MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

8.323: Relativistic Quantum Field Theory I

The Dirac Field
Part II

Thursday, May 1, 2008

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Announcements

Corrections to Problem Set 9:

Problem 2: Notational improvement. The two equations should read:

$$\Lambda_{\frac{1}{2}}\big(B_3(\eta)\big) \equiv e^{-i\eta K^3} = e^{-i\eta S^{03}} \ .$$

$$\Lambda_{\frac{1}{2}}\big(R_3(\theta)\big) \equiv e^{-i\theta J^3} = e^{-i\theta S^{12}} \ .$$

Problem 4: Should read $S^{\mu\nu} = \frac{i}{4}\left[\gamma^\mu\, , \, \gamma^\nu\right]$ in the preamble, and identity (vi) should read

$$\not{p}\,\not{q} = 2p \cdot q - \not{q}\,\not{p} = p \cdot q - 2iS^{\mu\nu}\, p_\mu\, q_\nu \ .$$

Summary of Where We Are

Poincaré symmetry implies

$$
\boxed{\begin{aligned} e^{iP_\mu x^\mu}\,\psi_a(y)\,e^{-iP_\mu x^\mu} &= \psi_a(x+y)\ , \\ U^{-1}(\Lambda)\psi_a(x)U(\Lambda) &= M_{ab}\psi_b(\Lambda^{-1}\,x)\quad , \end{aligned}} \tag{68}
$$

where $M(\Lambda)$ is a representation of the Lorentz group. Following Dirac we have chosen

$$
M_{ab} \equiv \Lambda_{\frac{1}{2},ab} = e^{-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}}\ ,\ \text{with } S^{\mu\nu} = \frac{i}{4}\left[\gamma^\mu\,,\,\gamma^\nu\right]\ , \tag{69}
$$

and following P&S we have made a particular choice of the γ-matrices. P^2 is a Casimir operator, which we take to equal m^2, the square of the particle mass, and we assume that $P^0 > 0$.

Creation operators transform as

$$
\boxed{U(\Lambda)\,a_s^\dagger(\vec p)\,U^{-1}(\Lambda) = \sqrt{\frac{E_{\Lambda\vec p}}{E_{\vec p}}}\sum_{s'} a_{s'}^\dagger(\Lambda\vec p)\,D_{s's}\big(R_W(\Lambda,\vec p)\big)\ .} \tag{70}
$$

Taking the adjoint:

$$
\boxed{\begin{aligned} U(\Lambda)\,a_s(\vec p)\,U^{-1}(\Lambda) &= \sqrt{\frac{E_{\Lambda\vec p}}{E_{\vec p}}}\sum_{s'} a_{s'}(\Lambda\vec p)\,D^*_{s's}\big(R_W(\Lambda,\vec p)\big) \\ &= \sqrt{\frac{E_{\Lambda\vec p}}{E_{\vec p}}}\sum_{s'} D^{-1}_{ss'}\big(R_W(\Lambda,\vec p)\big)\,a_{s'}(\Lambda\vec p)\ , \end{aligned}} \tag{71}
$$

Expand the field as

$$\psi_a(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_s \left\{ a_s(\vec{p})\, u^s_a(\vec{p})\, e^{-ip_\mu x^\mu} + b^\dagger_s(\vec{p})\, v^s_a(\vec{p})\, e^{ip_\mu x^\mu} \right\} \ , \tag{72}$$

where $p^0 = +\sqrt{|\vec{p}|^2 + m^2}$. The sign of p^0, which follows from our assumption that the operator P^0 was positive, will be important. The above equations imply that

$$u^s_a(\Lambda\vec{p}) = \Lambda_{\frac{1}{2},ab}\, u^{s'}_b(\vec{p}) D^{-1}_{s's}\big(R_W(\Lambda,\vec{p})\big) \ . \tag{73}$$

Writing $\psi_a(x)$ as

$$\psi_a(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix} = \begin{pmatrix} \psi_L(x) \\ \psi_R(x) \end{pmatrix} \ , \tag{74}$$

we found that rotations in the rest frame implied that

$$u^s_{L,a}(\vec{p} = 0) = \sqrt{m}\, \delta^s_a \ , \tag{75}$$

up to an arbitrary normalization. This is often written as

$$u^s_L(\vec{p} = 0) = \sqrt{m}\, \xi^s \ , \text{ where } \xi^+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \xi^- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ , \tag{76}$$

where ξ^+ corresponds to spin-up, and ξ^- corresponds to spin-down, along the z-axis. The constraints on u_R are identical, so we similarly choose

$$u^s_R(\vec{p} = 0) = \sqrt{m}\, \xi^s \ . \tag{77}$$

In both cases the normalization is an arbitrary convention, and we can choose them to have the same normalization without any loss of generality.

