MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

8.323: Relativistic Quantum Field Theory I

Prof. Alan Guth February 16, 2008

PROBLEM SET 2

DUE DATE: Tuesday, February 26, 2008, at 5:00 p.m. in the 8.323 homework box, at the intersection of the 3rd floor of building 8 and the 4th floor of building 16.

REFERENCES: Peskin and Schroeder, Section 2.3 and part of 2.4, through p. 29. Also Lecture Notes 1 (or Lecture Slides 1): *Quantization of the Free Scalar Field.*

Problem 1: Complex scalar fields *(10 points)*

Peskin and Schroeder, Problem 2.2. The problem as stated in the original printing asked you to find 4 conserved currents for the theory with two complex scalar fields. There are actually 6 conserved currents, as is indicated on the Peskin and Schroeder corrections web page,

http://www.slac.stanford.edu/~mpeskin/QFT.html

and in the newer printings of the book. You will get full credit for finding the same four that Peskin and Schroeder found, and their generalization for n fields. If you can find all six currents for two fields and their generalization for n fields, you will get extra credit, for a maximum of 12 points.

Problem 2: Lorentz transformations and Noether's theorem for scalar fields (continued) *(10 points)*

In Problem 4 of Problem Set 1, you showed that the Lorentz invariance of the theory of a single scalar field leads to a conserved current ($\partial_\mu j^{\mu\lambda\sigma} = 0$) that can be written as

$$j^{\mu\lambda\sigma} = x^\lambda T^{\mu\sigma} - x^\sigma T^{\mu\lambda} ,$$

where

$$T^{\mu\nu} = \partial^\mu\phi\,\partial^\nu\phi - \eta^{\mu\nu}\,\mathcal{L} .$$

The conserved quantities are then

$$M^{\lambda\sigma} \equiv \int \mathrm{d}^3x\, j^{0\lambda\sigma} .$$

Express these conserved quantities in terms of creation and annihilation operators. [Hint: For the case where λ and σ are both spacelike, the conserved quantity can be written as

$$M^{ij} \equiv \epsilon_{ijk} J^k ,$$

where ϵ_{ijk} is the fully antisymmetric Levi-Civita tensor. One then finds that

$$J^i = -i\epsilon_{ijk} \int \frac{\mathrm{d}^3p}{(2\pi)^3} p^j a^\dagger(\vec{p}) \frac{\partial a(\vec{p})}{\partial p^k} .$$

In the course of the derivation you will find ill-defined c-number contributions. You should verify the above expression for J^i, arguing that the c-number contributions vanish due to a symmetry argument. You should also calculate the answer for the case where $\lambda = 0$, $\sigma = i$.]

Problem 3: Lorentz transformations and Noether's theorem for the electromagetic potential $A_\mu(x)$ *(10 points)*

Consider again the electromagnetic potential $A_\mu(x)$, as discussed in problem 2.1 of Peskin and Schroeder. The Lagrangian is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \;, \tag{1}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \;. \tag{2}$$

In problem 2.1 we learned that translation invariance and Noether's theorem lead to a nonsymmetric energy-momentum tensor, which can be made symmetric by adding a piece that has the form of a total derivative that is automatically conserved regardless of the equations of motion. We discussed in lecture, however, how the conservation of angular momentum forces one to use a symmetric energy-momentum tensor, so that the cross product of $\vec{r}$ and the momentum density T^{0i} gives a conserved angular momentum density

$$\mathcal{J}_i = \epsilon_{ijk}\, x^j\, T^{0k} \;. \tag{3}$$

If $T^{\mu\nu}$ is both conserved and symmetric, then this angular momentum density can be written as the 0th component of the divergenceless current

$$K^{\mu\lambda\sigma} = x^\lambda\, T^{\mu\sigma} - x^\sigma\, T^{\mu\lambda} \;, \tag{4}$$

where

$$\partial_\mu K^{\mu\lambda\sigma} = 0 \qquad \text{and} \qquad \mathcal{J}_i = \epsilon_{ijk}\, K^{0jk} \;. \tag{5}$$

One might hope, therefore, that if one derived the conservation of angular momentum by using rotational symmetry and Noether's theorem, then one would be led directly to a symmetric energy-momentum tensor. This hope, however, is not realized, as will be shown in this problem.

