MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

8.323: Relativistic Quantum Field Theory I

Discrete Symmetries of the Dirac Field --- But first: some tidbits

Thursday, May 8, 2008

— Alan Guth

Canonical Anticommutation Relations For the Dirac Field

The Dirac Lagrangian was given as Eq. (120):

$$\mathcal{L}_{\text{Dirac}} = \bar{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi \ , \tag{120}$$

where

$$\bar{\psi}(x) = \psi^\dagger(x)\gamma^0 \qquad \left(\bar{\psi}_a(x) = \psi_b^\dagger(x)\gamma^0_{ba}\right) \ . \tag{121}$$

The canonical momentum is then

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = i\psi^\dagger \ ,$$

so we might expect canonical anticommutation relations of the form

$$\{\psi_a(\vec{x},0)\,,\,\pi_b(\vec{y},0)\} = i\delta^{(3)}(\vec{x}-\vec{y})\,\delta_{ab} \quad \Longrightarrow$$

$$\left\{\psi_a(\vec{x},0)\,,\,\psi_b^\dagger(\vec{y},0)\right\} = \delta^{(3)}(\vec{x}-\vec{y})\,\delta_{ab}$$

Remember Eq. (99):

$$\{\psi(\vec{x},0)\,,\,\bar{\psi}(\vec{y},0)\} = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}e^{i\vec{p}\cdot(\vec{x}-\vec{y})}\times$$
$$\begin{pmatrix} m\,(1-\beta_L\beta_R^*) & E_{\vec{p}}\,\left(1+|\beta_L|^2\right)-\vec{p}\cdot\vec{\sigma}\,\left(1-|\beta_L|^2\right) \\ E_{\vec{p}}\,\left(1+|\beta_R|^2\right)+\vec{p}\cdot\vec{\sigma}\,\left(1-|\beta_R|^2\right) & m\,(1-\beta_R\beta_L^*) \end{pmatrix} ,$$

which for $\beta_L = \beta_R = 1$ implies that

$$\{\psi(\vec{x},0)\,,\,\bar{\psi}(\vec{y},0)\} = \int \frac{\mathrm{d}^3p}{(2\pi)^3}e^{i\vec{p}\cdot(\vec{x}-\vec{y})}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \delta^{(3)}(\vec{x}-\vec{y})\,\gamma^0 ,$$

which is exactly right. The full canonical anticommutation relations are

$$\left\{\psi_a(\vec{x},0)\,,\,\psi_b^\dagger(\vec{y},0)\right\} = \delta^{(3)}(\vec{x}-\vec{y})\,\delta_{ab}$$

$$\{\psi_a(\vec{x},0)\,,\,\psi_b(\vec{y},0)\} = \left\{\psi_a^\dagger(\vec{x},0)\,,\,\psi_b^\dagger(\vec{y},0)\right\} = 0 .$$

Dirac Hole Theory

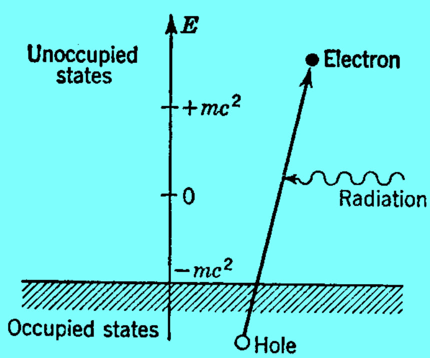
$$H = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} E_{\vec{p}} \sum_s \left\{ a_s^\dagger(\vec{p})\, a_s(\vec{p}) + b_s^\dagger(\vec{p})\, b_s(\vec{p}) \right\} + E_{\text{vac}} \ , \qquad (134)$$

where

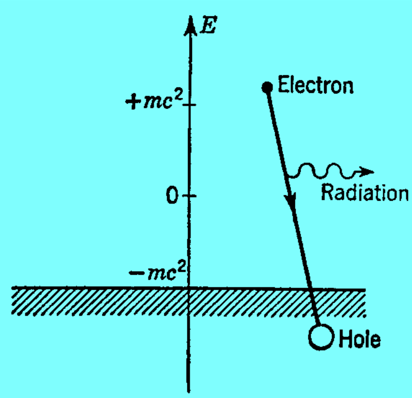
$$E_{\text{vac}} = -2 \int \mathrm{d}^3 p \, E_{\vec{p}} \, \delta^{(3)}(\vec{0}) = -2 \int \frac{\mathrm{d}^3 p}{(2\pi)^3} E_{\vec{p}} \times \text{Volume of space} \ . \qquad (135)$$

Note that Fermi statistics caused the antiparticle energy to be positive (good!), and the vacuum energy to be negative (surprising?). The negative vacuum energy, although ill-defined, is still welcome: allows at least the hope that one might get the positive (bosonic) contributions to cancel against the negative (fermionic) contributions, giving an answer that is finite and hopefully small. Note that if we had 4 free scalar fields with the same mass, the cancelation would be exact: this is what happens in EXACTLY supersymmetric models, but it is spoiled as soon as the supersymmetry is broken.

