does not mix different s values. The helicity h of a massless particle is Lorentz-invariant, so the photon can have $h = \pm 1$, but does not need an $h = 0$ state. The $h = \pm 1$ states are related by parity, but not by Lorentz transformations. If neutrinos were massless, they could have only $h = -\frac{1}{2}$ ("left-handed" only).

Dirac Field: Spacetime Dependence

We assume that $\psi_a(x)$ is linear in creation and annihilation operators.

Electrons are charged. The field should have a definite charge, to construct charge-conserving interactions. So let $\psi_a(x)$ contain annihilation operators for electrons. Then it must contain creation operators for antiparticles, i.e. positrons. (This is like the charged scalar field.) Then $\psi_a^\dagger(x)$ will create electrons and annihilate positrons.

The field must satisfy:

$$\boxed{\begin{aligned} e^{iP_\mu x^\mu}\,\psi_a(y)\,e^{-iP_\mu x^\mu} &= \psi_a(x+y)\;, \\ U^{-1}(\Lambda)\psi_a(x)U(\Lambda) &= \Lambda_{\frac{1}{2},ab}\psi_b(\Lambda^{-1}x)\quad, \end{aligned}} \qquad (62)$$

where

$$\Lambda_{\frac{1}{2},ab} \equiv M_{ab} = e^{-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}}\;,\;\text{with}\; S^{\mu\nu} = \frac{i}{4}\left[\gamma^\mu\,,\,\gamma^\nu\right]\;. \qquad (63)$$

$$\psi_a(x) = e^{iP_\mu x^\mu}\,\psi_a(0)\,e^{-iP_\mu x^\mu}\ .$$

But $\psi_a(0)$ is linear in electron annihilation operators $a_s(\vec p)$ and positron creation operators $b_s^\dagger(\vec p)$, so write

$$\psi_a(0) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\,\frac{1}{\sqrt{2E_{\vec p}}}\sum_s \left\{a_s(\vec p)\,u_a^s(\vec p) + b_s^\dagger(\vec p)\,v_a^s(\vec p)\right\}\ .$$

But

$$e^{iP_\mu x^\mu}\,a_s(\vec p)\,e^{-iP_\mu x^\mu} = e^{-ip_\mu x^\mu}\,a_s(\vec p)\ ,$$

$$e^{iP_\mu x^\mu}\,b_s^\dagger(\vec p)\,e^{-iP_\mu x^\mu} = e^{ip_\mu x^\mu}\,b_s^\dagger(\vec p)\ ,$$

so

$$\boxed{\psi_a(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\,\frac{1}{\sqrt{2E_{\vec p}}}\sum_s \left\{a_s(\vec p)\,u_a^s(\vec p)\,e^{-ip_\mu x^\mu} + b_s^\dagger(\vec p)\,v_a^s(\vec p)\,e^{ip_\mu x^\mu}\right\}\ ,} \tag{64}$$

where $p^0 = +\sqrt{|\vec p|^2 + m^2}$. The choice $p^0 > 0$ will be important.

Lorentz Transformation of $\psi_a(x)$

$$U^{-1}(\Lambda)\psi_a(x)U(\Lambda) = \Lambda_{\frac{1}{2},ab}\psi_b(\Lambda^{-1}x)\ ,$$

so

$$U(\Lambda)\psi_a(x)U^{-1}(\Lambda) = \Lambda^{-1}_{\frac{1}{2},ab}\psi_b(\Lambda\, x)$$

$$= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\,\frac{1}{\sqrt{2E_{\vec p}}}\sum_{s,s'}\left\{\sqrt{\frac{E_{\Lambda\vec p}}{E_{\vec p}}}D^{-1}_{ss'}\big(R_W(\Lambda,\vec p)\big)a_{s'}(\Lambda\vec p)u_a^s(\vec p)e^{-ip\cdot x} + \dots\right\}\ .$$

