MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

8.323: Relativistic Quantum Field Theory I

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PROBLEM SET 8

DUE DATE: Thursday, April 24, 2008, at 5:00 p.m. in the 8.323 homework box, at the intersection of the 3rd floor of building 8 and the 4th floor of building 16.

DESCRIPTION: This problem set contains only the three problems held over from the previous problem set. However, an extra part was added to Problem 2, and Problem 3 has been changed from extra credit to a required problem.

Problem 1: Commutation relations for the Lorentz group *(10 points)*

Infinitesimal Lorentz transformations can be described by

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} - iG^{\mu}{}_{\nu} \ ,$$

where

$$G^{\mu\nu} = -G^{\nu\mu} \ .$$

There are therefore 6 generators, since there are 6 linearly independent antisymmetric 4×4 matrices. One convenient way to choose a basis of 6 independent generators is to label them by two antisymmetric spacetime indices, $J^{\mu\nu} \equiv -J^{\nu\mu}$, with the explicit matrix definition

$$J^{\mu\nu}_{\alpha\beta} \equiv i\left(\delta^{\mu}_{\alpha}\,\delta^{\nu}_{\beta} - \delta^{\mu}_{\beta}\,\delta^{\nu}_{\alpha}\right) \ .$$

Here μ and ν label the generator, and for each μ and ν (with $\mu \neq \nu$) the formula above describes a matrix with indices α and β. For the usual rules of matrix multiplication to apply, the index α should be raised, which is done with the Minkowski metric $g^{\mu\nu}$:

$$J^{\mu\nu\alpha}{}_{\beta} = i\left(g^{\mu\alpha}\,\delta^{\nu}_{\beta} - g^{\nu\alpha}\,\delta^{\mu}_{\beta}\right) \ .$$

(a) Show that the commutator is given by

$$[J^{\mu\nu}\, ,\, J^{\rho\sigma}] = i\left(g^{\nu\rho}\,J^{\mu\sigma} - g^{\mu\rho}\,J^{\nu\sigma} - g^{\nu\sigma}\,J^{\mu\rho} + g^{\mu\sigma}\,J^{\nu\rho}\right) \ .$$

To minimize the number of terms that you have to write, I recommend adopting the convention that $\{\}_{\mu\nu}$ denotes antisymmetrization, so

$$\{\qquad\}_{\mu\nu} \equiv \frac{1}{2}\Big[\{\qquad\} - \{\ \mu \leftrightarrow \nu\ \}\Big] \ .$$

With this notation, the commutator can be written

$$[J^{\mu\nu}\,,\,J^{\rho\sigma}] = 4i\left\{\,\{\,g^{\nu\rho}\,J^{\mu\sigma}\,\}_{\mu\nu}\right\}_{\lambda\sigma}\quad .$$

You might even want to adopt a more abbreviated notation, writing

$$[J^{\mu\nu}\,,\,J^{\rho\sigma}] = 4i\,\{\,g^{\nu\rho}\,J^{\mu\sigma}\,\}_{\substack{\mu\nu\\ \lambda\sigma}}\quad .$$

(b) Construct a Lorentz transformation matrix $\Lambda^{\alpha}{}_{\beta}$ corresponding to an infinitesimal boost in the positive z-direction, and use this to show that the generator of such a boost is given by $K^3 \equiv J^{03}$. Signs are important here.

Problem 2: Representations of the Lorentz group *(15 points)*

(a) Using the definitions

$$J^i = \frac{1}{2}\epsilon^{ijk}\,J^{jk} \text{ and } K^i = J^{0i}\ ,$$

for the generators of rotations (J^i) and boosts (K^i), with the general commutation relations found in Problem 3,

$$[J^{\mu\nu}\,,\,J^{\rho\sigma}] = i\,(g^{\nu\rho}\,J^{\mu\sigma} - g^{\mu\rho}\,J^{\nu\sigma} - g^{\nu\sigma}\,J^{\mu\rho} + g^{\mu\sigma}\,J^{\nu\rho})\ ,$$

show that the rotation and boost operators obey the commutation relations

$$\begin{aligned}
\left[J^i\,,\,J^j\right] &= i\,\epsilon^{ijk}\,J^k\\
\left[K^i\,,\,K^j\right] &= -i\,\epsilon^{ijk}\,J^k\\
\left[J^i\,,\,K^j\right] &= i\,\epsilon^{ijk}\,K^k\ .
\end{aligned}$$

