MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

*8.323: Relativistic Quantum Field Theory I*

# Interacting Field Theories (Executive Summary Version)

*Thursday, May 15, 2008*

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## Interacting Field Theories: Outline

1. Calculate Green's functions: expectation values of time-ordered Heisenberg fields in the true vacuum state.

2. Define $S$-matrix, with $S \equiv 1 + iT$, and $T = (2\pi)^4 \delta^{(4)}(\Delta p_{\rm tot}) i\mathcal{M}$ . Express cross sections and lifetimes in terms of $\mathcal{M}$.

3. Express $S$-matrix in terms of Green's functions.

# Time-Dependent Perturbation Theory

**Sample theory: $\lambda\phi^4$ —**

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \;.$$

Then

$$H = H_0 + H_{\rm int} \;,$$

where

$$H_{\rm int} = \int \mathrm{d}^3x \, \frac{\lambda}{4!}\phi^4(\vec{x}) \;.$$

Goal: to perturbatively calculate matrix elements of the Heisenberg field

$$\phi(\vec{x}, t) = e^{iHt}\phi(\vec{x}, 0)e^{-iHt}$$

in the state $|\Omega\rangle$, the true ground state of the interacting theory.



# Interaction Picture

**Schrödinger Picture:**

Field operators are time-independent, states evolve.

**Heisenberg Picture:**

Fields evolve, states are time-independent.

**Interaction Picture:**

Fields evolve according to free field theory. Matrix elements, propagators, etc. can be calculated as in free field theory.

States evolve as necessary, with the evolution driven by the interaction Hamiltonian.

**Interaction picture:**

Choose any reference time $t_0$, at which the interaction picture operators and Heisenberg operators will coincide. Define

$$\boxed{\phi_I(\vec{x},t) \equiv e^{iH_0(t-t_0)}\phi(\vec{x},t_0)e^{-iH_0(t-t_0)} \ .}$$

At $t=t_0$, can expand Heisenberg $\phi$ and $\pi$ in creation and annihilation operators:

$$\phi(\vec{x},t_0) = \int \frac{\mathrm{d}^3 p}{(2\pi)^2}\frac{1}{\sqrt{2E_p}}\left(a_{\vec{p}}\,e^{i\vec{p}\cdot\vec{x}} + a^\dagger_{\vec{p}}\,e^{-i\vec{p}\cdot\vec{x}}\right) \ .$$

$$\pi(\vec{x},t_0) = \dot{\phi}(\vec{x},t_0) = \int \frac{\mathrm{d}^3 p}{(2\pi)^2}\frac{1}{\sqrt{2E_p}}\left(-iE_p a_{\vec{p}}\,e^{i\vec{p}\cdot\vec{x}} + iE_p a^\dagger_{\vec{p}}\,e^{-i\vec{p}\cdot\vec{x}}\right) \ .$$

$a^\dagger_{\vec{p}}$ creates state of momentum $\vec{p}$, but not energy $E_p$ — not single particle. But

$$[\phi(\vec{x},t_0)\,,\,\pi(\vec{y},t_0)] = i\delta^3(\vec{x}-\vec{y}) \quad \Longrightarrow \quad \left[a_{\vec{p}}\,,\,a^\dagger_{\vec{q}}\right] = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q}) \ .$$

–4–

Can write $\phi_I(\vec{x},t)$ for all $t$:

$$\phi_I(\vec{x},t) = \int \frac{\mathrm{d}^3 p}{(2\pi)^2}\frac{1}{\sqrt{2E_p}}\left(a_{\vec{p}}\,e^{-ip\cdot x} + a^\dagger_{\vec{p}}\,e^{ip\cdot x}\right)\Big|_{x^0=t-t_0} \ .$$

How to express $\phi(\vec{x},t)$:

$$\begin{aligned}\phi(\vec{x},t) &= e^{iH(t-t_0)}e^{-iH_0(t-t_0)}\phi_I(\vec{x},t)e^{iH_0(t-t_0)}e^{-iH(t-t_0)} \\ &\equiv U^\dagger(t,t_0)\phi_I(\vec{x},t)U(t,t_0) \ ,\end{aligned}$$

where

$$U(t,t_0) = e^{iH_0(t-t_0)}e^{-iH(t-t_0)} \ .$$

Differential equation for $U$:

$$\begin{aligned} i\frac{\partial}{\partial t}U(t,t_0) &= e^{iH_0(t-t_0)}(H-H_0)e^{-iH(t-t_0)} \\ &= e^{iH_0(t-t_0)}H_{\rm int}\,e^{-iH(t-t_0)} \\ &= e^{iH_0(t-t_0)}H_{\rm int}\,e^{-iH_0(t-t_0)}e^{iH_0(t-t_0)}e^{-iH(t-t_0)} \\ &= H_I(t)U(t,t_0) \ ,\end{aligned}$$

where

$$H_I(t) = e^{iH_0(t-t_0)}H_{\rm int}\,e^{-iH_0(t-t_0)} = \int \mathrm{d}^3x\,\frac{\lambda}{4!}\phi_I^4(\vec{x},t) \ .$$

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**Solution to differential equation:**

$$i\frac{\partial}{\partial t}U(t,t_0) = H_I(t)U(t,t_0) \qquad \text{with} \qquad U(t_0,t_0) = I$$

implies the integral equation

$$U(t,t_0) = I - i\int_{t_0}^{t} \mathrm{d}t'\, H_I(t')U(t',t_0)\ .$$

To first order in $H_I$,

$$U(t,t_0) = I - i\int_{t_0}^{t} \mathrm{d}t_1\, H_I(t_1)\ .$$

To second order in $H_I$,

$$U(t,t_0) = I - i\int_{t_0}^{t} \mathrm{d}t_1 H_I(t_1) + (-i)^2\int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t_1} \mathrm{d}t_2\, H_I(t_1)H_I(t_2)\ .$$

To third order,

$$U(t,t_0) = \ldots + (-i)^3\int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t_1} \mathrm{d}t_2 \int_{t_0}^{t_2} \mathrm{d}t_3\, H_I(t_1)H_I(t_2)H_I(t_3)\ .$$

–6–

Note that $t_1 \geq t_2 \geq t_3$. Can rewrite 3rd order term as

$$\begin{aligned} U(t,t_0) &= \ldots + (-i)^3\int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t_1} \mathrm{d}t_2 \int_{t_0}^{t_2} \mathrm{d}t_3\, H_I(t_1)H_I(t_2)H_I(t_3) \\ &= \ldots + \frac{(-i)^3}{3!}\int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t} \mathrm{d}t_2 \int_{t_0}^{t} \mathrm{d}t_3\, T\{H_I(t_1)H_I(t_2)H_I(t_3)\}\ , \end{aligned}$$

where $T\{\}$ is time-ordered product (earliest time to right). Finally,

$$\begin{aligned} U(t,t_0) &= I + (-i)\int_{t_0}^{t} \mathrm{d}t_1 H_I(t_1) + \frac{(-i)^2}{2!}\int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t} \mathrm{d}t_2\, T\{H_I(t_1)H_I(t_2)\} + \ldots \\ &\equiv T\left\{\exp\left[-i\int_{t_0}^{t} \mathrm{d}t'\, H_I(t')\right]\right\}\ . \end{aligned}$$

–7–

Generalize to arbitrary $t_0$:

$$\boxed{U(t_2, t_1) \equiv T\left\{\exp\left[-i\int_{t_1}^{t_2} \mathrm{d}t'\, H_I(t')\right]\right\}\ ,}$$

where (for $t_1 < t_0 < t_2$)

$$\begin{aligned} U(t_2, t_1) &= T\left\{\exp\left[-i\int_{t_0}^{t_2} \mathrm{d}t'\, H_I(t')\right]\right\} T\left\{\exp\left[-i\int_{t_1}^{t_0} \mathrm{d}t'\, H_I(t')\right]\right\} \\ &= U(t_2, t_0)U^{-1}(t_1, t_0)\ . \end{aligned}$$

Given

$$U(t, t_0) = e^{iH_0(t-t_0)}e^{-iH(t-t_0)}\ ,$$

have

$$\begin{aligned} U(t_2, t_1) &= e^{iH_0(t_2-t_0)}e^{-iH(t_2-t_0)}e^{iH(t_1-t_0)}e^{-iH_0(t_1-t_0)} \\ &= e^{iH_0(t_2-t_0)}e^{-iH(t_2-t_1)}e^{-iH_0(t_1-t_0)}\ . \end{aligned}$$



$$U(t_2, t_1) \equiv T\left\{\exp\left[-i\int_{t_1}^{t_2} \mathrm{d}t'\, H_I(t')\right]\right\}\ .$$

Properties:

- $U$ is unitary.
- $U(t_3, t_2)U(t_2, t_1) = U(t_3, t_1)$ .
- $U(t_2, t_1)^{-1} = U(t_1, t_2)$ .