We are interested mainly in rotations, but for the sake of generality we will consider arbitrary Lorentz transformations, which include rotations as a special case. Since $A_\mu(x)$ is a Lorentz vector, under a Lorentz transformation $x'^\mu = \Lambda^\mu{}_\nu\, x^\nu$ it transforms as

$$A'^\mu(x') = \Lambda^\mu{}_\nu\, A^\nu(x) \;. \tag{6}$$

For an infinitesimal Lorentz transformation $\Lambda^\mu{}_\nu = \delta^\mu_\nu - \Sigma^\mu{}_\nu$, where $\Sigma_{\mu\nu}$ is antisymmetric as discussed in the previous problem, the symmetry tranformation becomes

$$A'^\mu(x) = A^\mu(x) + \Sigma_{\lambda\sigma} \left\{ x^\sigma\, \partial^\lambda A^\mu(x) - \eta^{\mu\lambda}\, A^\sigma(x) \right\} \;. \tag{7}$$

(a) Show that the above symmetry leads via Noether's theorem to the conserved current

$$j^{\mu\lambda\sigma} = x^\sigma \left\{ F^{\mu\kappa}\, \partial^\lambda A_\kappa + \eta^{\mu\lambda}\, \mathcal{L} \right\} - F^{\mu\lambda}\, A^\sigma - (\lambda \leftrightarrow \sigma) \ . \tag{8}$$

Here "$-(\lambda \leftrightarrow \sigma)$" means to subtract an expression identical to everything previously written on the right-hand side, except that the subscripts λ and σ are interchanged.

(b) Show directly from the equations of motion that the above current is conserved ($\partial_\mu j^{\mu\lambda\sigma} = 0$). Thus Noether's theorem leads to a conserved current, as it must, but since Eq. (8) does not match the form of Eq. (4), the conserved angular momentum current can be constructed without using a symmetric energy-momentum tensor.

(c) As in Peskin and Schroeder's problem 2.1, we can construct a modified form of the conserved current by adding a derivative term:

$$\hat{\jmath}^{\,\mu\lambda\sigma} = j^{\mu\lambda\sigma} + \partial_\kappa N^{\kappa\mu\lambda\sigma} \ , \tag{9}$$

where $N^{\kappa\mu\lambda\sigma}$ is antisymmetric in its first two indices and in its last two indices. Show that if

$$N^{\kappa\mu\lambda\sigma} = x^\lambda\, F^{\mu\kappa}\, A^\sigma - x^\sigma\, F^{\mu\kappa}\, A^\lambda \ , \tag{10}$$

then $\hat{\jmath}^{\,\mu\lambda\sigma}$ can be written in the form of Eq. (4), with a symmetric energy-momentum tensor.

Problem 4: Non-uniqueness of the harmonic oscillator quantization *(10 points)*

On p. 20 of Peskin and Schroeder, the authors show that the Fourier expansion function $\phi(\vec{p}, t)$ obeys the equation of motion of a harmonic oscillator, and they then reflexively invoke the standard quantization procedure. While the results they obtain are certainly correct, we show in this problem that their logic is insufficient. The classical equations of motion do not imply a unique quantization. Rather, one needs a classical canonical formulation to determine the corresponding quantum theory.

Consider for example a classical quantity $x(t)$ that obeys the simple harmonic equation of motion

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\omega^2\, x \ . \tag{1}$$

We will show in this problem that there are nonstandard ways in which such a quantity can appear in a quantum theory.

