PROBLEM SET 5

DUE DATE: Thursday, March 20, 2008, at 5:00 p.m. in the 8.323 homework box, at the intersection of the 3rd floor of building 8 and the 4th floor of building 16.

REFERENCES: Lecture Notes #4: Dirac Delta Function as a Distribution, on the website. Peskin and Schroeder, Sec. 2.4; Optional reference: Quantum Field Theory, by Lowell Brown, section 1.7 (coherent states).

Problem 1: Subtleties of delta functions (10 points)

(a) Consider $g_1(t)$ and $g_2(t)$ defined by

$$g_1(t) \equiv f(t) \delta(t - a)$$

and

$$g_2(t) \equiv f(a) \delta(t - a),$$

where $f(t)$ is an arbitrary smooth function, $a$ is a constant, and $\delta$ denotes the Dirac delta function. By “smooth,” I mean that $f(t)$ is continuous, and differentiable as many times as might be necessary for any issue that arises. Note that $g_1(t)$ and $g_2(t)$ are distributions, defined more explicitly as

$$T_{g_i}[\varphi] \equiv \int_{-\infty}^{\infty} dt \, g_i(t) \varphi(t),$$

where $\varphi(t)$ is a test function. By evaluating the functionals shown in Eq. (1.3) for an arbitrary allowed test function, find out if $g_1(t)$ and $g_2(t)$ are equal to each other as distributions.

(b) Using a prime to indicate that a function is differentiated with respect to its argument, consider the distributions $h_1(t)$ and $h_2(t)$ defined by

$$h_1(t) \equiv f(t) \delta'(t - a)$$

and

$$h_2(t) \equiv f(a) \delta'(t - a),$$

where again $f(t)$ is an arbitrary smooth function and $\delta$ denotes the Dirac delta function. The derivative of a distribution was defined in Lecture Notes 4. By
evaluating these two distributions for an arbitrary allowed test function \( \varphi(t) \), find out if they are equal to each other as distributions.

(c) Now consider the derivatives of the distributions defined above in Eqs. (1.1) and (1.2):

\[
g_1'(t) = f'(t) \delta(t - a) + f(t) \delta'(t - a)
\]  
(1.6)

and

\[
g_2'(t) = f(a) \delta'(t - a)
\]  
(1.7)

**IF** you concluded that \( g_1(t) \) and \( g_2(t) \) are equal, then you should certainly expect that their derivatives should be equal, even if they do not appear to be identical. Find out if \( g_1'(t) \) and \( g_2'(t) \) are equal.

(d) Let the function \( \theta(t) \) be defined in the standard way,

\[
\theta(t) = \begin{cases} 
1 & \text{if } t > 0 \\
0 & \text{otherwise},
\end{cases}
\]  
(1.8)

and consider the corresponding distribution

\[
T_\theta[\varphi] \equiv \int_{-\infty}^{\infty} dt \theta(t) \varphi(t) .
\]  
(1.9)

Define the derivative of a \( \theta \)-function as a distribution, so

\[
\int_{-\infty}^{\infty} dt \, \theta'(t) \varphi(t) \equiv T'_\theta[\varphi] \equiv -T_\theta \left[ \frac{d\varphi}{dt} \right] .
\]  
(1.10)

Show that

\[
\theta'(t) = \delta(t) .
\]  
(1.11)

**Warning:** The product of two distributions cannot be defined in general, nor can the square of a distribution. As a function, it is clear from the definition (1.8) that

\[
\theta^n(t) = \theta(t) ,
\]  
(1.12)

where \( n \) denotes any positive integer. Any function that is piecewise continuous and bounded by a power can be promoted to a corresponding distribution, so \( T_\theta[\varphi] \) and \( T_{\theta^n}[\varphi] \) are both well-defined distributions and are equal to each other. One might conjecture that

\[
\frac{d}{dt} \theta(t) = \delta(t) ,
\]  
(1.13a)

\[
\frac{d}{dt} \theta^2(t) = 2 \theta(t) \delta(t) ,
\]  
(1.13b)

and
\[
\frac{d}{dt} \theta^3(t) = 3 \theta^2(t) \delta(t) .
\]

Eq. (1.13a) is identical to Eq. (1.11) and is correct, but the right-hand sides of Eqs. (1.13b) and (1.13c) are ill-defined. Since \( \theta^n(t) = \theta(t) \), the left-hand sides of Eqs. (1.13a) – (1.13c) are all well-defined and are equal to each other. It is hard to imagine any consistent definition that would make the right-hand sides equal, so the standard approach is to consider them undefined.