**Construction of true ground state $|\Omega\rangle$:**

Assume that $|0\rangle$ has nonzero overlap with $|\Omega\rangle$:

$$e^{-iHT}|0\rangle = \sum_n e^{-iE_nT}|n\rangle\langle n|0\rangle \ .$$

If $T$ had large negative imaginary part, all other states would be suppressed relative to $|\Omega\rangle$.

$$|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} e^{-iH(T+t_0)}|0\rangle\left(e^{-iE_0(T+t_0)}\langle\Omega|0\rangle\right)^{-1}$$

$$= \lim_{T\to\infty(1-i\epsilon)} e^{-iH(T+t_0)}e^{iH_0(T+t_0)}|0\rangle\left(e^{-iE_0(T+t_0)}\langle\Omega|0\rangle\right)^{-1} \ .$$

Recall

$$U(t_2,t_1) = e^{iH_0(t_2-t_0)}e^{-iH(t_2-t_1)}e^{-iH_0(t_1-t_0)} \ ,$$

so

$$|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} U(t_0,-T)|0\rangle\left(e^{-iE_0(T+t_0)}\langle\Omega|0\rangle\right)^{-1} \ .$$



Similarly,

$$\langle\Omega| = \lim_{T\to\infty(1-i\epsilon)} \left(e^{-iE_0(T-t_0)}\langle\Omega|0\rangle\right)^{-1}\langle 0|U(T,t_0) \ .$$

Recall

$$\phi(\vec{x},x^0) = U^\dagger(x^0,t_0)\phi_I(\vec{x},x^0)U(x^0,t_0) \ .$$

So, for $x^0 > y^0$,

$$\begin{aligned}\langle\Omega|\phi(x)\phi(y)|\Omega\rangle &= \lim_{T\to\infty(1-i\epsilon)} \langle 0|U(T,t_0)U(t_0,x^0)\phi_I(\vec{x},x^0)U(x^0,t_0)\\ &\quad\times U(t_0,y^0)\phi_I(\vec{y},y^0)U(y^0,t_0)U(t_0,-T)|0\rangle\\ &\quad\times \text{Normalization factor}\\ &= \lim_{T\to\infty(1-i\epsilon)} \langle 0|U(T,x^0)\phi_I(\vec{x},x^0)U(x^0,y^0)\\ &\quad\times \phi_I(\vec{y},y^0)U(y^0,-T)|0\rangle\\ &\quad\times \text{Normalization factor} \ .\end{aligned}$$

But

$$\text{Normalization factor} = \langle\Omega|\Omega\rangle^{-1} \ ,$$

so

$$\langle \Omega \left| \phi(x)\phi(y) \right| \Omega \rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\left\langle 0 \left| T\left\{ \phi_I(x)\phi_I(y) \exp\left[ -i \int_{-T}^{T} \mathrm{d}t\, H_I(t) \right] \right\} \right| 0 \right\rangle}{\left\langle 0 \left| T\left\{ \exp\left[ -i \int_{-T}^{T} \mathrm{d}t\, H_I(t) \right] \right\} \right| 0 \right\rangle} .$$



# Path Integral Connection

Recall from last screen:

$$\langle \Omega \left| \phi(x)\phi(y) \right| \Omega \rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\left\langle 0 \left| T\left\{ \phi_I(x)\phi_I(y) \exp\left[ -i \int_{-T}^{T} \mathrm{d}t\, H_I(t) \right] \right\} \right| 0 \right\rangle}{\left\langle 0 \left| T\left\{ \exp\left[ -i \int_{-T}^{T} \mathrm{d}t\, H_I(t) \right] \right\} \right| 0 \right\rangle} .$$

Recall from Lecture Notes on Path Integrals and Green's Functions:

$$G(t_N, \ldots, t_1) = \lim_{T\to\infty(1-i\epsilon)} \frac{\displaystyle\int_{x(-T)=x_0}^{x(T)=x_0} \mathcal{D}x(t)\, e^{\frac{i}{\hbar}S[x(t)]}\, x(t_N) \ldots x(t_1)}{\displaystyle\int_{x(-T)=x_0}^{x(T)=x_0} \mathcal{D}x(t)\, e^{\frac{i}{\hbar}S[x(t)]}} , \quad (6.20)$$

with an obvious generalization to field theories.

Generalizing path integrals to field theories,

$$\langle\Omega\,|\phi(x)\phi(y)|\,\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\displaystyle\int_{\phi(\vec{x},-T)=\phi_0}^{\phi(\vec{x},T)=\phi_0} \mathcal{D}\phi(x)\, e^{\frac{i}{\hbar}S[\phi(t)]}\,\phi(x)\,\phi(y)}{\displaystyle\int_{\phi(\vec{x},-T)=\phi_0}^{\phi(\vec{x},T)=\phi_0} \mathcal{D}\phi(x)\, e^{\frac{i}{\hbar}S[\phi(t)]}} ,$$

To calculate perturbatively, just write

$$S[\phi(t)] = S_0[\phi(t)] + \frac{1}{4!}\int d^4x\lambda\phi^4(t) ,$$

and expand in powers of $\lambda$.



**Causality:**

$$H_I(t) = \int \mathrm{d}^3x\,\mathcal{H}(\vec{x},t) ,$$

so

$$\langle\Omega\,|\phi(x)\phi(y)|\,\Omega\rangle = \frac{\left\langle 0\left|T\left\{\phi_I(x)\phi_I(y)\exp\left[-i\int \mathrm{d}^4z\,\mathcal{H}_I(z)\right]\right\}\right|0\right\rangle}{\left\langle 0\left|T\left\{\exp\left[-i\int \mathrm{d}^4z\,\mathcal{H}_I(z)\right]\right\}\right|0\right\rangle} .$$

If $z_1$ and $z_2$ are spacelike-separated, their time ordering is frame-dependent. Need $\Big[\mathcal{H}_I(z_1),\,\mathcal{H}_I(z_2)\Big] = 0$ to get same answer in all frames.

**Time-Dependent Perturbation Theory:**

$$\langle \Omega \,|\phi(x)\phi(y)|\, \Omega \rangle = \frac{\left\langle 0 \left| T \left\{ \phi_I(x)\phi_I(y) \exp\left[ -i \int \mathrm{d}^4 z\, \mathcal{H}_I(z) \right] \right\} \right| 0 \right\rangle}{\left\langle 0 \left| T \left\{ \exp\left[ -i \int \mathrm{d}^4 z\, \mathcal{H}_I(z) \right] \right\} \right| 0 \right\rangle} .$$



# Wick's Theorem

Gian-Carlo Wick
October 15, 1909 – April 20, 1992

http://stills.nap.edu/html/biomems/gwick.html

$$T\{\phi(x_1)\phi(x_2)\ldots\phi(x_m)\} = N\{\phi(x_1)\phi(x_2)\ldots\phi(x_m) + \text{ all possible contractions }.$$

Example:

$$\begin{aligned} T\{\phi_1\phi_2\phi_3\phi_4\} = N\{&\phi_1\phi_2\phi_3\phi_4 + \wick{\c1\phi_1\c1\phi_2\phi_3\phi_4} + \wick{\c1\phi_1\phi_2\c1\phi_3\phi_4} + \wick{\c1\phi_1\phi_2\phi_3\c1\phi_4} \\ &+ \wick{\phi_1\c1\phi_2\c1\phi_3\phi_4} + \wick{\phi_1\c1\phi_2\phi_3\c1\phi_4} + \wick{\phi_1\phi_2\c1\phi_3\c1\phi_4} \\ &+ \wick{\c1\phi_1\c1\phi_2\c2\phi_3\c2\phi_4} + \wick{\c1\phi_1\c2\phi_2\c1\phi_3\c2\phi_4} + \wick{\c1\phi_1\c2\phi_2\c2\phi_3\c1\phi_4}\}. \end{aligned}$$

Corollary:

$$\langle 0\,|T\{\phi(x_1)\phi(x_2)\ldots\phi(x_m)\}|\,0\rangle = \text{ all possible FULL contractions }.$$

Example:

$$\begin{aligned} \langle 0\,|T\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\}|\,0\rangle &= \Delta_F(x_1-x_2)\Delta_F(x_3-x_4) \\ &+ \Delta_F(x_1-x_3)\Delta_F(x_2-x_4) + \Delta_F(x_1-x_4)\Delta_F(x_2-x_3)\ . \end{aligned}$$



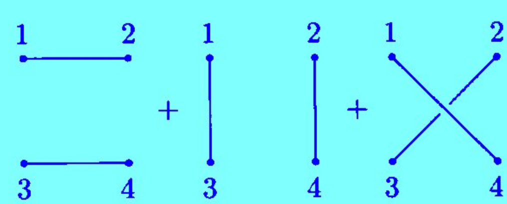
# Feynman Diagrams

Example:

$$\begin{aligned} \langle 0\,|T\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\}|\,0\rangle &= \Delta_F(x_1-x_2)\Delta_F(x_3-x_4) \\ &+ \Delta_F(x_1-x_3)\Delta_F(x_2-x_4) + \Delta_F(x_1-x_4)\Delta_F(x_2-x_3)\ . \end{aligned}$$

Feynman diagrams:

$$\langle 0|\,T\{\phi_1\phi_2\phi_3\phi_4\}\,|0\rangle =$$




[Note: the diagrams and some equations on this and the next 12 pages were taken from *An Introduction to Quantum Field Theory*, by Michael Peskin and Daniel Schroeder.]

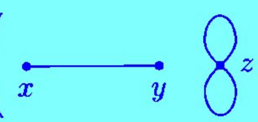
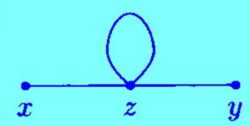
**Nontrivial Example:** $\mathcal{H}_I(z) = \dfrac{\lambda}{4!}\phi^4(z)$

$$\langle \Omega\,| T\{\phi(x)\phi(y)\}|\,\Omega\rangle = \left\langle 0 \left| T\left\{\phi(x)\phi(y) + \phi(x)\phi(y)\left[-i\int \mathrm{d}^4 z\,\mathcal{H}_I(z)\right] + \ldots\right\}\right| 0\right\rangle .$$

$$\left\langle 0 \left| T\left\{\phi(x)\phi(y)\left[-i\int \mathrm{d}^4 z\,\mathcal{H}_I(z)\right]\right\}\right| 0\right\rangle$$

$$= 3\cdot\left(\frac{-i\lambda}{4!}\right) D_F(x-y)\int \mathrm{d}^4 z D_F(z-z)D_F(z-z)$$

$$+\, 12\cdot\left(\frac{-i\lambda}{4!}\right)\int \mathrm{d}^4 z D_F(x-z)D_F(y-z)D_F(z-z)$$

$=$ (  ) + (  )





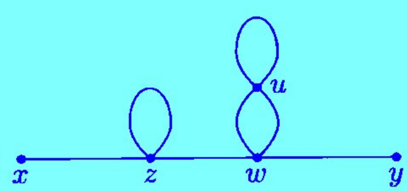


**Very Nontrivial Example:**

$$\langle 0|\,\phi(x)\phi(y)\,\tfrac{1}{3!}\left(\tfrac{-i\lambda}{4!}\right)^3 \int d^4z\,\phi\phi\phi\phi\,\int d^4w\,\phi\phi\phi\phi\,\int d^4u\,\phi\phi\phi\phi\,|0\rangle$$

$$= \frac{1}{3!}\left(\frac{-i\lambda}{4!}\right)^3 \int d^4z\,d^4w\,d^4u\; D_F(x-z)D_F(z-z)D_F(z-w)$$

$$\times\, D_F(w-y)D_F^2(w-u)D_F(u-u).$$

$=$ 

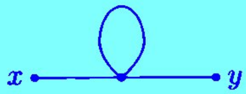


How many identical contractions are there?

