MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

8.323: Relativistic Quantum Field Theory I

Prof. Alan Guth March 4, 2008

PROBLEM SET 4

DUE DATE: Tuesday, March 11, 2008, at 5:00 p.m. in the 8.323 homework box, at the intersection of the 3rd floor of building 8 and the 4th floor of building 16.

REFERENCES: *Lecture Notes #3: Distributions and the Fourier Transform*, on the website.

Problem 1: Evaluation of $\langle 0|\phi(x)\phi(y)|0\rangle$ for spacelike separations *(10 points)*

Problem 2.3 of Peskin and Schroeder. Alternatively — and this alternative might be more useful to your education — you can evaluate the expectation value by numerical integration, and draw a graph of the result. If you scale your axes with the appropriate powers of the mass m, the same graph can be valid for all values of m.

Problem 2: A tale of three cutoffs *(15 points)*

Consider the function

$$g(t) = \theta(t) e^{-i\omega_0 t} \ . \tag{1.1}$$

In this problem we will examine the Fourier transform of this function, which is defined formally by the ill-defined integral

$$\tilde{g}(\omega) = \int_{-\infty}^{\infty} \mathrm{d}t \, e^{i\omega t} \, g(t) \ . \tag{1.2}$$

While this integral does not exist, it should be possible to define the Fourier transform of $g(t)$ in the sense of distributions.

The Fourier transform will then not be a function, but instead will be a distribution, which we will denote generically as $F[\varphi]$, where $\varphi(t)$ is the test function on which the distribution will act. The function $g(t)$ itself is replaced by the distribution

$$F_g[\varphi] \equiv \int_{-\infty}^{\infty} \mathrm{d}t \, g(t) \, \varphi(t) \ . \tag{1.3}$$

(a) Let $F^{(1)}[\varphi]$ denote the Fourier transform of $F_g[\varphi]$, in the formal sense of distributions, which can also be denoted by $\tilde{F}_g[\varphi]$. Evaluate $F^{(1)}[\varphi]$ for the specific choice

$$\varphi(\omega) = \varphi_0(\omega) \equiv \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(\omega-\omega_1)^2/(2\sigma^2)} \ . \tag{1.4}$$

You may leave your answer in the form of a definite integral, or you can evaluate it explicitly in terms of the error function (also called the Fresnel integral)

$$\Phi(x) \equiv \text{erf}(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x \mathrm{d}t\, e^{-t^2} \ . \tag{1.5}$$

(b) Another way to give meaning to expression (1.2) is to insert a convergence factor, replacing the original function by

$$g_\epsilon(t) = \theta(t) e^{-i\omega_0 t} e^{-\epsilon t} \ . \tag{1.6}$$

This function can be Fourier-transformed by the usual definition,

$$\tilde{g}_\epsilon(\omega) = \int_{-\infty}^{\infty} \mathrm{d}t\, e^{i\omega t}\, g_\epsilon(t) \ . \tag{1.7}$$

As a function there is no way to compare this with the answer in part (a), but we can promote it to a distribution by defining

$$F^{(2)}[\varphi] \equiv \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \mathrm{d}\omega\, \tilde{g}_\epsilon(\omega)\, \varphi(\omega) \ . \tag{1.8}$$

Evaluate this functional for $\varphi_0(\omega)$ given by Eq. (1.4). [*Hint:* you may want to make use of the fact that if an integral is absolutely convergent, you can exchange orders of integration without worry.]

(c) The distribution theory approach guarantees us that we can use any cutoff function that we want, provided only that

$$|g_\epsilon(t)| < g(t) \qquad \text{for } \epsilon > 0 \tag{1.9a}$$

and

$$\lim_{\epsilon \to 0} g_\epsilon(t) = g(t) \qquad \text{for each } t, \tag{1.9b}$$

where the limit is not required to be uniform in t. Another possible cutoff, therefore, would be to simply truncate the integration at $\Lambda = 1/\epsilon$. So let

$$g_\epsilon^{(3)}(t) = \theta(t)\, e^{-i\omega_0 t}\, \theta\left(\frac{1}{\epsilon} - t\right) \ . \tag{1.10}$$

This function can also be Fourier-transformed, in analogy to Eq. (1.7), but the result $\tilde{g}_\epsilon^{(3)}(\omega)$ looks very little like $\tilde{g}_\epsilon(\omega)$. However, we can define the distribution

$$F^{(3)}[\varphi] \equiv \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \mathrm{d}\omega\, \tilde{g}_\epsilon^{(3)}(\omega)\, \varphi(\omega) \ . \tag{1.11}$$

Evaluate this functional for $\varphi_0(\omega)$ given by Eq. (1.4). If all goes well you should be able to show that it is equal to the previous two cases.

(d) Small violations of the condition (1.9a) do not usually matter, but if one violates it grossly one can indeed construct cutoff functions $g_\epsilon(t)$ which are still consistent with condition (1.9b), but which lead to distributions that are not equivalent to $F^{(1)}[\varphi]$. Construct such a cutoff function. You need not evaluate $F[\varphi_0]$ for your case, but carry it far enough to show that the answer is different from the previous cases.