Dirac Hole Theory



Electron-positron pair creation



Electron-positron pair annihilation

In the 1-particle quantum mechanics formulation, positrons show up as negative energy states. Dirac proposed that in the vacuum, the negative energy "sea" was filled. Physical positrons, in this view, are holes in the Dirac sea. In QFT, on the other hand, particles and antiparticles are on equal footing. Nonetheless, the Dirac sea allows an intuitive way to understand the negative vacuum energy.

From Bjorken & Drell, vol. 1, p. 65

Vacuum Energy

- In quantum field theory (no gravity), vacuum energy is meaningless and can be dropped.
- In "semiclassical gravity," in which the expectation value of the energy-momentum tensor is taken as the source of a classical gravitational field, the vacuum energy matters, but it can be subtracted. The subtraction, however, does not appear to be theoretically well-motivated.
- In string theory, any subtraction would destroy the consistency of the theory. The vacuum energy density is exactly zero in supersymmetric vacua, but of order the Planck scale ($\rho \sim G^{-2}$ with $\hbar = c = 1$) for typical vacua, which is about 120 orders of magnitude too large. There are believed to be maybe 10^{500} different vacua (the "landscape" of string theory), and anthropic arguments are sometimes used to explain why we find ourselves in one of the unusual vacua with a very small but nonzero vacuum energy.

The Dirac Propagator

This is straightforward, so I will only summarize the results.

$$
\begin{aligned}
\langle 0\,|\psi_a(x)\,\bar\psi_b(y)|\,0\rangle &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\vec p}}\sum_s u_a^s(\vec p)\,\bar u_b^s(\vec p)\,e^{-ip\cdot(x-y)} \\
&= (i\,\partial\!\!\!/_x + m)_{ab}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\vec p}}e^{-ip\cdot(x-y)} \\
&= (i\,\partial\!\!\!/_x + m)_{ab}\,D(x-y) \\
\langle 0\,|\bar\psi_b(x)\,\psi_a(y)|\,0\rangle &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\vec p}}\sum_s v_a^s(\vec p)\,\bar v_b^s(\vec p)\,e^{-ip\cdot(y-x)} \\
&= -(i\,\partial\!\!\!/_x + m)_{ab}\,D(y-x)\ ,
\end{aligned}
\tag{136}
$$

where $\partial\!\!\!/ = \gamma^\mu\,\partial_\mu$ and $D(x)$ is the scalar 2-point function $\langle 0\,|\phi(x)\phi(0)|\,0\rangle$.

The Retarded Dirac Propagator

$$S_R^{ab}(x-y) \equiv \theta(x^0-y^0)\,\left\langle 0\left|\psi_a(x)\,\bar\psi_b(y)\right|0\right\rangle = (i\,\partial\!\!\!/_x + m)\,D_R(x-y)\ , \qquad (137)$$

where $D_R(x-y)$ is the scalar retarded propagator. One can show

$$(i\,\partial\!\!\!/_x - m)S_R(x-y) = i\delta^{(4)}(x-y)\cdot 1_{4\times 4}\ . \qquad (138)$$

The Fourier expansion is

$$S_R(x) = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\,e^{-ip\cdot x}\,\tilde S_R(p)\ ,\ \text{where}\ \ \tilde S_R(p) = \frac{i(\,p\!\!\!/+m)}{(p^0+i\epsilon)^2 - |\vec p\,|^2 - m^2}\ . \qquad (139)$$

The Feynman Propagator

$$S_F(x-y) \equiv \begin{cases} \left\langle 0\left|\psi(x)\,\bar\psi(y)\right|0\right\rangle & \text{for } x^0 > y^0 \\ -\left\langle 0\left|\bar\psi(y)\,\psi(x)\right|0\right\rangle & \text{for } y^0 > x^0 \end{cases} \equiv \left\langle 0\left|T\left\{\psi(x)\,\bar\psi(y)\right\}\right|0\right\rangle\ . \qquad (140)$$

The Feynman propagator also satisfies Eq. (138). The Fourier expansion is

$$S_F(x) = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\,e^{-ip\cdot x}\,\tilde S_F(p)\ ,\ \text{where}\ \ \tilde S_F(p) = \frac{i(\,p\!\!\!/+m)}{p^2-m^2+i\epsilon}\ . \qquad (141)$$

This differs from the scalar field Feynman propagator by the factor $(\,p\!\!\!/+m)$.

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To continue, look for Lecture Slides 6

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