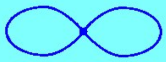
$$\underbrace{3!}_{\text{interchange of vertices}} \times \underbrace{4\cdot 3}_{\text{placement of contractions into } z \text{ vertex}} \times \underbrace{4\cdot 3\cdot 2}_{\text{placement of contractions into } w \text{ vertex}} \times \underbrace{4\cdot 3}_{\text{placement of contractions into } u \text{ vertex}} \times \underbrace{1/2}_{\text{interchange of } w\text{–}u \text{ contractions}}$$

Overall factor: $\frac{1}{3!}\times\left(\frac{1}{4!}\right)^3\times 3!\times 4\cdot 3\times 4!\times 4\cdot 3\times\frac{1}{2} = \frac{1}{8} \equiv \frac{1}{\text{symmetry factor}}$ .

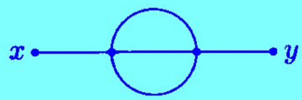
## Symmetry Factors:



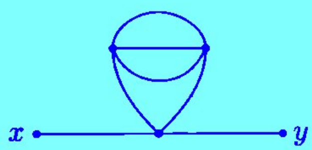

$S = 2$ Equivalence of 2 line endings



$S = 2 \cdot 2 \cdot 2 = 8$ Equivalence of 2 line endings (twice)
Equivalence of 2 lines




$S = 3! = 6$ Equivalence of 3 lines



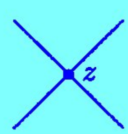

$S = 3! \cdot 2 = 12$ Equivalence of 2 lines
Equivalence of 3 lines



# Feynman Rules for $\lambda\phi^4$ Theory

$$\left\langle 0 \left| T \left\{ \phi(x)\phi(y) \exp\left[ -i \int d^4z\, \mathfrak{H}_I(z) \right] \right\} \right| 0 \right\rangle = \left( \begin{array}{c}\text{sum of all possible diagrams} \\ \text{with two external points}\end{array} \right) .$$

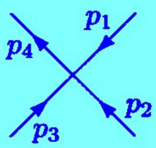
Rules:

1. For each propagator,  $= D_F(x-y)$;
2. For each vertex,  $= (-i\lambda)\int d^4z$;
3. For each external point,  $= 1$;
4. Divide by the symmetry factor.

## Momentum Space Feynman Rules:

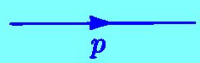
$$D_F(x-y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} \, e^{-ip\cdot(x-y)} \; .$$

Vertex:

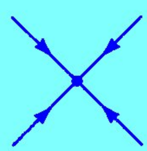


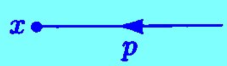

$$\longleftrightarrow \quad \int \mathrm{d}^4 z e^{-ip_1 z} e^{-ip_2 z} e^{-ip_3 z} e^{-ip_4 z}$$
$$= (2\pi)^4 \delta^{(4)}(p_1 + p_2 + p_3 + p_4) \; .$$



1. **For each propagator,** 

$= \dfrac{i}{p^2 - m^2 + i\epsilon}$;

2. **For each vertex,** 
$= -i\lambda$;

3. **For each external point,** 

$= e^{-ip\cdot x}$;

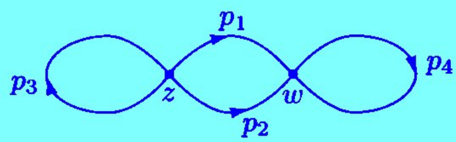
4. **Impose momentum conservation at each vertex;**

5. **Integrate over each undetermined momentum:** $\displaystyle\int \frac{d^4 p}{(2\pi)^4}$;

6. **Divide by the symmetry factor.**

# Disconnected Diagrams

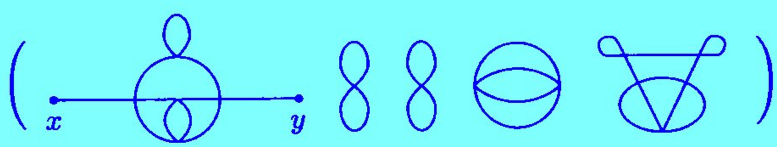
Consider




Momentum conservation at one vertex implies conservation at the other. Graph is proportional to $(2\pi)^4\delta^{(4)}(0)$ Use

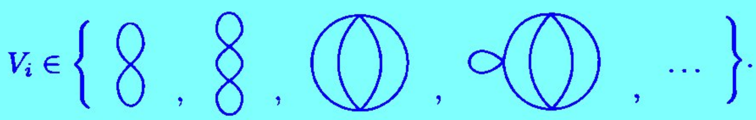
$$(2\pi)^4\delta^{(4)}(0) = (\text{volume of space}) \times 2T \ .$$

Disconnected diagrams:






Give each disconnected piece a name:




Then

$$\text{Diagram} = (\text{value of connected piece}) \cdot \prod_i \frac{1}{n_i!}(V_i)^{n_i} \ .$$

So, the sum of all diagrams is:

$$\sum_{\substack{\text{all possible}\\ \text{connected}\\ \text{pieces}}} \ \sum_{\text{all } \{n_i\}} \begin{pmatrix} \text{value of} \\ \text{connected piece} \end{pmatrix} \times \left(\prod_i \frac{1}{n_i!}(V_i)^{n_i}\right),$$

Factoring,

$$= \left(\sum \text{connected}\right) \times \sum_{\text{all } \{n_i\}} \left(\prod_i \frac{1}{n_i!}(V_i)^{n_i}\right),$$

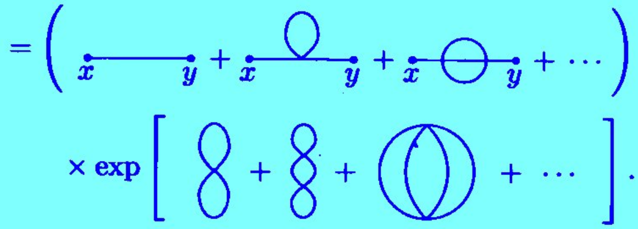
Factoring even more:

$$
\begin{aligned}
&= \Big(\sum \text{connected}\Big) \times \Big(\sum_{n_1} \frac{1}{n_1!} V_1^{n_1}\Big)\Big(\sum_{n_2} \frac{1}{n_2!} V_2^{n_2}\Big)\Big(\sum_{n_3} \frac{1}{n_3!} V_3^{n_3}\Big)\cdots \\
&= \Big(\sum \text{connected}\Big) \times \prod_i \Big(\sum_{n_i} \frac{1}{n_i!} V_i^{n_i}\Big) \\
&= \Big(\sum \text{connected}\Big) \times \prod_i \exp(V_i) \\
&= \Big(\sum \text{connected}\Big) \times \exp\Big(\sum_i V_i\Big).
\end{aligned}
$$



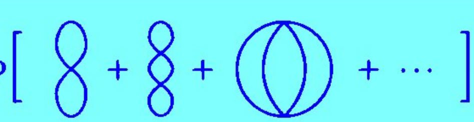
For our example,

$$
\lim_{T\to\infty(1-i\epsilon)} \langle 0| T\Big\{\phi_I(x)\phi_I(y)\exp\Big[-i\int_{-T}^{T} dt\, H_I(t)\Big]\Big\}|0\rangle
$$



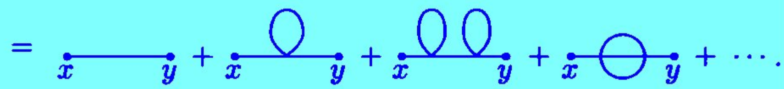

Now look at denominator of matrix element:

$$
\langle 0| T\Big\{\exp\Big[-i\int_{-T}^{T} dt\, H_I(t)\Big]\Big\}|0\rangle = \exp\Big[
$$



Finally,

$$\langle\Omega|\, T\big[\phi(x)\phi(y)\big]\,|\Omega\rangle$$
$$= \text{sum of all connected diagrams with two external points}$$

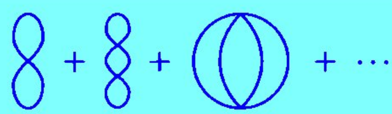



Summarizing,

$$\langle\Omega|\, T\big[\phi(x_1)\cdots\phi(x_n)\big]\,|\Omega\rangle = \begin{pmatrix}\text{sum of all connected diagrams}\\ \text{with } n \text{ external points}\end{pmatrix}.$$