To discuss nonzero momenta, we boost from the rest frame:

$$\boxed{u^s(\vec{q}) = \sqrt{m}\,\Lambda_{\frac{1}{2}}(B_{\vec{q}}) \begin{pmatrix} \xi^s \\ \xi^s \end{pmatrix} \ .} \tag{78}$$

We will calculate this explicitly in the canonical basis:

$$\Lambda_{\frac{1}{2}}(B_{\vec{p}}) = e^{-i\eta\hat{p}\cdot\vec{K}} \ , \tag{79}$$

where η is the rapidity, with $\cosh\eta = E/m$, $\sinh\eta = |\vec{p}|/m$, and $\tanh\eta = v$. (Earlier I called the rapidity ξ, but ξ is now being used for the basis spinors.) In our basis

$$K^j = J^{0j} = -\frac{i}{2}\begin{pmatrix} \sigma^j & 0 \\ 0 & -\sigma^j \end{pmatrix} \ .$$

$$\Lambda_{\frac{1}{2}}(B_{\vec{p}}) = \exp\left\{\frac{1}{2}\eta\begin{pmatrix} -\hat{p}\cdot\vec{\sigma} & 0 \\ 0 & \hat{p}\cdot\vec{\sigma} \end{pmatrix}\right\} \ .$$

Note that $(\hat{p}\cdot\vec{\sigma})^2 = 1$, so

$$\begin{aligned}
\Lambda_{\frac{1}{2}}(B_{\vec{p}}) &= \exp\left\{\frac{1}{2}\eta\begin{pmatrix} -\hat{p}\cdot\vec{\sigma} & 0 \\ 0 & \hat{p}\cdot\vec{\sigma} \end{pmatrix}\right\} \\
&= 1 + \left(\frac{\eta}{2}\right)\begin{pmatrix} -\hat{p}\cdot\vec{\sigma} & 0 \\ 0 & \hat{p}\cdot\vec{\sigma} \end{pmatrix} + \frac{1}{2!}\left(\frac{\eta}{2}\right)^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \\
&\qquad + \frac{1}{3!}\left(\frac{\eta}{2}\right)^3\begin{pmatrix} -\hat{p}\cdot\vec{\sigma} & 0 \\ 0 & \hat{p}\cdot\vec{\sigma} \end{pmatrix} + \ldots \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cosh\left(\frac{\eta}{2}\right) + \begin{pmatrix} -\hat{p}\cdot\vec{\sigma} & 0 \\ 0 & \hat{p}\cdot\vec{\sigma} \end{pmatrix}\sinh\left(\frac{\eta}{2}\right) \ .
\end{aligned}$$

To simplify, look at 2×2 blocks:

$$\Lambda_{\frac{1}{2}}(B_{\vec{p}}) = \begin{pmatrix} \Lambda_L & 0 \\ 0 & \Lambda_R \end{pmatrix}, \text{ where } \Lambda_{\substack{L \\ R}} = \cosh\left(\frac{\eta}{2}\right) \mp \hat{p}\cdot\vec{\sigma}\,\sinh\left(\frac{\eta}{2}\right) \ .$$

Then remember that $e^{\pm\eta} = (E \pm |\vec{p}|)/m$, so

$$\Lambda_L = \frac{1}{2}\left\{\sqrt{\frac{E+|\vec{p}|}{m}}\,[1-\hat{p}\cdot\vec{\sigma}] + \sqrt{\frac{E-|\vec{p}|}{m}}\,[1+\hat{p}\cdot\vec{\sigma}]\right\} \ .$$

$$\Lambda_L = \frac{1}{2}\left\{\sqrt{\frac{E+|\vec{p}|}{m}}\,[1-\hat{p}\cdot\vec{\sigma}] + \sqrt{\frac{E-|\vec{p}|}{m}}\,[1+\hat{p}\cdot\vec{\sigma}]\right\} .$$