**Problem 2:** \((\Box + m^2) D_F(x - y) = -i\delta^{(4)}(x - y)\) \((10 \text{ points})\)

The Feynman propagator for a free scalar field \(\phi(x)\) is defined by

\[
D_F(x - y) = \theta(x^0 - y^0) \langle 0 | \phi(x) \phi(y) | 0 \rangle + \theta(y^0 - x^0) \langle 0 | \phi(y) \phi(x) | 0 \rangle ,
\]

where \(|0\rangle\) denotes the vacuum state. Use the canonical commutation relations to show that

\[
(\Box + m^2) D_F(x - y) = -i\delta^{(4)}(x - y) ,
\]

where

\[
\Box \equiv \eta^{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu} .
\]

[Suggestion: First calculate

\[
\frac{\partial}{\partial x^0} D_F(x - y) ,
\]

and then differentiate this expression again to find

\[
\frac{\partial^2}{\partial (x^0)^2} D_F(x - y) .
\]

Then add in the other terms to find the final answer.]

**Problem 3: Coherent states \((15 \text{ points})\)**

In lecture we solved the problem of a quantized scalar field \(\phi(x)\) interacting with a fixed classical source \(j(x)\),

\[
(\Box + m^2)\phi(x) = j(x) .
\]

We found that the in and out operators are related by

\[
\phi_{\text{out}}(x) = S^{-1} \phi_{\text{in}}(x) S
\]

\[
a_{\text{out}}(\vec{p}) = S^{-1} a_{\text{in}}(\vec{p}) S ,
\]
where $S$ can be written

$$ S = e^{-\frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} |\tilde{j}(p)|^2} e^F e^G , $$

where

$$ F = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \tilde{j}(p) a^\dagger_{in}(\vec{p}) $$

$$ G = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \tilde{j}(-p) a_{in}(\vec{p}) , $$

and

$$ \tilde{j}(p) \equiv \int d^4y e^{ip \cdot y} j(y) . $$

(a) Show that $S$ can also be written as

$$ S = e^{-\frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} |\tilde{j}(p)|^2} e^{F'} e^{G'} , $$

where

$$ F' = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \tilde{j}(p) a^\dagger_{out}(\vec{p}) $$

$$ G' = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \tilde{j}(-p) a_{out}(\vec{p}) , $$

and hence that

$$ |0_{in}\rangle = e^{-\frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} |\tilde{j}(p)|^2} e^{F'} |0_{out}\rangle . $$

Note that the right-hand-side of the above equation gives a useful description of the final state, since the out operators have a straightforward interpretation at late times. States of this form — exponentials of creation operators acting on the vacuum — are called coherent states.

(b) To study further the properties of coherent states, it is useful to consider a single harmonic oscillator,

$$ H = \frac{p^2}{2m} + \frac{\omega^2m}{2} q^2 , $$

which is simplified by the canonical transformation

$$ p = \sqrt{m\omega} \tilde{p} , \quad q = \frac{1}{\sqrt{m\omega}} \tilde{q} . $$
Dropping the overbars, the creation and annihilation operators are then given by

\[ a^\dagger = \frac{1}{\sqrt{2}}(q - ip) \]
\[ a = \frac{1}{\sqrt{2}}(q + ip) . \]

A coherent state \(|z\rangle\) can be defined by

\[ |z\rangle \equiv e^{za^\dagger} |0\rangle . \]

Show that \(|z\rangle\) is an eigenstate of the annihilation operator, and find its eigenvalue.

(c) Show that \(\langle z_2 | z_1 \rangle = e^{z_2 z_1}\). (If you look at Lowell Brown’s book, note that I am defining \(\langle z |\) to be the bra vector that corresponds to the ket \(|z\rangle\), so my \(\langle z |\) is equal to Brown’s \(\langle z^* |\).)

(d) Find

\[ \langle q \rangle_z \equiv \frac{\langle z | q | z \rangle}{\langle z | z \rangle} \]

and

\[ \langle p \rangle_z \equiv \frac{\langle z | p | z \rangle}{\langle z | z \rangle} . \]

(e) Compute the standard deviations of \(q\) and \(p\),

\[ \Delta q^2 = \langle (q - \langle q \rangle)^2 \rangle \]
\[ \Delta p^2 = \langle (p - \langle p \rangle)^2 \rangle \]

and show that \(|z\rangle\) is a minimal-uncertainty state, in the sense that

\[ \Delta q \Delta p = \frac{1}{2} . \]