Vacuum energy density:

$$\frac{E_0}{\text{volume}} = i\Big[$$

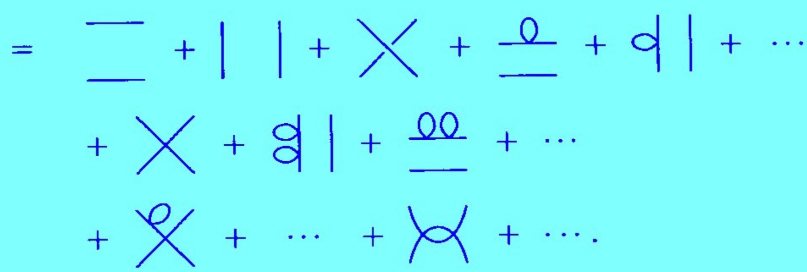



$$\Big]\Big/\Big[(2\pi)^4\delta^{(4)}(0)\Big].$$



Final sum for four-point function:

$$\langle\Omega|\, T\phi_1\phi_2\phi_3\phi_4\,|\Omega\rangle$$

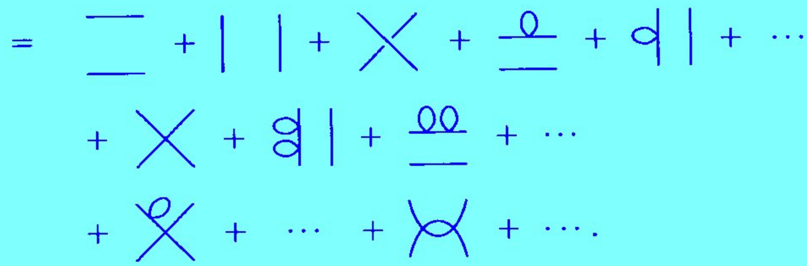


# Summary of Time-Dependent Perturbation Theory

$$\langle\Omega|\,T\big[\phi(x_1)\cdots\phi(x_n)\big]\,|\Omega\rangle = \begin{pmatrix}\text{sum of all connected diagrams}\\ \text{with } n \text{ external points}\end{pmatrix}.$$

Example: The four-point function:

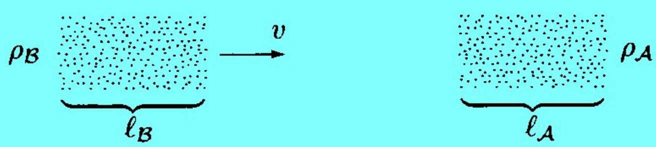
$\langle\Omega|\,T\phi_1\phi_2\phi_3\phi_4\,|\Omega\rangle$





# Cross Sections and Decay Rates

## Definition of Cross Section:




where

$\rho_{\mathcal{A}}$ and $\rho_{\mathcal{B}}$ = number density of particles

$A$ = cross sectional area of beams

$\ell_{\mathcal{A}}$ and $\ell_{\mathcal{B}}$ = lengths of particle packets

Then
$$\boxed{\sigma \equiv \frac{\text{Number of scattering events of specified type}}{\rho_{\mathcal{A}}\,\ell_{\mathcal{A}}\,\rho_{\mathcal{B}}\,\ell_{\mathcal{B}}\,A}\ .}$$

Note that $\sigma$ depends on frame.

Special case: 1 particle in each beam, so $\rho_{\mathcal{A}}\ell_{\mathcal{A}}A = 1$ and $\rho_{\mathcal{B}}\ell_{\mathcal{B}}A = 1$. Then

$$\text{Number of events} = \sigma/A\ .$$

## Definition of Decay Rate:

$$\Gamma \equiv \frac{\text{Number of decays per unit time}}{\text{Number of particles present}} \ .$$

Number of surviving particles at time $t$:

$$N(t) = N_0 e^{-\Gamma t} \ .$$

Mean lifetime:

$$\tau = \frac{1}{N_0} \int_0^\infty \mathrm{d}t \left( -\frac{\mathrm{d}N}{\mathrm{d}t} \right) t = 1/\Gamma \ .$$

Half-life:

$$e^{-\Gamma t} = \frac{1}{2} \quad \Longrightarrow \quad t_{1/2} = \tau \ln 2 \ .$$



## Unstable Particles as Resonances (Preview):

Unstable particles are not eigenstates of $H$; they are resonances in scattering experiments.

In nonrelativistic quantum mechanics, the Breit-Wigner formula

$$f(E) \propto \frac{1}{E - E_0 + i\Gamma/2} \quad \Longrightarrow \quad \sigma \propto \frac{1}{(E-E_0)^2 + \Gamma^2/4} \ .$$

The "full width at half max" of the resonance $= \Gamma$.

In the relativistic theory, the Breit-Wigner formula is replaced by a modified (Lorentz-invariant) propagator:

$$\frac{1}{p^2 - m^2 + im\Gamma} \approx \frac{1}{2E_{\vec{p}}\,(p^0 - E_{\vec{p}} + i(m/E_{\vec{p}})\Gamma/2)} \ ,$$

which can be seen using

$$(p^0)^2 - |\vec{p}|^2 - m^2 = (p^0)^2 - E_{\vec{p}}^2 = \left(p^0 + E_{\vec{p}}\right)\left(p^0 - E_{\vec{p}}\right) \approx 2E_{\vec{p}}\left(p^0 - E_{\vec{p}}\right) \ .$$

# Cross Sections and the S-Matrix

## Initial and Final States— In- and Out-States:

Recall our discussion of particle creation by an external source,

$$(\Box + m^2)\phi(x) = j(x) \ ,$$

where $j$ was assumed to be nonzero only during a finite interval $t_1 < t < t_2$.

- In that case, the Fock space of the free theory for $t < t_1$ defined the in-states, the Fock space of the free theory for $t > t_2$ defined the out-states, and we could calculate exactly the relationship between the two.
- We started in the in-vacuum and stayed there. The amplitude

$$\langle \vec{p}_1 \, \vec{p}_2 \, \ldots \, \vec{p}_N, \text{out} \, | 0, in \rangle$$

was then interpreted as the amplitude for producing a set of final particles with momenta $\vec{p}_1 \, \ldots \, \vec{p}_N$.

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For interacting QFT's, it is more complicated. The interactions do not turn off, and affect even the 1-particle states. It is still possible to define in- and out-states $|\vec{p}_1 \, \ldots \, \vec{p}_N, \text{in}\rangle$ and $|\vec{p}_1 \, \ldots \, \vec{p}_N, \text{out}\rangle$ with the following properties:

- They are exact eigenstates of the full Hamiltonian.
- At asymptotically early times, wavepackets constructed from $|\vec{p}_1 \, \ldots \, \vec{p}_N, \text{in}\rangle$ evolve as free wavepackets. (The pieces of this ket that describe the scattering vanish in stationary phase approximation at early times.) These states are used to describe the initial state of the scattering.
- At asymptotically late times, wavepackets constructed from $|\vec{p}_1 \, \ldots \, \vec{p}_N, \text{out}\rangle$ evolve as free wavepackets. These states are used to describe the final state.

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## Wavepacket States:

One-particle incoming wave packet:

$$|\phi\rangle = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{k}}}} \phi(\vec{k}) \left|\vec{k}, in\right\rangle \;,$$

where

$$\langle \phi\,|\phi\,\rangle = 1 \quad \Longrightarrow \quad \int \frac{d^3k}{(2\pi)^3} \left|\phi(\vec{k})\right|^2 = 1 \;.$$

Two-particle initial state:

$$\left|\phi_{\mathcal{A}}\phi_{\mathcal{B}}, \vec{b}, \mathrm{in}\right\rangle = \int \frac{d^3k_{\mathcal{A}}}{(2\pi)^3} \frac{d^3k_{\mathcal{B}}}{(2\pi)^3} \frac{\phi_{\mathcal{A}}(\vec{k}_{\mathcal{A}})\,\phi_{\mathcal{B}}(\vec{k}_{\mathcal{B}})e^{-i\vec{b}\cdot\vec{k}_{\mathcal{B}}}}{\sqrt{(2E_{\mathcal{A}})(2E_{\mathcal{B}})}} \left|\vec{k}_{\mathcal{A}}\,\vec{k}_{\mathcal{B}}, \mathrm{in}\right\rangle \;,$$

where $\vec{b}$ is a vector which translates particle $\mathcal{B}$ orthogonal to the beam, so that we can construct collisions with different impact parameters.

Multiparticle final state:

$$\langle \phi_1 \, \ldots \, \phi_n, \mathrm{out}| = \left( \prod_{f=1}^{n} \int \frac{d^3p_f}{(2\pi)^3} \frac{\phi_f(\vec{p}_f)}{\sqrt{2E_f}} \right) \langle \vec{p}_1 \, \ldots \, \vec{p}_n, \mathrm{out}| \;.$$



## The S-Matrix:

Definition:

$$\boxed{S\,|\Psi, \mathrm{out}\rangle = |\Psi, \mathrm{in}\rangle \;.}$$

Therefore

$$\langle \Psi, \mathrm{out}\,|\Psi, \mathrm{in}\,\rangle = \langle \Psi, \mathrm{out}\,|S|\,\Psi, \mathrm{out}\rangle \;.$$

But $S$ maps a complete set of orthonormal states onto a complete set of orthonormal states, so $S$ is unitary. Therefore

$$\begin{aligned} \langle \Psi, \mathrm{out}\,|S|\,\Psi, \mathrm{out}\rangle &= \left\langle \Psi, \mathrm{out}\,\left|S^\dagger\, S\, S\right|\,\Psi, \mathrm{out}\right\rangle \\ &= \langle \Psi, \mathrm{in}\,|S|\,\Psi, \mathrm{in}\rangle \;, \end{aligned}$$

so P&S often do not label the states as in or out.