(b) Using linear combinations of the generators defined by

$$\begin{aligned}
\vec{J}_+ &= \frac{1}{2}\left(\vec{J} + i\vec{K}\right)\\
\vec{J}_- &= \frac{1}{2}\left(\vec{J} - i\vec{K}\right)\ ,
\end{aligned}$$

show that $\vec{J}_+$ and $\vec{J}_-$ obey the commutation relations

$$\begin{aligned}
\left[J^i_+\,,\,J^j_+\right] &= i\,\epsilon^{ijk}\,J^k_+\\
\left[J^i_-\,,\,J^j_-\right] &= i\,\epsilon^{ijk}\,J^k_-\\
\left[J^i_+\,,\,J^j_-\right] &= 0\ .
\end{aligned}$$

[*Discussion:* Since $\vec{J}_+$ and $\vec{J}_-$ commute, they can be simultaneously diagonalized, so a representation of the Lorentz group is obtained by combining a representation of $\vec{J}_+$ with a representation of $\vec{J}_-$. Furthermore, $\vec{J}_+$ and $\vec{J}_-$ each have the commutation relations of the three-dimensional rotation group (or SU(2)), so we already know the finite-dimensional representations: they are labeled by a "spin" j, which is an integer or half-integer, with $\vec{J}^{\,2} = j(j+1)$. The spin-j representation has dimension $2j+1$. The finite-dimensional representations of the Lorentz group can then be described by the pair (j_1, j_2), with $\vec{J}_+^{\,2} = j_1(j_1+1)$ and $\vec{J}_-^{\,2} = j_2(j_2+1)$. The dimension of the (j_1, j_2) representation is then $(2j_1+1)(2j_2+1)$.]

(c) Consider the $(\frac{1}{2}, \frac{1}{2})$ representation of the Lorentz group, which has dimension $(2j_1+1)(2j_2+1) = 4$. The basis for this representation can be written as $|m_1, m_2\rangle$, where m_1 and m_2 denote the eigenvalues of J_+^z and J_-^z, respectively. The names of the four basis states can be abbreviated as $|{+}{+}\rangle$, $|{+}{-}\rangle$, $|{-}{+}\rangle$, and $|{-}{-}\rangle$, where "$\pm$" denotes "$J^z = \pm\frac{1}{2}$". One can represent both J_+^i and J_-^i by the usual Pauli spin matrices, $J^i = \frac{1}{2}\sigma^i$, where

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \text{ and } \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ,$$

where J_+^i operates on the first index (m_1), and J_-^i operates on the second (m_2). This representation is equivalent to the usual four-vector representation of the Lorentz group, by which we mean that the $|m_1, m_2\rangle$ states can be expressed in terms of a new basis of states $|\alpha\rangle$, where $\alpha = 0$, 1, 2, or 3, such that

$$\langle \alpha \,| J^{\mu\nu} |\, \beta \rangle = (J^{\mu\nu})^{\alpha}{}_{\beta} = i\left(g^{\mu\alpha}\,\delta^{\nu}_{\beta} - g^{\nu\alpha}\,\delta^{\mu}_{\beta} \right) .$$

Find expressions for the basis states $|0\rangle$, $|1\rangle$, $|2\rangle$, and $|3\rangle$ in terms of the $|m_1, m_2\rangle$ states. It will be sufficient to find an expression for each $|\alpha\rangle$ without any general expression, and you need not verify that the above expression for $\langle \alpha \,| J^{\mu\nu} |\, \beta \rangle$ holds for all values of the indices.

Problem 3: The Baker-Campbell-Hausdorff Theorem *(20 points)*

The Baker-Campbell-Hausdorff theorem states that if A and B are linear operators (including the possibility of finite-dimensional matrices), then one can write

$$e^A\, e^B = e^{A+B+C} , \tag{3.1}$$

where

$$C = \frac{1}{2}\,[A\,,\, B] + \ldots , \tag{3.2}$$

where every term in the infinite series denoted by ... can be expressed as an iterated commutator of A and B. When I say that the series is infinite, I mean that the general theorem requires an infinite series. In a particular application, it is possible that only a finite number of terms will be nonzero. In class, for example, we used this theorem for a case where $[A\,,\,B]$ was a c-number, so all the higher iterated commutators vanished, and only the terms written explicitly above contributed.

The set of iterated commutators is defined by starting with A and B, and then adding to the set the commutator of any two elements in the set, ad infinitum. Then A and B are removed from the set, as they do not by themselves count as iterated commutators.