The eigenvalues of $\hat{p}\cdot\vec{\sigma}$ are ± 1, so

$$\Lambda_L = \begin{cases} \sqrt{\frac{E-|\vec{p}|}{m}} & \text{if } \hat{p}\cdot\vec{\sigma} = 1 \\ \sqrt{\frac{E+|\vec{p}|}{m}} & \text{if } \hat{p}\cdot\vec{\sigma} = -1 \end{cases}$$

$$= \sqrt{\frac{E-\vec{p}\cdot\vec{\sigma}}{m}} .$$

If the eigenvalues of a matrix are positive, we define its square root by diagonalizing the matrix and taking the positive square root of its eigenvalues.

Similarly,

$$\Lambda_R = \sqrt{\frac{E+\vec{p}\cdot\vec{\sigma}}{m}} .$$

The notation can be simplified further by defining 4-vectors $\sigma^\mu = (1, \sigma^i)$ and $\bar{\sigma}^\mu = (1, -\sigma^i)$, so

$$\Lambda_L = \sqrt{\frac{p\cdot\sigma}{m}},\ \Lambda_R = \sqrt{\frac{p\cdot\bar{\sigma}}{m}} ,$$

and finally

$$\boxed{u^s(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi^s \end{pmatrix} .} \qquad (80)$$

Useful Identities for $u^s(\vec{p})$

$$(p \cdot \sigma)(p \cdot \bar{\sigma}) = p^2 = m^2 \; . \tag{81}$$

Normalization:

$$\begin{aligned} u^\dagger(\vec{p})\, u(\vec{p}) &= \left(\xi^\dagger \sqrt{p\cdot\sigma}\;\; \xi^\dagger \sqrt{p\cdot\bar{\sigma}}\right) \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi^s \end{pmatrix} \; . \\ &= \xi^\dagger (p\cdot\sigma + p\cdot\bar{\sigma})\xi \\ &= 2E_{\vec{p}}\, \xi^\dagger \xi \; . \end{aligned} \tag{82}$$

$$\begin{aligned} u^\dagger(\vec{p})\,\gamma^0\, u(\vec{p}) &\equiv \bar{u}(\vec{p})\, u(\vec{p}) \\ &= \xi^\dagger \sqrt{p\cdot\sigma}\sqrt{p\cdot\bar{\sigma}}\xi + \xi^\dagger \sqrt{p\cdot\bar{\sigma}}\sqrt{p\cdot\sigma}\xi \\ &= 2m\, \xi^\dagger \xi \; . \end{aligned} \tag{83}$$

$$u^s(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi^s \end{pmatrix} \quad \Longrightarrow \quad \boxed{\sum_s u^s(\vec{p})\, \bar{u}^s(\vec{p}) = \gamma\cdot p + m \; ,} \tag{84}$$

where $\bar{u}^s(\vec{p}) \equiv u^{s\dagger}(\vec{p})\gamma^0$. For the $v's$,

$$\boxed{u_a^s(\Lambda\vec{p}) = \Lambda_{\frac{1}{2},ab}\, u_b^{s'}(\vec{p}) D^{-1}_{s's}\big(R_W(\Lambda,\vec{p})\big) \; .} \tag{73}$$

is replaced by

$$\boxed{v_a^s(\Lambda\vec{p}) = \Lambda_{\frac{1}{2},ab}\, v_b^{s'}(\vec{p}) D^{-1\,*}_{s's}\big(R_W(\Lambda,\vec{p})\big) \; .} \tag{85}$$

From looking at the rest frame, we use

$$D^*(R) = \sigma^2 D(R) \sigma^2 \ ,$$

and following the same logic that we used to show that ${u_a}^s(\vec{p})$ must be a multiple of the identity, here we show that $(v\sigma^2)_a{}^s$ must be a multiple of the identity. The normalization is arbitrary, so we can write

$$v^s_{L,a}(\vec{p}=0) = -i\sqrt{m}\,\sigma^2_{as}\beta_L, \quad v^s_{R,a}(\vec{p}=0) = i\sqrt{m}\,\sigma^2_{as}\,\beta_R \ , \tag{86}$$

where β_L and β_R are used here to indicate our freedom to choose the normalization of these terms arbitrarily, as far as the Lorentz transformations of the field are concerned. We will soon find, however, that these constants are constrained.