## $S$, $T$, and $\mathcal{M}$:

No scattering $\Longrightarrow$ final $=$ initial, so separate this part of $S$:

$$\boxed{S \equiv 1 + iT\ .}$$

But $T$ must contain a momentum-conserving $\delta$-function, so define

$$\boxed{\langle \vec{p}_1 \ldots \vec{p}_n \, | iT | \, k_{\mathcal{A}}\, k_{\mathcal{B}} \rangle \equiv (2\pi)^4 \delta^{(4)} \left( k_{\mathcal{A}} + k_{\mathcal{B}} - \sum p_f \right) \cdot i\mathcal{M}(k_{\mathcal{A}} k_{\mathcal{B}} \to \{p_f\})\ .}$$

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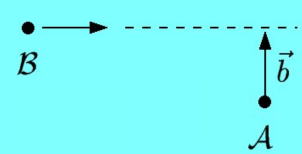
## Evaluation of the Cross Section:

The probability of scattering into the specified final states is just the square of the $S$-matrix element, summed over the final states:

$$\mathcal{P}\left(\{\mathcal{AB}, \vec{b}\} \to \{p_1 \ldots p_n\}\right) = \left( \prod_{f=1}^{n} \int \frac{\mathrm{d}^3 p_f}{(2\pi)^3} \frac{1}{2E_f} \right) \left| \left\langle \vec{p}_1 \ldots \vec{p}_n \, | S | \, \phi_{\mathcal{A}} \phi_{\mathcal{B}}, \vec{b} \right\rangle \right|^2 .$$

To relate to the cross section, think of a single particle $\mathcal{B}$ scattering off of a particle $\mathcal{A}$, with impact parameter vector $\vec{b}$:




Remembering that the cross section can be viewed as the cross sectional area blocked off by the target particle,

$$\mathrm{d}\sigma = \int \mathrm{d}^2 b \; \mathcal{P}\left(\{\mathcal{AB}, \vec{b}\} \to \{p_1 \ldots p_n\}\right)\ .$$

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Substituting the expression for $\mathcal{P}$ and writing out the wavepacket integrals describing the initial state,

$$d\sigma = \int d^2 b \left( \prod_{f=1}^{n} \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f} \right) \int \frac{d^3 k_{\mathcal{A}}}{(2\pi)^3} \int \frac{d^3 k_{\mathcal{B}}}{(2\pi)^3} \frac{\phi_{\mathcal{A}}(\vec{k}_{\mathcal{A}})\, \phi_{\mathcal{B}}(\vec{k}_{\mathcal{B}})}{\sqrt{(2E_{\mathcal{A}})(2E_{\mathcal{B}})}}$$

$$\times \int \frac{d^3 \bar{k}_{\mathcal{A}}}{(2\pi)^3} \int \frac{d^3 \bar{k}_{\mathcal{B}}}{(2\pi)^3} \frac{\phi^*_{\mathcal{A}}(\vec{\bar{k}}_{\mathcal{A}})\, \phi^*_{\mathcal{B}}(\vec{\bar{k}}_{\mathcal{B}})}{\sqrt{(2\bar{E}_{\mathcal{A}})(2\bar{E}_{\mathcal{B}})}} e^{i\vec{b}\cdot(\vec{\bar{k}}_{\mathcal{B}} - \vec{k}_{\mathcal{B}})}$$

$$\times \left\langle \vec{p}_1 \ldots \vec{p}_n \left| S \right| \vec{k}_{\mathcal{A}} \vec{k}_{\mathcal{B}} \right\rangle \left\langle \vec{p}_1 \ldots \vec{p}_n \left| S \right| \vec{\bar{k}}_{\mathcal{A}} \vec{\bar{k}}_{\mathcal{B}} \right\rangle^* .$$

This can be simplified by using

$$\int d^2 b\, e^{i\vec{b}\cdot(\vec{\bar{k}}_{\mathcal{B}} - \vec{k}_{\mathcal{B}})} = (2\pi)^2 \delta^{(2)} \left( \vec{\bar{k}}^{\perp}_{\mathcal{B}} - \vec{k}^{\perp}_{\mathcal{B}} \right) ,$$

$$\left\langle \vec{p}_1 \ldots \vec{p}_n \left| S \right| \vec{k}_{\mathcal{A}} \vec{k}_{\mathcal{B}} \right\rangle = i\mathcal{M} \left( \vec{k}_{\mathcal{A}} \vec{k}_{\mathcal{B}} \to \{\vec{p}_f\} \right) (2\pi)^4 \delta^{(4)} \left( k_{\mathcal{A}} + k_{\mathcal{B}} - \sum p_f \right) ,$$

$$\left\langle \vec{p}_1 \ldots \vec{p}_n \left| S \right| \vec{\bar{k}}_{\mathcal{A}} \vec{\bar{k}}_{\mathcal{B}} \right\rangle^* = -i\mathcal{M}^* \left( \vec{\bar{k}}_{\mathcal{A}} \vec{\bar{k}}_{\mathcal{B}} \to \{\vec{p}_f\} \right) (2\pi)^4 \delta^{(4)} \left( \bar{k}_{\mathcal{A}} + \bar{k}_{\mathcal{B}} - \sum p_f \right) .$$

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We first integrate over $\vec{\bar{k}}_{\mathcal{A}}$ and $\vec{\bar{k}}_{\mathcal{B}}$ using $\delta^{(2)} \left( \vec{\bar{k}}^{\perp}_{\mathcal{B}} - \vec{k}^{\perp}_{\mathcal{B}} \right)$ and

$$\delta^{(4)} \left( \bar{k}_{\mathcal{A}} + \bar{k}_{\mathcal{B}} - \sum p_f \right) = \delta^{(2)} \left( \vec{\bar{k}}^{\perp}_{\mathcal{A}} + \vec{\bar{k}}^{\perp}_{\mathcal{B}} - \sum p^{\perp}_f \right) \delta \left( \bar{k}^z_{\mathcal{A}} + \bar{k}^z_{\mathcal{B}} - \sum p^z_f \right)$$

$$\times \delta \left( \bar{E}_{\mathcal{A}} + \bar{E}_{\mathcal{B}} - \sum E_f \right) ,$$

where the beam is taken along the $z$-axis. After integrating over $\vec{\bar{k}}^{\perp}_{\mathcal{A}}$ and $\vec{\bar{k}}^{\perp}_{\mathcal{B}}$, we are left with

$$\int d\bar{k}^z_{\mathcal{A}} d\bar{k}^z_{\mathcal{B}}\, \delta \left( \bar{k}^z_{\mathcal{A}} + \bar{k}^z_{\mathcal{B}} - \sum p^z_f \right) \delta \left( \bar{E}_{\mathcal{A}} + \bar{E}_{\mathcal{B}} - \sum E_f \right) = \int d\bar{k}^z_{\mathcal{A}}\, \delta \left( F(\bar{k}^z_{\mathcal{A}}) \right) ,$$

where the first $\delta$-function was used to integrate $\bar{k}^z_{\mathcal{B}}$, and

$$F(\bar{k}^z_{\mathcal{A}}) = \sqrt{\left|\bar{k}^{\perp}_{\mathcal{A}}\right|^2 + \left(\bar{k}^z_{\mathcal{A}}\right)^2 + m^2} + \sqrt{\left|\bar{k}^{\perp}_{\mathcal{B}}\right|^2 + \left(\sum p^z_f - \bar{k}^z_{\mathcal{A}}\right)^2 + m^2} - \sum E_f .$$

Then

$$\int d\bar{k}^z_{\mathcal{A}}\, \delta \left( F(\bar{k}^z_{\mathcal{A}}) \right) = \frac{1}{\left| \dfrac{dF}{d\bar{k}^z_{\mathcal{A}}} \right|} , \text{ evaluated where } F(\bar{k}^z_{\mathcal{A}}) = 0 .$$

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Rewriting

$$F(\bar{k}^z_{\mathcal{A}}) = \sqrt{|\bar{k}^\perp_{\mathcal{A}}|^2 + \left(\bar{k}^z_{\mathcal{A}}\right)^2 + m^2} + \sqrt{|\bar{k}^\perp_{\mathcal{B}}|^2 + \left(\sum p^z_f - \bar{k}^z_{\mathcal{A}}\right)^2 + m^2} - \sum E_f \ ,$$

one finds

$$\left|\frac{\mathrm{d}F}{\mathrm{d}\bar{k}^z_{\mathcal{A}}}\right| = \left|\frac{\bar{k}^z_{\mathcal{A}}}{\bar{E}_{\mathcal{A}}} - \frac{\sum p^z_f - \bar{k}^z_{\mathcal{A}}}{\bar{E}_{\mathcal{B}}}\right|$$

Remembering the $\delta$-function constraint $\delta\left(\bar{k}^z_{\mathcal{A}} + \bar{k}^z_{\mathcal{B}} - \sum p^z_f\right)$ from the previous slide, one has

$$\left|\frac{\mathrm{d}F}{\mathrm{d}\bar{k}^z_{\mathcal{A}}}\right| = \left|\frac{\bar{k}^z_{\mathcal{A}}}{\bar{E}_{\mathcal{A}}} - \frac{\bar{k}^z_{\mathcal{B}}}{\bar{E}_{\mathcal{B}}}\right| = |\bar{v}^z_{\mathcal{A}} - \bar{v}^z_{\mathcal{B}}| \ .$$

What values of $\bar{k}^z_{\mathcal{A}}$ satisfy the constraint $F(\bar{k}^z_{\mathcal{A}}) = 0$? There are two solutions, since $F(\bar{k}^z_{\mathcal{A}}) = 0$ can be manipulated into a simple quadratic equation. (To see this, move one square root to the RHS of the equation and square both sides. The $(\bar{k}^z_{\mathcal{A}})^2$ term on each side cancels, leaving only linear terms and a square root on the LHS. Isolate the square root and square both sides again, obtaining a quadratic equation.) One solution gives $\vec{\bar{k}}_{\mathcal{A}} = \vec{k}_{\mathcal{A}}$, and the other corresponds to $\mathcal{A}$ and $\mathcal{B}$ approaching each other from opposite directions. Assume that the initial wavepacket is too narrow to overlap the 2nd solution.

