MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

8.323: Relativistic Quantum Field Theory I

Prof. Alan Guth — February 10, 2008

PROBLEM SET 1

Corrected Version†

DUE DATE: Thursday, February 14, 2008, at 5:00 p.m. in the 8.323 homework box, at the intersection of the 3rd floor of building 8 and the 4th floor of building 16.

REFERENCES: Peskin and Schroeder, Sections 2.1 and 2.2.

Problem 1: The energy-momentum tensor for source-free electrodynamics *(10 points)*

Peskin and Schroeder, problem 2.1. (Note that when you are asked to "derive Maxwell's equations as the Euler-Lagrange equations of this action," you should be aware that only two of Maxwell's four equations are obtained in this way. You should show that the remaining two Maxwell equations follow directly from the definition of $F_{\mu\nu}$ in terms of the vector potential $A_\mu(x)$.)

Besides parts (a) and (b) as stated in the textbook, please answer the following additional part:

(c) Consider the transformation

$$A^\mu(x) \to A'^\mu(x) = A^\mu(x) + a^\nu F_\nu{}^\mu(x) \ .$$

Show that this transformation is equivalent to a translation in in x, as in part (b), combined with a gauge transformation. Show that when Noether's theorem is applied to this symmetry, it gives directly the symmetric form of the energy-momentum tensor.

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† This version replaces the February 7 version, which had a typographical error in Problem 2. The second expression for the Lagrangian $\boldsymbol{L}$ was corrected by replacing

$$\left(\frac{n\pi}{2}\right)^2 \quad \text{by} \quad \left(\frac{n\pi}{a}\right)^2 \ .$$

Problem 2: Waves on a string *(10 points)**

A string of length a, mass per unit length σ and under tension T is fixed at each end. The Lagrangian governing the time evolution of the transverse displacement $y(x,t)$ is

$$L = \int_0^a dx \left[\frac{\sigma}{2} \left(\frac{\partial y}{\partial t} \right)^2 - \frac{T}{2} \left(\frac{\partial y}{\partial x} \right)^2 \right] ,$$

where x identifies position along the string from one end point. By expressing the displacement as a sine series Fourier expansion in the form

$$y(x,t) = \sqrt{\frac{2}{a}} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{a} \right) q_n(t) ,$$

show that the Lagrangian becomes

$$L = \sum_{n=1}^{\infty} \left[\frac{\sigma}{2} \dot{q}_n^2 - \frac{T}{2} \left(\frac{n\pi}{a} \right)^2 q_n^2 \right] .$$

Derive the equations of motion. Hence show that the string is equivalent to an infinite set of decoupled harmonic oscillators with frequencies

$$\omega_n = \sqrt{\frac{T}{\sigma}} \left(\frac{n\pi}{a} \right) .$$

Problem 3: Fields with SO(3) symmetry *(10 points)*†

Verify that the Lagrangian density

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi_a \, \partial^\mu \phi_a - \frac{1}{2} m^2 \phi_a \phi_a$$

for a triplet of real fields ϕ_a $(a = 1, 2, 3)$ is invariant under an infinitesimal SO(3) rotation by $\delta\theta$,

$$\phi_a \to \phi_a' = \phi_a + \delta\theta \, \epsilon_{abc} \, n_b \, \phi_c ,$$

where n_a is a unit vector and ϵ_{abc} is the fully antisymmetric Livi-Civita symbol, with $\epsilon_{123} = 1$. Compute the Noether current j^μ. Deduce that the three quantities

$$Q_a = \int d^3x \, \epsilon_{abc} \, \dot{\phi}_b \, \phi_c$$

are all conserved and verify this directly using the field equations satisfied by ϕ_a.

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* Problem taken from David Tong's *Lectures on Quantum Field Theory,* http://www.damtp.cam.ac.uk/user/dt281/qft.html.

† Problem taken from David Tong's *Lectures on Quantum Field Theory,* http://www.damtp.cam.ac.uk/user/dt281/qft.html, with minor changes in notation.

Problem 4: Lorentz transformations and Noether's theorem for scalar fields *(15 points)*

Infinitesimal Lorentz transformations, including both rotations and boosts, can be described by

$$x'^\lambda = x^\lambda - \Sigma^\lambda{}_\sigma \, x^\sigma \ ,$$

where

$$\Sigma_{\lambda\sigma} = -\Sigma_{\sigma\lambda} \qquad \text{and} \qquad \Sigma^\lambda{}_\sigma \equiv \eta^{\lambda\kappa}\Sigma_{\kappa\sigma} \ .$$

(a) Show that to first order in Σ, $x'^2 \equiv x'^\lambda \, x'_\lambda = x^\lambda x_\lambda$. That is, show that the length of a vector is invariant under this transformation.

Now consider a scalar field $\phi(x)$, with Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi \, \partial^\mu\phi - V(\phi) \ ,$$

which transforms as

$$\phi'(x') = \phi(x) \ .$$

Expanding to lowest order,

$$\phi'(x') = \phi(x'^\lambda + \Sigma^\lambda{}_\sigma \, x'^\sigma) = \phi(x'^\lambda) + \Sigma^\lambda{}_\sigma x'^\sigma \, \partial_\lambda \, \phi(x') \ .$$

Dropping the primes on the coordinates,

$$\phi'(x) = \phi(x) + \Sigma_{\lambda\sigma} \, x^\sigma \partial^\lambda \phi(x) \ .$$

(b) Show that the conserved Noether current for this transformation can be written as

$$j^{\mu\lambda\sigma} = x^\lambda \, T^{\mu\sigma} - x^\sigma \, T^{\mu\lambda} \ , \qquad \text{where} \qquad \partial_\mu \, j^{\mu\lambda\sigma} = 0 \ ,$$

where

$$T^{\mu\nu} = \partial^\mu\phi \, \partial^\nu\phi - \eta^{\mu\nu} \, \mathcal{L}$$

is the conserved energy-momentum tensor which we have already derived. You should apply Noether's theorem, and then verify from the equations of motion that your expression for the current is actually conserved.

(c) When λ and σ are spacelike, the conserved quantity is the angular momentum (e.g., $j^{012} = j_z$, the density of angular momentum in the z direction). What is the significance of the conserved quantity when $\lambda = 0$, $\sigma = i$?