This theorem is sometimes useful to simplify calculations, as we found in class when we used it to normal order the S-matrix for the problem of particle production by an external source. However, it is much more important in the context of Lie groups. We have all learned, for example, that if we can find any three matrices with commutation relations

$$[J_i\,,\,J_j] = i\epsilon_{ijk}J_k \,, \tag{3.3}$$

where ϵ_{ijk} denotes the completely antisymmetric Levi-Civita tensor, then we can use them to construct a representation of the rotation group. If we let

$$R(\hat{n},\theta) = e^{-i\theta\hat{n}\cdot\vec{J}} \,, \tag{3.4}$$

then we know that this matrix can be used to represent a counterclockwise rotation about a unit vector $\hat{n}$ by an angle θ. The representation describes the rotation group in the sense that if a rotation about $\hat{n}_1$ by an angle θ_1, followed by a rotation about $\hat{n}_2$ by an angle θ_2, is equivalent to a rotation about $\hat{n}_3$ by an angle θ_3, then we expect

$$R(\hat{n}_2,\theta_2)\,R(\hat{n}_1,\theta_1) = R(\hat{n}_3,\theta_3) \,. \tag{3.5}$$

But how do we know that this relation will hold? The answer is that it follows as a consequence of the Baker-Campbell-Hausdorff theorem, which assures us that the product $R(\hat{n}_2,\theta_2)\,R(\hat{n}_1,\theta_1)$ can be written as

$$R(\hat{n}_2,\theta_2)\,R(\hat{n}_1,\theta_1) = \exp\left\{-i\theta_1\hat{n}_1\cdot\vec{J} - i\theta_2\hat{n}_2\cdot\vec{J} - \frac{1}{2}\,\theta_1\theta_2\left[\hat{n}_1\cdot\vec{J}\,,\,\hat{n}_2\cdot\vec{J}\right] + (\text{iterated commutators})\right\} \,. \tag{3.6}$$

Thus, the commutation relations are enough to completely determine the exponent appearing on the right-hand side, so any matrices with the right commutation relations will produce the right group multiplication law.

In this problem we will construct a proof of the Baker-Campbell-Hausdorff theorem.

Note, by the way, that there are complications in the applications of Eq. (3.6), as we know from the spin-$\frac{1}{2}$ representation of the rotation group. In that case, the matrices $R(\hat{n}, \theta)$ and $R(\hat{n}, \theta + 2\pi)$ exponentiate to give different matrices, differing by a sign, even though the two rotations are identical. In this case there are two matrices corresponding to every rotation. The matrices themselves form the group SU(2), for which there is a 2:1 map into the rotation group. In general, it is also possible for the series expansions of the exponentials in Eq. (3.1) to diverge — the Baker-Campbell-Hausdorff theorem only guarantees that the terms on the left- and right-hand sides will match term by term. The best way to use Eq. (3.6) is to restrict oneself to transformations that are near the identity, and generators that are near zero. For any Lie group, the exponentiation of generators in some finite neighborhood of the origin gives a 1:1 mapping into the group within some finite neighborhood of the identity.

(a) The first step is to derive a result that is best known to physicists in the context of time-dependent perturbation theory. We will therefore describe this part in terms of two operators that I will call H_0 and ΔH, which are intended to suggest that we are talking about an unperturbed Hamiltonian and a perturbation that could be added to it. However, you should also keep in mind that the derivation will not rely on any special properties of Hamiltonians, so the result will hold for any two linear operators.

The goal is to consider the operators

$$U_0(t) \equiv e^{-iH_0 t} \tag{3.7}$$

and

$$U(t) \equiv e^{-i(H_0 + \epsilon \Delta H)t} \ , \tag{3.8}$$

and to calculate the effects of the perturbation ΔH to first order in ϵ. We want to express the answer in the form

$$U(t) = U_0(t)\,[1 + \epsilon\,\Delta U_1(t)] \quad , \tag{3.9}$$

where $\Delta U_1(t)$ is the quantity that must be found. Define

$$U_I(t) \equiv U_0^{-1}(t)\,U(t) \ , \tag{3.10}$$

so

$$U_I(t) = 1 + \epsilon\,\Delta U_1(t) + \mathcal{O}(\epsilon^2) \ . \tag{3.11}$$

Derive a differential equation for $U_I(t)$ of the form

$$\frac{\mathrm{d}U_I}{\mathrm{d}t} = \epsilon\, Q(t)\, U_I(t) \ , \tag{3.12}$$

where $Q(t)$ is an operator that you must determine.

(b) What is $U_I(0)$? To zeroth order in ϵ, what is $U_I(t)$? Using these answers and Eq. (3.12), find $U_I(t)$ to first order in ϵ.