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Then

$$\mathrm{d}\sigma = \left(\prod_{f=1}^{n} \frac{\mathrm{d}^3 p_f}{(2\pi)^3} \frac{1}{2E_f}\right) \int \frac{\mathrm{d}^3 k_{\mathcal{A}}}{(2\pi)^3} \int \frac{\mathrm{d}^3 k_{\mathcal{B}}}{(2\pi)^3} \frac{|\phi_{\mathcal{A}}(\vec{k}_{\mathcal{A}})|^2 \, |\phi_{\mathcal{B}}(\vec{k}_{\mathcal{B}})|^2}{(2E_{\mathcal{A}})(2E_{\mathcal{B}})\, |v^z_{\mathcal{A}} - v^z_{\mathcal{B}}|}$$

$$\times \left|\mathcal{M}\left(\vec{k}_{\mathcal{A}}\vec{k}_{\mathcal{B}} \to \{\vec{p}_f\}\right)\right|^2 (2\pi)^4 \delta^{(4)}\left(k_{\mathcal{A}} + k_{\mathcal{B}} - \sum p_f\right) \ .$$

Define the relativistically invariant n-body phase space measure

$$\boxed{\mathrm{d}\Pi_n(P) \equiv \left(\prod_{f=1}^{n} \frac{\mathrm{d}^3 p_f}{(2\pi)^3} \frac{1}{2E_f}\right) (2\pi)^4 \delta^{(4)}\left(P - \sum p_f\right) \ ,}$$

and assume that $E_{\mathcal{A}}(\vec{k}_{\mathcal{A}})$, $E_{\mathcal{B}}(\vec{k}_{\mathcal{B}})$, $|v^z_{\mathcal{A}} - v^z_{\mathcal{B}}|$, $\left|\mathcal{M}\left(\vec{k}_{\mathcal{A}}\vec{k}_{\mathcal{B}} \to \{\vec{p}_f\}\right)\right|^2$, and $\mathrm{d}\Pi_n\left(k_{\mathcal{A}} + k_{\mathcal{B}}\right)$ are all sufficiently slowly varying that they can be evaluated at the central momenta of the two intial wavepackets, $\vec{k}_{\mathcal{A}} = \vec{p}_{\mathcal{A}}$ and $\vec{k}_{\mathcal{B}} = \vec{p}_{\mathcal{B}}$. Then the normalization of the wavepackets implies that

$$\int \frac{\mathrm{d}^3 k_{\mathcal{A}}}{(2\pi)^3} \int \frac{\mathrm{d}^3 k_{\mathcal{B}}}{(2\pi)^3} \, |\phi_{\mathcal{A}}(\vec{k}_{\mathcal{A}})|^2 \, |\phi_{\mathcal{B}}(\vec{k}_{\mathcal{B}})|^2 = 1 \ ,$$

–45–

so finally

$$\boxed{d\sigma = \frac{\left|\mathcal{M}\left(\vec{p}_{\mathcal{A}}\vec{p}_{\mathcal{B}} \to \{\vec{p}_f\}\right)\right|^2}{(2E_{\mathcal{A}})(2E_{\mathcal{B}})\left|v^z_{\mathcal{A}} - v^z_{\mathcal{B}}\right|}\, d\Pi_n(p_{\mathcal{A}} + p_{\mathcal{B}}) \ .}$$

This formula holds whether the final state particles are distinguishable or not. In calculating a *total* cross section, however, one must not double-count final states. If the final state contains $n$ identical particles, one must either restrict the integration or divide the answer by $n!$.



**Special Case: Two-Particle Final State:**

In the center of mass (CM) frame, $\vec{p}_{\mathcal{A}} = -\vec{p}_{\mathcal{B}}$ and $E_{\rm cm} = E_{\mathcal{A}} + E_{\mathcal{B}}$, so

$$\begin{aligned}
d\Pi_2(p_{\mathcal{A}} + p_{\mathcal{B}}) &= \left(\prod_{f=1}^{2} \frac{d^3 p_f}{(2\pi)^3}\frac{1}{2E_f}\right)(2\pi)^4\delta^{(4)}\left(p_{\mathcal{A}} + p_{\mathcal{B}} - \sum p_f\right) \\
&= \frac{d^3 p_1}{(2\pi)^3}\frac{d^3 p_2}{(2\pi)^3}\frac{1}{(2E_1)(2E_2)}(2\pi)^4\delta^{(4)}\left(p_{\mathcal{A}} + p_{\mathcal{B}} - p_1 - p_2\right) \\
&= \frac{d^3 p_1}{(2\pi)^3}\frac{1}{(2E_1)(2E_2)}(2\pi)\delta(E_{\rm cm} - E_1 - E_2) \\
&= d\Omega\, \frac{p_1^2\, dp_1}{(2\pi)^3}\, \frac{1}{(2E_1)(2E_2)}(2\pi)\delta\left(E_{\rm cm} - \sqrt{p_1^2 + m_1^2} - \sqrt{p_1^2 + m_2^2}\right) \\
&= d\Omega\, \frac{p_1^2}{(2\pi)^2}\, \frac{1}{(2E_1)(2E_2)}\left|\frac{p_1}{E_1} + \frac{p_1}{E_2}\right|^{-1} \\
&= d\Omega\, \frac{p_1}{16\pi^2 E_{\rm cm}} \ .
\end{aligned}$$

The two-particle final state, center-of-mass cross section is then

$$\boxed{\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{cm}} = \frac{|\vec{p}_{\mathcal{A}}|\, |\mathcal{M}(\vec{p}_{\mathcal{A}}\vec{p}_{\mathcal{B}} \to \vec{p}_1\vec{p}_2)|^2}{64\pi^2 E_{\mathcal{A}} E_{\mathcal{B}}(E_{\mathcal{A}}+E_{\mathcal{B}})\, |v^z_{\mathcal{A}} - v^z_{\mathcal{B}}|}\ .}$$

If all four masses are equal, then

$$E_{\mathcal{A}} = E_{\mathcal{B}} = \frac{1}{2}E_{\mathrm{cm}}$$

and

$$|v^z_{\mathcal{A}} - v^z_{\mathcal{B}}| = \frac{2\,|\vec{p}_{\mathcal{A}}|}{E_{\mathcal{A}}} = \frac{4\,|\vec{p}_{\mathcal{A}}|}{E_{\mathrm{cm}}}\ ,$$

so

$$\boxed{\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{cm}} = \frac{|\mathcal{M}|^2}{64\pi^2 E^2_{\mathrm{cm}}} \qquad \text{(all masses equal)}\ .}$$



**Evaluation of the Decay Rate:**

The formula for decay rates is more difficult to justify, since decaying particles have to be viewed as resonances in a scattering experiment. For now we just state the result. By analogy with the formula for cross sections,

$$\mathrm{d}\sigma = \frac{|\mathcal{M}(\vec{p}_{\mathcal{A}}\vec{p}_{\mathcal{B}} \to \{\vec{p}_f\})|^2}{(2E_{\mathcal{A}})(2E_{\mathcal{B}})\, |v^z_{\mathcal{A}} - v^z_{\mathcal{B}}|}\, \mathrm{d}\Pi_n(p_{\mathcal{A}} + p_{\mathcal{B}})\ ,$$

we write

$$\boxed{\mathrm{d}\Gamma = \frac{|\mathcal{M}(\vec{p}_{\mathcal{A}} \to \{\vec{p}_f\})|^2}{2E_{\mathcal{A}}}\, \mathrm{d}\Pi_n(p_{\mathcal{A}})\ .}$$

Here $\mathcal{M}$ cannot be defined in terms of an $S$-matrix, since decaying particles cannot be described by wavepackets constructed in the asymptotic past. $\mathcal{M}$ can be calculated, however, by the Feynman rules that Peskin & Schroeder describe in Section 4.6. If some or all of the final state particles are identical, then the same comments that were made about cross sections apply here.

# Summary of Key Results So Far

**Time-dependent perturbation theory:**

$$\begin{aligned}&\langle\Omega\,|T\,\{\phi(x_1)\ldots\phi(x_n)\}|\,\Omega\rangle\\ &\quad=\left\langle 0\left|T\left\{\phi_I(x_1)\ldots\phi_I(x_n)\exp\left[-i\int \mathrm{d}^4z\,\mathcal{H}_I(z)\right]\right\}\right|0\right\rangle_{\text{connected}}\\ &\quad=\text{Sum of all connected diagrams with external points } x_1\ldots x_n\ .\end{aligned}$$

Status: derivation was more or less rigorous, except for ignoring problems connected with renormalization: evaluation of integrals in this expression will lead to divergences. These questions will be dealt with next term. If the theory is regulated, for example by defining it on a lattice of finite size, the formula above would be exactly true for the regulated theory. One finds, however, that the limit as the lattice spacing goes to zero cannot be taken unless the parameters $m$, $\lambda$, etc., are allowed to vary as the limit is taken, and in addition the field operators must be rescaled.



**Cross sections from $S$-matrix elements:**

$$S=1+iT\ ,$$

where

$$\langle\vec{p}_1\ \ldots\ \vec{p}_n\,|iT|\,k_{\mathcal{A}}\,k_{\mathcal{B}}\rangle\equiv(2\pi)^4\delta^{(4)}\left(k_{\mathcal{A}}+k_{\mathcal{B}}-\sum p_f\right)\cdot i\mathcal{M}(k_{\mathcal{A}}k_{\mathcal{B}}\to\{p_f\})\ .$$

The relativistically invariant n-body phase space measure is

$$\mathrm{d}\Pi_n(P)\equiv\left(\prod_{f=1}^{n}\frac{\mathrm{d}^3p_f}{(2\pi)^3}\frac{1}{2E_f}\right)(2\pi)^4\delta^{(4)}\left(P-\sum p_f\right)\ ,$$

and the differential cross section is

$$\mathrm{d}\sigma=\frac{\left|\mathcal{M}\left(\vec{p}_{\mathcal{A}}\vec{p}_{\mathcal{B}}\to\{\vec{p}_f\}\right)\right|^2}{(2E_{\mathcal{A}})(2E_{\mathcal{B}})\,|v^z_{\mathcal{A}}-v^z_{\mathcal{B}}|}\,\mathrm{d}\Pi_n(p_{\mathcal{A}}+p_{\mathcal{B}})\ .$$

Status: this derivation was more or less rigorous, making mild assumptions about in- and out-states. These assumptions, and the formula above, will need to be modified slightly when massless particles are present, since the resulting long-range forces modify particle trajectories even in the asymptotic past. These modifications arise only in higher order perturbation theory, and are part of the renormalization issue.