Boosting to an arbitrary momentum,

$$\boxed{v^s(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}(-i\sigma^2\,\xi^s)\beta_L \\ -\sqrt{p\cdot\bar{\sigma}}(-i\sigma^2\,\xi^s)\beta_R \end{pmatrix} \ ,} \tag{87}$$

where as before

$$\xi^+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \xi^- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

describe spin-up and spin-down particles. We can also define

$$\boxed{\eta^s \equiv -i\sigma^2 \xi^s \ ,} \tag{88}$$

and the two spinors η^s, for $s=+$ and $s=-$, will form a pair of orthonormal vectors in the space of 2-component spinors. Then one has

$$\boxed{v^s(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\eta^s\,\beta_L \\ -\sqrt{p\cdot\bar{\sigma}}\,\eta^s\,\beta_R \end{pmatrix} \ ,} \tag{89}$$

The spin sum for the v's then becomes

$$\boxed{\sum_s v^s(\vec{p})\,\bar{v}^s(\vec{p}) = \begin{pmatrix} -m\beta_L\beta_R^* & p\cdot\sigma|\beta_L|^2 \\ p\cdot\bar{\sigma}|\beta_R|^2 & -m\beta_R\beta_L^* \end{pmatrix} \ .} \tag{90}$$

Spin and Statistics

- A key result of quantum field theory is the **Spin-Statistics Theorem**, which says that particles of integer spin must be bosons, while particles of half-integer spin must be fermions. (Note that there is no such theorem in nonrelativistic quantum theory.)
- A full proof is beyond this course. See, for example, *PCT, Spin & Statistics, and All That,* by R.F. Streater and Arthur S. Wightman, 1964. Despite the informal title and modest length, this book is not easy reading for most of us.
- We will, however, demonstrate a limited version of this theorem. We will consider the possibility that the Dirac particles are either bosons or fermions, and show that only the second choice gives a consistent theory.

The Boson Hypothesis: Dirac Field Commutators

Consider first the boson possibility. Bosons are like the scalar particles that we have studied, with creation/annihilation operator commutators

$$\begin{aligned} \left[a^s(\vec{p})\,,\,a^{r\dagger}(\vec{q})\right] &= (2\pi)^3\,\delta^{rs}\,\delta^{(3)}(\vec{p}-\vec{q})\;. \\ \left[b^s(\vec{p})\,,\,b^{r\dagger}(\vec{q})\right] &= (2\pi)^3\,\delta^{rs}\,\delta^{(3)}(\vec{p}-\vec{q})\;, \end{aligned} \tag{91h}$$

with other commutators ($[a\,,\,a]$, $[b\,,\,b]$, and their adjoints) vanishing. The "h" in the equation number stands for "hypothesis". It will turn out NOT to be consistent. We will now compute the equal-time commutator

$$\left[\psi(\vec{x},0)\,,\,\psi^\dagger(\vec{y},0)\right]$$

which is clearly a spacelike separation. There are two reasons why this should vanish.

Why the Commutator Should Vanish at Spacelike Separations

1) The fields should represent measurable quantities, so a nonzero commutator implies an uncertainty principle — a precise measurement of one causes the other to become completely uncertain. But if this happens at spacelike separations, then information can be transmitted faster than light.

2) Later we will use these fields to describe terms in an interaction Hamiltonian, and perturbation theory will allow us to express evolution (and hence scattering cross sections) in terms of time-ordered products. However, the time-ordering between 2 spacetime points is ambiguous if they are spacelike separated, so a unique answer is found only if the operators commute. So, the Lorentz-invariance of interactions calculated in QFT would be violated if fields did not commute at spacelike separations.