Now let $\vec q \equiv \Lambda\vec p$, and use

$$\int \frac{\mathrm{d}^3p}{(2\pi)^3}\,\frac{1}{2E_{\vec p}} = \int \frac{\mathrm{d}^3q}{(2\pi)^3}\,\frac{1}{2E_{\vec q}}\ ,\ \text{so}$$

$$U(\Lambda)\psi_a(x)U^{-1}(\Lambda) =$$

$$\int \frac{\mathrm{d}^3q}{(2\pi)^3}\,\frac{1}{\sqrt{2E_{\vec q}}}\sum_{s,s'}\left\{D^{-1}_{ss'}\big(R_W(\Lambda,\vec p)\big)a_{s'}(\vec q)u_a^s(\vec p)e^{-i(\Lambda^{-1}q)\cdot x} + \dots\right\}\ .$$

$$U(\Lambda)\psi_a(x)U^{-1}(\Lambda) =$$

$$\int \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec q}}}\sum_{s,s'}\left\{D^{-1}_{ss'}\big(R_W(\Lambda,\vec p)\big)a_{s'}(\vec q)u^s_a(\vec p)e^{-i(\Lambda^{-1}q)\cdot x}+\ldots\right\} \ .$$

But this must equal

$$\Lambda^{-1}_{\frac{1}{2},ab}\psi_b(\Lambda\, x) =$$

$$\int \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec q}}}\sum_{s'}\left\{a_{s'}(\vec q)\Lambda^{-1}_{\frac{1}{2},ab}u^{s'}_b(\vec q)e^{-iq\cdot(\Lambda x)}+\ldots\right\} \ .$$

These two expressions match if

$$\Lambda^{-1}_{\frac{1}{2},ab}u^{s'}_b(\Lambda\vec p) = \sum_s u^s_a(\vec p)D^{-1}_{ss'}\big(R_W(\Lambda,\vec p)\big) \ ,$$

or equivalently,

$$\boxed{u^s_a(\Lambda\vec p) = \Lambda_{\frac{1}{2},ab}\,u^{s'}_b(\vec p)D^{-1}_{s's}\big(R_W(\Lambda,\vec p)\big) \ .} \qquad (65)$$

Rotations in Rest Frame ($\vec p = 0$):

$$\boxed{u^s_a(\Lambda\vec p) = \Lambda_{\frac{1}{2},ab}\,u^{s'}_b(\vec p)D^{-1}_{s's}\big(R_W(\Lambda,\vec p)\big) \ .} \qquad (65)$$

In the rest frame $\vec p = 0$, $\Lambda = R$ (rotation matrix), so $\Lambda\vec p = 0$, and $B_{\vec p} = B_{\Lambda\vec p} = I$, so $R_W = B^{-1}_{\Lambda\vec p}\Lambda B_{\vec p} = R$. So

$$u^s_a(\vec p = 0) = \Lambda_{\frac{1}{2},ab}(R)u^{s'}_b(\vec p = 0)\,D^{-1}_{s's}(R) \ .$$

Look separately at upper and lower spinor of ψ:

$$\boxed{\psi_a(x) = \begin{pmatrix}\psi_1(x)\\ \psi_2(x)\\ \psi_3(x)\\ \psi_4(x)\end{pmatrix} = \begin{pmatrix}\psi_L(x)\\ \psi_R(x)\end{pmatrix} \ .} \qquad (43)$$

We know that for rotations, $\Lambda_{\frac{1}{2},ab}$ are just the matrices generated by

$$J^i = \frac{1}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} ,$$

and the $D_{s's}(R)$ matrices are the same! So the above equation can be rewritten as

$$u_{L,a}{}^s = D_{ab}(R)\, u_{L,b}{}^{s'}\, D^{-1}_{s's}(R) \ , \text{ or } u_L\, D(R) = D(R)\, u_L$$

for every R. Since the spin-$\frac{1}{2}$ representation is irreducible, the only matrices that commute with all $D(R)$ are multiples of the identity. Choose

$$\boxed{u^s_{L,a}(\vec{p}=0) = \sqrt{m}\delta^s_a \ .} \tag{66}$$

Boosting from the rest frame: For $\vec{p}=0$, let $\Lambda = B_{\vec{q}}$, so $\Lambda\vec{p} = \vec{q}$. Then

$$R_W(\Lambda,\vec{p}) = B^{-1}_{\Lambda\vec{p}}\Lambda B_{\vec{p}} = B^{-1}_{\vec{q}}\, B_{\vec{q}}\, I = I \ ,$$

So

$$\boxed{u^s_a(\vec{q}) = \Lambda_{\frac{1}{2},ab}(B_{\vec{q}})\, u^s_b(\vec{p}=0) \ .} \tag{67}$$