We will start by formulating a standard harmonic oscillator, described by the Lagrangian

$$L = \frac{1}{2}\dot{q}^2 - \frac{1}{2}\omega_0^2 q^2 \ . \tag{2}$$

Then $p = \partial L/\partial \dot{q} = \dot{q}$, and we as usual require that $[q, p] = i$. The creation and annihilation operators are defined in the usual way,

$$\begin{aligned} a &= \sqrt{\frac{\omega_0}{2}} q + \frac{i}{\sqrt{2\omega_0}} p \\ a^\dagger &= \sqrt{\frac{\omega_0}{2}} q - \frac{i}{\sqrt{2\omega_0}} p \ , \end{aligned} \tag{3}$$

and can then be shown to satisfy the usual commutation relation $[a, a^\dagger] = 1$. The Hamiltonian can be written as $H = \omega_0 \left(a^\dagger a + \frac{1}{2}\right)$, and the coordinate q can then be written in the Heisenberg picture as

$$q(t) = e^{iHt} q e^{-iHt} = \frac{1}{\sqrt{2\omega_0}} \left(a\, e^{-i\omega_0 t} + a^\dagger\, e^{i\omega_0 t} \right) \ . \tag{4}$$

Everything up to here has been standard, but using this formalism we can now explore some alternatives to the standard quantization.

(a) Consider the "first harmonic" operator x_2, which we will define in the Schrödinger picture by

$$x_2 = \frac{1}{\omega_0} \left(a^2 + a^{\dagger 2} \right) \ . \tag{5}$$

(Note that we are still talking about the system defined by the Lagrangian of Eq. (2)—we are simply considering a new operator defined on the same quantum mechanical Hilbert space.) Find an expression for the Heisenberg operator $x_2(t)$ analogous to Eq. (4). Use this expression to show that

$$\frac{\mathrm{d}^2 x_2}{\mathrm{d}t^2} = -\omega^2\, x_2, \qquad \text{where } \omega = 2\omega_0 \ . \tag{6}$$

Thus x_2 has the equation of motion of a harmonic oscillator of angular frequency $2\omega_0$, yet it is defined on the Hilbert space of a harmonic oscillator of angular frequency ω_0, and it is not expressed in terms of creation and annihilation operators in the usual way.

(b) Show that x_2 can be expressed in terms of the original operators p and q by

$$x_2 = q^2 - \frac{p^2}{\omega_0^2}. \tag{7}$$

Now use the Heisenberg equations of motion for q and p, namely $\dot{q} = p$ and $\dot{p} = -\omega_0^2 q$, to confirm that $x_2(t)$ obeys Eq. (6).

(c) Construct a second harmonic operator x_3, which oscillates with angular frequency $3\omega_0$. Construct it first in terms of creation and annihilation operators, analogous to Eq. (5), and then re-express it in terms of q and p. Note that the overall normalization of x_3 is arbitrary, so don't worry about how to fix it.

(d) As an alternative way of seeing that there are ambiguities in the quantization of an operator obeying the harmonic oscillator equation of motion, consider the one-parameter class of Hermitian operators defined in the Schrödinger picture by

$$z_\lambda \equiv \frac{1}{\sqrt{2\omega_0}} \left[a(1+\lambda N) + (1+\lambda N)a^\dagger \right] \ , \tag{8}$$

where $N \equiv a^\dagger a$ is the number operator (i.e., $N\,|n\rangle = n\,|n\rangle$). Find an expression for the Heisenberg operator $z_\lambda(t)$, and show that it obeys

$$\frac{\mathrm{d}^2 z_\lambda}{\mathrm{d}t^2} = -\omega_0^2\, z_\lambda \ . \tag{9}$$

Problem 5: Space-time translations of $a_{\boldsymbol{k}}$ *(10 points)*

(a) Show that for any operators A and B,

$$e^A B e^{-A} = B + [A\,,\,B] + \frac{1}{2}\,[A\,,\,[A\,,\,B]] + \frac{1}{3!}\,[A\,,\,[A\,,\,[A\,,\,B]]] + \ldots \ .$$

(b) Now apply this relation to the space-time translation operator on the Fock space of free scalar particles. These translations are generated by the 4-momentum operator,

$$\hat{P}^\mu = \int \frac{d^3k}{(2\pi)^3}\, k^\mu a_k^\dagger a_k \ ,$$

where $k_0 \equiv E_k = \sqrt{\vec{k}^2 + m^2}$, and $a_k^\dagger$ and a_k are the single particle creation and annihilation operators, normalized as in Peskin and Schroeder. Show that

$$e^{i\hat{P}\cdot x} a_k e^{-i\hat{P}\cdot x} = a_k e^{-ik\cdot x} \ .$$