The requirement that commutators vanish at spacelike separations is called **CAUSALITY.**

Calculation of the Commutator

From Eq. (72),

$$\begin{aligned}\psi_a(\vec{x},0) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\sum_s \left\{ a_s(\vec{p})\,u_a^s(\vec{p})\,e^{-ip_\mu x^\mu} + b_s^\dagger(\vec{p})\,v_a^s(\vec{p})\,e^{ip_\mu x^\mu}\right\} \\ &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\sum_s \left\{ a_s(\vec{p})\,u_a^s(\vec{p}) + b_s^\dagger(-\vec{p})\,v_a^s(-\vec{p})\right\} e^{i\vec{p}\cdot\vec{x}} \ .\end{aligned} \tag{92}$$

So

$$\begin{aligned}\left[\psi(\vec{x},0)\,,\,\bar{\psi}(\vec{y},0)\right] &= \int \frac{d^3p\,d^3q}{(2\pi)^6}\frac{1}{\sqrt{(2E_{\vec{p}})\,(2E_{\vec{q}})}} \\ &\times e^{i(\vec{p}\cdot\vec{x}-\vec{q}\cdot\vec{y})}\sum_{r,s}\Big(\left[a^r(\vec{p})\,,\,a^{s\dagger}(\vec{q})\right]u^r(\vec{p})\,\bar{u}^s(\vec{q}) \\ &+ \left[b^{r\dagger}(-\vec{p})\,,\,b^s(-\vec{q})\right]v^r(-\vec{p})\,\bar{v}^s(-\vec{q})\Big)\ .\end{aligned} \tag{93}$$

Using the creation/annihilation operator commutators (91h) and the spin sums (84) and (90), this becomes

$$\left[\psi(\vec{x},0)\,,\,\bar{\psi}(\vec{y},0)\right] = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}e^{i\vec{p}\cdot(\vec{x}-\vec{y})}\times$$
$$\begin{pmatrix} m\,(1+\beta_L\beta_R^*) & E_{\vec{p}}\left(1-|\beta_L|^2\right)-\vec{p}\cdot\vec{\sigma}\left(1+|\beta_L|^2\right) \\ E_{\vec{p}}\left(1-|\beta_R|^2\right)+\vec{p}\cdot\vec{\sigma}\left(1+|\beta_R|^2\right) & m\,(1+\beta_R\beta_L^*) \end{pmatrix} . \tag{94}$$

The expression has particle contributions, with no β's, and antiparticle contributions, with 2 β's. The killer terms are the ones in $\vec{p}\cdot\vec{\sigma}$, since the coefficients of particles and antiparticles add. Looking back to Eq. (93), one sees that the antiparticle contribution has 2 sign changes relative to the particle contribution: the signs of $\vec{p}$ and $\vec{q}$ are reversed, and the commutator has the creation operator first.

The term proportional to $\vec{p}\cdot\vec{\sigma}$ does not vanish when integrated, as it is proportional to

$$-i\vec{\sigma}\cdot\vec{\nabla}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}e^{i\vec{p}\cdot(\vec{x}-\vec{y})}\ , \tag{95}$$

where the integral does not vanish.

The Fermion Hypothesis: Dirac Field Anti-commutators

This time let us assume that the particles are fermions, which means that only **one** particle can have a specified momentum and spin, and that the multi-particle basis state

$$\boxed{|\vec{p}_1\, s_1, \ldots\ , \vec{p}_N\, s_N\rangle}$$

is antisymmetric in its particle labels. For example,

$$\boxed{|\vec{p}_1\, s_1, \vec{p}_2\, s_2\rangle = -\,|\vec{p}_2\, s_2, \vec{p}_1\, s_1\rangle\ .} \tag{96}$$

For fermions the creation and annihilation operators satisfy **anti**commutation relations

$$\begin{aligned} \left\{a^s(\vec{p})\,,\, a^{r\dagger}(\vec{q})\right\} &= (2\pi)^3\,\delta^{rs}\,\delta^{(3)}(\vec{p}-\vec{q})\ . \\ \left\{b^s(\vec{p})\,,\, b^{r\dagger}(\vec{q})\right\} &= (2\pi)^3\,\delta^{rs}\,\delta^{(3)}(\vec{p}-\vec{q})\ , \end{aligned} \tag{97}$$

- Note that the antisymmetry of states, as in Eq. (96), implies that

$$a^{s_1\dagger}(\vec{p}_1)\, a^{s_2\dagger}(\vec{p}_2) = -a^{s_2\dagger}(\vec{p}_2)\, a^{s_1\dagger}(\vec{p}_1)\ , \tag{98}$$

since the creation operators create multiparticle states with the particle labels in the same order as the operators. Thus there is no possibility that fermion creation/annihilation operators could obey commutation rather than anticommutation relations.