(c) On Problem 5 of Problem Set 2, you showed that

$$e^A B e^{-A} = B + [A\,,\, B] + \frac{1}{2}\,[A\,,\, [A\,,\, B]] + \frac{1}{3!}\,[A\,,\, [A\,,\, [A\,,\, B]]] + \dots \tag{3.13}$$

Here we would like to use that result, and we would like a more compact notation for the terms that appear on the right-hand side. We let $\Omega_n(A, B)$ denote the nth order iterated commutator of A with B, defined by

$$\begin{aligned} \Omega_0(A,B) &= B \ , \\ \Omega_1(A,B) &= [A,B] \ , \\ \Omega_2(A,B) &= [A,[A,B]] \ , \\ \Omega_n(A,B) &= [A,\Omega_{n-1}(A,B)] \ , \end{aligned} \tag{3.14}$$

or equivalently

$$\Omega_n(A,B) = \underbrace{[A,[A,\ \dots\ ,[A,}_{n \text{ iterations of } A}\ B] \ . \tag{3.15}$$

Using this definition and the answer to (b), show that

$$\Delta U_1(t) = \sum_{n=0}^{\infty} A_n(t)\Omega_n(H_0, \Delta H) \ , \tag{3.16}$$

where the $A_n(t)$ are coefficients that you must calculate.

Note, by the way, that the iterated commutators that are used in the construction of C in Eq. (3.2) form a much more general class than the $\Omega_n(A, B)$.

(d) In the context of time-dependent perturbation theory, the dependence of these expressions on t is very important. Here, however, we want to use this formalism to learn how the exponential of an operator changes if the operator is perturbed, so time plays no role. We can set $t = 1$, and then Eq. (3.16) with the earlier definitions gives us a prescription for expanding the operator

$$U(1) = e^{-i(H_0 + \epsilon\, \Delta H)} \tag{3.17}$$

to first order in ϵ. Using this line of reasoning, consider an arbitrary linear operator $M(s)$ that depends on some parameter s. Show that

$$\frac{d}{ds} e^{M(s)} = e^{M(s)} \sum_{n=0}^{\infty} B_n \Omega_n \left(M, \frac{dM}{ds} \right) , \tag{3.18}$$

where the B_n are coefficients that you must calculate. *Hint: remember that to first order*

$$M(s + \epsilon \Delta s) = M(s) + \epsilon \Delta s \frac{\mathrm{d}M}{\mathrm{d}s} . \tag{3.19}$$

Eq. (3.18) can be very useful, so once you derive it you should save it in a notebook.

(e) We wish to construct an operator C which satisfies Eq. (3.1), and we wish to show that it can be constructed entirely from iterated commutators. The trick is to realize that we want to assemble the terms on the right-hand-side of Eq. (3.2) in order of increasing numbers of operators A and B. The first term $\frac{1}{2}[A, B]$ has two operators, the second will have three, etc. To arrange the terms in this order, it is useful to introduce a parameter s, and write Eq. (3.1) as

$$e^{sA} e^{sB} = e^{s(A+B)+C(s)} . \tag{3.20}$$

Then if we write $C(s)$ as a power series in s, the terms will appear in the desired order. Once we have found the desired terms, we can set $s = 1$ to give an answer to the original question.

To proceed, one can define

$$F(s) \equiv e^{-sB} e^{-sA} e^{s(A+B)+C(s)} , \tag{3.21}$$

where the desired solution for $C(s)$ should be determined by the condition $F(s) = 1$. Using Eq. (3.18) show that

$$\begin{aligned} \frac{\mathrm{d}F}{\mathrm{d}s} &= -BF(s) - e^{-Bs} A e^{Bs} F(s) \\ &\quad + F(s) \sum_{n=0}^{\infty} B_n \Omega_n \left((A+B)s + C(s), A + B + \frac{dC}{ds} \right) . \end{aligned} \tag{3.22}$$

(f) Use the fact that $F(s) = 1$, and $\mathrm{d}F/\mathrm{d}s = 0$, to write an expression of the form

$$\frac{\mathrm{d}C}{\mathrm{d}s} = \sum_{n=1}^{\infty} s^n E_n(A, B) - \sum_{n=1}^{\infty} B_n \Omega_n \left((A+B)s + C(s), A + B + \frac{dC}{ds} \right) , \tag{3.23}$$

where $E_n(A, B)$ is an expression involving iterated commutators of A and B which you must find.

(g) From Eq. (3.20), one sees that $C(0) = 0$. We can therefore write $C(s)$ as a power series, with the zero-order term omitted:

$$C(s) = C_1\, s + C_2\, s^2 + C_3\, s^3 + \ldots\ . \tag{3.24}$$

Use Eq. (3.23) to show that $C_1 = 0$, and that $C_2 = \frac{1}{2}\,[A\,,\,B]$. Find C_3.

(h) Use Eq. (3.23) to argue that every C_n can be determined, and that every C_n will be expressed solely in terms of iterated commutators, in the general sense defined at the beginning of this problem.