Special case— two-particle final states, in the center-of-mass frame:

$$\left(\frac{d\sigma}{d\Omega}\right)_{\rm cm} = \frac{|\vec p_{\mathcal A}|\,\left|\mathcal M\left(\vec p_{\mathcal A}\vec p_{\mathcal B} \to \vec p_1 \vec p_2\right)\right|^2}{64\pi^2 E_{\mathcal A} E_{\mathcal B}(E_{\mathcal A}+E_{\mathcal B})\,|v^z_{\mathcal A} - v^z_{\mathcal B}|}$$

$$= \frac{|\mathcal M|^2}{64\pi^2 E^2_{\rm cm}} \qquad \text{(if all masses are equal)} \ .$$

**Decay rate from $S$-matrix elements:**

$$d\Gamma = \frac{\left|\mathcal M\left(\vec p_{\mathcal A} \to \{\vec p_f\}\right)\right|^2}{2E_{\mathcal A}}\, d\Pi_n(p_{\mathcal A}) \ .$$

Status: completely nonrigorous at this point. Unstable particles should be treated as resonances, an issue which is discussed in Peskin & Schroeder in Chapter 7.

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# Computing the S-Matrix From Feynman Diagrams

Complication:

$$\langle \vec p_1 \ldots \vec p_n \,| S |\, \vec p_{\mathcal A} \vec p_{\mathcal B} \rangle \equiv \langle \vec p_1 \ldots \vec p_n, {\rm out} \,|\, \vec p_{\mathcal A} \vec p_{\mathcal B}, {\rm in} \rangle \ ,$$

but the in- and out-states are hard to construct: even single-particle states are modified by interactions. The solution will make use of the fact that

$$\langle \Omega \,|\phi(x)|\, \vec p \rangle = \langle \Omega \,| e^{iP\cdot x}\phi(0) e^{-iP\cdot x} |\, \vec p \rangle = e^{-ip\cdot x}\langle \Omega \,|\phi(0)|\, \vec p\rangle$$

is an exact expression for the interacting fields, with the full operator $P^\mu$ and the exact eigenstate $|\vec p\rangle$. By generalizing this to in- and out-states, it will be possible to manipulate the correlation functions $\langle \Omega \,| T(\phi_1 \ldots \phi_n) |\, \Omega \rangle$ by inserting complete sets of in- and out-states at various places. When the correlation function is Fourier-transformed in its variables $x_1 \ldots x_n$ to produce a function of $p_1 \ldots p_n$, one can show that it contains poles when any $p_i$ is on its mass shell, $p_i^2 = m_i^2$, and that the residue when all the $p_i$ are on mass shell is the $S$-matrix element.

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A derivation will be given in Chapter 7, but for now we accept the intuitive notion that $U(t_2, t_1)$ describes time evolution in the interaction picture, and that the $S$-matrix describes time evolution from minus infinity to infinity. So we write

$$\boxed{\begin{aligned}&\langle \vec{p}_1 \ldots \vec{p}_n \,| S |\, \vec{p}_{\mathcal{A}} \vec{p}_{\mathcal{B}} \rangle \\ &\qquad = {}_I\!\left\langle \vec{p}_1 \ldots \vec{p}_n \left| T \left\{ \exp\left[ -i \int \mathrm{d}^4 x\, \mathcal{H}_I(x) \right] \right\} \right| \vec{p}_{\mathcal{A}} \vec{p}_{\mathcal{B}} \right\rangle_I \Bigg|_{\substack{\text{connected,}\\ \text{amputated}}} ,\end{aligned}}$$

where "connected" means that the disconnected diagrams will cancel out as before, and the meaning of "amputated" will be discussed below. It will be shown in Chapter 7 that this formula is valid, up to an overall multiplicative factor that arises only in higher-order perturbation theory, and is associated with the rescaling of field operators required by renormalization.



**Calculation of 2 ⟶ 2 Scattering:**

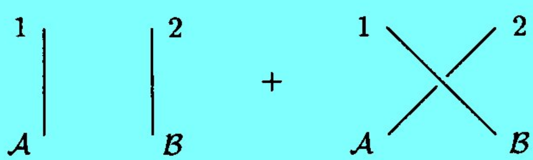
Normalization conventions:

$$|\vec{p}\rangle = \sqrt{2E_{\vec{p}}}\, a^\dagger(\vec{p})\, |0\rangle \ , \qquad \left[ a(\vec{q})\, ,\, a^\dagger \vec{p} \right] = (2\pi)^3 \delta^{(3)}(\vec{q} - \vec{p}) \ .$$

To zeroth order in $\mathcal{H}_I$,

$$\begin{aligned}\langle \vec{p}_1 \vec{p}_2 \,| S |\, \vec{p}_{\mathcal{A}} \vec{p}_{\mathcal{B}} \rangle &= {}_I\langle \vec{p}_1 \vec{p}_2 \,|\, \vec{p}_{\mathcal{A}} \vec{p}_{\mathcal{B}} \rangle_I \\ &= \sqrt{(2E_1)(2E_2)(2E_{\mathcal{A}})(2E_{\mathcal{B}})} \left\langle 0 \left| a_1 a_2 a_{\mathcal{A}}^\dagger a_{\mathcal{B}}^\dagger \right| 0 \right\rangle \\ &= (2E_{\mathcal{A}})(2E_{\mathcal{B}})(2\pi)^6 \left[ \delta^{(3)}(\vec{p}_{\mathcal{A}} - \vec{p}_1) \delta^{(3)}(\vec{p}_{\mathcal{B}} - \vec{p}_2) \right. \\ &\qquad \left. + \, \delta^{(3)}(\vec{p}_{\mathcal{A}} - \vec{p}_2) \delta^{(3)}(\vec{p}_{\mathcal{B}} - \vec{p}_1) \right] \ .\end{aligned}$$

Graphically,




Contributes only to "1" of $S = 1 + iT$.

To first order in $\mathscr{H}_I$:

$$
{}_I\left\langle \vec p_1 \vec p_2 \left| T\left\{ -i\frac{\lambda}{4!}\int \mathrm{d}^4x\, \phi_I^4(x) \right\} \right| \vec p_{\mathcal{A}} \vec p_{\mathcal{B}} \right\rangle_I
$$

$$
= {}_I\left\langle \vec p_1 \vec p_2 \left| N\left\{ -i\frac{\lambda}{4!}\int \mathrm{d}^4x\, \phi_I^4(x) + \text{contractions} \right\} \right| \vec p_{\mathcal{A}} \vec p_{\mathcal{B}} \right\rangle_I \ ,
$$

where

$$
\text{contractions} = -i\frac{\lambda}{4!}\left[ 6\ \overbracket{\phi\phi}\ \phi(x)\phi(x) + 3\ \overbracket{\phi\phi}\ \overbracket{\phi\phi} \right] \ .
$$

Uncontracted fields can destroy particles in initial state or create them in the final state:

$$
\phi_I(x) = \phi_I^+(x) + \phi_I^-(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec k}}}\left[ a_I(\vec k)e^{-ik\cdot x} + a_I^\dagger(\vec k)e^{ik\cdot x} \right] \ ,
$$

so

$$
\phi_I^+(x)\, |\vec p\rangle_I = e^{-ip\cdot x}\ |0\rangle_I \ ,
$$

where $\phi_I^+$ and $\phi_I^-$ refer to the parts of $\phi_I(x)$ containing annihilation and creation operators, respectively.

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This leads to a new type of contraction:

$$
\overbracket{\phi_I(x)\, |\vec p}\rangle_I = e^{-ip\cdot x} \ , \qquad {}_I\langle \overbracket{\vec p|\, \phi_I}(x) = e^{+ip\cdot x} \ .
$$

We show this kind of contraction in a Feynman diagram as an external line.

Looking at the contracted terms from the Wick expansion, the fully contracted term produces a multiple of the identity matrix element,

$$
-i\frac{\lambda}{4!}\int \mathrm{d}^4x\ {}_I\left\langle \vec p_1 \vec p_2 \left| \overbracket{\phi\phi}\,\overbracket{\phi\phi} \right| \vec p_{\mathcal{A}} \vec p_{\mathcal{B}} \right\rangle_I = -i\frac{\lambda}{4!}\int \mathrm{d}^4x\, \overbracket{\phi\phi}\,\overbracket{\phi\phi} \times {}_I\langle \vec p_1 \vec p_2\, | \vec p_{\mathcal{A}} \vec p_{\mathcal{B}}\, \rangle_I \ ,
$$

so this term also contributes only to the uninteresting $1$ part of $S = 1 + iT$.

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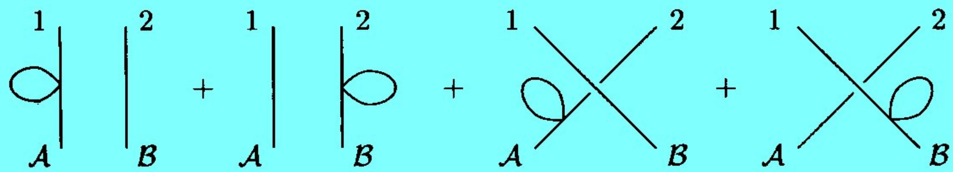
The singly contracted term

$$-i\frac{6\lambda}{4!}\int \mathrm{d}^4x_I \left\langle \vec{p}_1\vec{p}_2 \left| \overline{\phi\phi}\,\phi(x)\phi(x) \right| \vec{p}_{\mathcal{A}}\vec{p}_{\mathcal{B}} \right\rangle_I$$

contains terms where one $\phi(x)$ contracts with an incoming particle and the other contracts with an outgoing particle, giving the Feynman diagrams

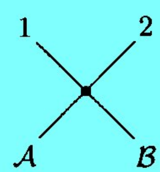



The integration over $x$ gives an energy-momentum conserving $\delta$-function, and the uncontracted inner product produces another, so these diagrams are again a contribution to the $1$ of $S = 1 + iT$.