- Since the fields are linear in creation and annihilation operators, if the creation and annihilation operators satisfy anticommutation relations, then so must the fields.

- If the fields anticommute at spacelike separations, that is okay. It means that any measurable quantity — any quantity that can appear in the Hamiltonian — must be bilinear in the fermi fields, and then they will commute. We might have expected this anyway for a fermi field, since under a 360° rotation it turns into minus itself! If such a field were measurable, what would it look like?

Calculation of the Anticommutator

The calculation looks much like the previous one, with one key difference. In the boson calculation we found that the antiparticle contribution was proportional to $\left[b^{s\dagger}(-\vec{p})\,,\,b^{s}(-\vec{q})\right]$, which produced a minus sign because the creation and annihilation operators were not in their usual order. This time, however, we find an anticommutator instead, so there is no change in sign. Our final answer will be identical to the previous one, except that the entire antiparticle contribution will change sign:

$$\left\{\psi(\vec{x},0)\,,\,\bar{\psi}(\vec{y},0)\right\} = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}e^{i\vec{p}\cdot(\vec{x}-\vec{y})}\times \left(\begin{array}{cc} m\,(1-\beta_L\beta_R^*) & E_{\vec{p}}\,\left(1+|\beta_L|^2\right)-\vec{p}\cdot\vec{\sigma}\,\left(1-|\beta_L|^2\right) \\ E_{\vec{p}}\,\left(1+|\beta_R|^2\right)+\vec{p}\cdot\vec{\sigma}\,\left(1-|\beta_R|^2\right) & m\,(1-\beta_R\beta_L^*) \end{array}\right) . \tag{99}$$

The change in sign makes all the difference!

Consequences of the Calculation

$$\left\{\psi(\vec{x},0)\,,\,\bar{\psi}(\vec{y},0)\right\} = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}e^{i\vec{p}\cdot(\vec{x}-\vec{y})}\times \left(\begin{array}{cc} m\,(1-\beta_L\beta_R^*) & E_{\vec{p}}\,\left(1+|\beta_L|^2\right)-\vec{p}\cdot\vec{\sigma}\,\left(1-|\beta_L|^2\right) \\ E_{\vec{p}}\,\left(1+|\beta_R|^2\right)+\vec{p}\cdot\vec{\sigma}\,\left(1-|\beta_R|^2\right) & m\,(1-\beta_R\beta_L^*) \end{array}\right) . \tag{99}$$

- ☆ The previously fatal $\vec{p}\cdot\vec{\sigma}$ terms will cancel if

$$|\beta_L|^2 = |\beta_R|^2 = 1 . \tag{100}$$

- ☆ The terms in m will cancel as long as β_L and β_R have the same phase. Thus, we require

$$\beta_L = \beta_R . \tag{101}$$

From now on we will adopt the standard phase convention,

$$\beta_L = \beta_R = 1 . \tag{102}$$

$$\left\{\psi(\vec{x},0)\,,\,\bar{\psi}(\vec{y},0)\right\} = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}e^{i\vec{p}\cdot(\vec{x}-\vec{y})}\times$$

$$\begin{pmatrix} m\,(1-\beta_L\beta_R^*) & E_{\vec{p}}\,\left(1+|\beta_L|^2\right)-\vec{p}\cdot\vec{\sigma}\,\left(1-|\beta_L|^2\right) \\ E_{\vec{p}}\,\left(1+|\beta_R|^2\right)+\vec{p}\cdot\vec{\sigma}\,\left(1-|\beta_R|^2\right) & m\,(1-\beta_R\beta_L^*) \end{pmatrix} \ . \qquad (99)$$

- The terms in $E_{\vec{p}}$ do not cancel, but notice that the $E_{\vec{p}}$ cancels the same factor in the denominator, producing a Dirac δ-function which vanishes when $\vec{x} \neq \vec{y}$.