The contributions to $T$ come from **fully connected** diagrams, where all external lines are connected to each other.



Nontrivial contribution to T:




There are 4! ways of contracting the 4 fields with the 4 external lines, so the contribution is

$$\begin{aligned} 4! \cdot \left(-i\frac{\lambda}{4!}\right) \int \mathrm{d}^4x\, e^{-i(p_{\mathcal{A}}+p_{\mathcal{B}}-p_1-p_2)\cdot x} \\ = -i\lambda(2\pi)^4\delta^{(4)}(p_{\mathcal{A}}+p_{\mathcal{B}}-p_1-p_2) \\ \equiv i\mathcal{M}(2\pi)^4\delta^{(4)}(p_{\mathcal{A}}+p_{\mathcal{B}}-p_1-p_2)\ , \end{aligned}$$

so

$$\boxed{\mathcal{M} = -\lambda\ .}$$

Repeating,

$$\boxed{\mathcal{M} = -\lambda\ .}$$

By our previous rules, this implies

$$\boxed{\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{cm}} = \frac{\lambda^2}{64\pi^2 E_{\mathrm{cm}}^2}\ .}$$

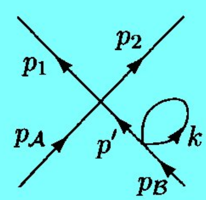
For $\sigma_{\mathrm{total}}$ one uses the fact that the two final particles are identical. If we integrate over all final angles we have double-counted, so we divide the answer by 2!.

$$\sigma_{\mathrm{total}} = \frac{\lambda^2}{64\pi^2 E_{\mathrm{cm}}^2} \times 4\pi \times \frac{1}{2!} = \boxed{\frac{\lambda^2}{32\pi E_{\mathrm{cm}}^2}\ .}$$



**Amputation:**

Consider the 2nd order diagram




Contribution is

$$\frac{1}{2}\int \frac{\mathrm{d}^4 p'}{(2\pi)^4}\frac{i}{p'^2 - m^2}\int \frac{\mathrm{d}^4 k}{(2\pi)^4}\frac{i}{k^2 - m^2}$$
$$\times (-i\lambda)(2\pi)^4\delta^{(4)}(p_{\mathcal{A}} + p_{\mathcal{B}} - p_1 - p_2) \times (-i\lambda)(2\pi)^4\delta^{(4)}(p_{\mathcal{B}} - p')\ .$$

Note that $\delta^{(4)}(p_{\mathcal{B}} - p') \quad \Longrightarrow \quad p'^2 = m^2$, so

$$\frac{1}{p'^2 - m^2} = \frac{1}{0}\ ,$$

which is infinite. Any diagram in which all the momentum from one external line is channeled through a single internal line will produce an infinite propagator.

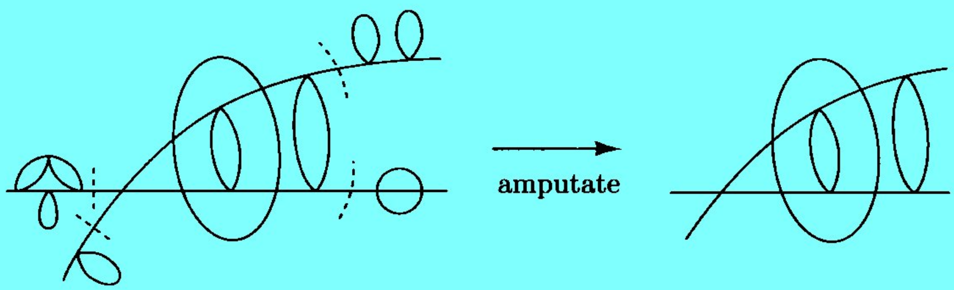
Note that $\delta^{(4)}(p_\mathcal{B} - p') \quad \Longrightarrow \quad p'^2 = m^2$, so

$$\frac{1}{p'^2 - m^2} = \frac{1}{0} \,,$$

which is infinite. Any diagram in which all the momentum from one external line is channeled through a single internal line will produce an infinite propagator.

**Amputation:** Eliminate all diagrams for which cutting a single line results in separating a single leg from the rest of the diagram.

For example,



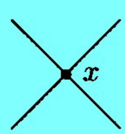
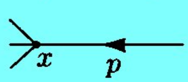



**Feynman Rules for $\lambda\phi^4$ in Position Space:**

$$i\mathcal{M} \cdot (2\pi)^4 \delta^{(4)}(p_\mathcal{A} + p_\mathcal{B} - \sum p_f)$$

$$= (\text{sum of all connected, amputated diagrams}) \,,$$

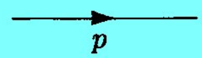
where the diagrams are constructed by the following rules:

1. For each propagator,  $= D_F(x - y)$;
2. For each vertex,  $= (-i\lambda) \int d^4x$;
3. For each external line,  $= e^{-ip\cdot x}$;
4. Divide by the symmetry factor.

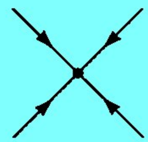
### Feynman Rules for $\lambda\phi^4$ in Momentum Space:

$$i\mathcal{M} = \text{(sum of all connected, amputated diagrams)}\ ,$$

where the diagrams are constructed by the following rules:

1. For each propagator, 

$= \dfrac{i}{p^2 - m^2 + i\epsilon}$;

2. For each vertex,  $= -i\lambda$;

3. For each external line,  $= 1$;

4. Impose momentum conservation at each vertex;

5. Integrate over each undetermined loop momentum: $\displaystyle\int \frac{d^4p}{(2\pi)^4}$;

6. Divide by the symmetry factor.



# Feynman Rules for Fermions

**Time-dependent perturbation theory:**

Generalizes easily, since $\mathcal{H}_I$ is bilinear in Fermi fields, so it obeys commutation relations:

$$\langle \Omega \left| T\left\{\phi \ldots \psi \ldots \bar{\psi} \ldots \right\} \right| \Omega \rangle$$

$$= {}_I\left\langle 0 \left| T\left\{ \phi_I \ldots \psi_I \ldots \bar{\psi}_I \ldots \exp\left[ -i \int \mathrm{d}^4 z\, \mathcal{H}_I(z) \right] \right\} \right| 0 \right\rangle_{I,\text{connected}}$$

$= $ Sum of all connected diagrams with specified external points .

But, to use Wick's theorem, we must define time-ordering and normal ordering for fermion operators.

**Time-ordered products of Fermi fields:**

Suppose $x$ and $y$ are spacelike separated, with $y^0 > x^0$. Then

$$\psi(x)\psi(y) = -\psi(y)\psi(x) \ .$$

The RHS is already time-ordered, so $T\{-\psi(y)\psi(x)\} = -\psi(y)\psi(x)$. If $T$ is to act consistently on both sides, then

$$T\{\psi(x)\psi(y)\} = -\psi(y)\psi(x) \ .$$

Generalizing,

$$T\{\psi_1\psi_2\ldots\psi_n\}$$
$$= (\text{product of } \psi\text{'s ordered by time, earliest to right}) \times (-1)^N \ ,$$

where $N$ is the number of interchanges necessary to bring the ordering on the LHS to the ordering on the RHS. (Here $\psi$ represents a general Fermi field, $\psi$ or $\bar{\psi}$.)



**The Fermion Propagator:**

By this definition

$$T\{\psi(x)\bar{\psi}(y)\} \equiv \begin{cases} \psi(x)\bar{\psi}(y) & \text{for } x^0 > y^0 \\ -\bar{\psi}(y)\psi(x) & \text{for } y^0 > x^0 \end{cases} \ .$$

In free field theory we have already learned that

$$\left\langle 0 \left| T\{\psi(x)\bar{\psi}(y)\} \right| 0 \right\rangle = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon} \, e^{-ip\cdot(x-y)} \equiv S_F(x-y) \ .$$

**Normal-ordered products of Fermi operators:**

For $\vec{p} \neq \vec{q}$,

$$a^s(\vec{q})a^{s'\dagger}(\vec{p}) = -a^{s'\dagger}(\vec{p})a^s(\vec{q}) \ .$$

The RHS is normal-ordered, so one presumably defines $N\{-a^{s'\dagger}(\vec{p})a^s(\vec{q})\} = -a^{s'\dagger}(\vec{p})a^s(\vec{q})$. If $N$ is to act consistently on both sides, then

$$N\{a^s(\vec{q})a^{s'\dagger}(\vec{p})\} = -a^{s'\dagger}(\vec{p})a^s(\vec{q}) \ .$$

Generalizing,

$$\boxed{\begin{aligned} &N\{\text{product of } a\text{'s and } a^\dagger\text{'s}\} \\ &\qquad = (\text{product with all } a\text{'s to the right}) \times (-1)^N \ , \end{aligned}}$$

where $N$ is the number of interchanges necessary to bring the ordering on the LHS to the ordering on the RHS.



**Wick's Theorem:**

$$\boxed{T\{\psi_1\bar{\psi}_2\psi_3\ldots\} = N\{\psi_1\bar{\psi}_2\psi_3\ldots + (\text{all possible contractions})\} \ .}$$

A sample contraction would be

$$N\{\overbracket{\psi_1\bar{\psi}_2\psi_3\bar{\psi}_4}\} = -\overbracket{\psi_1\bar{\psi}_4}N\{\psi_3\bar{\psi}_2\} = -S_F(x_1 - x_4)N\{\psi_3\bar{\psi}_2\} \ .$$