MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

8.323: Relativistic Quantum Field Theory I

Prof. Alan Guth May 2, 2008

PROBLEM SET 9

Corrected Version†

DUE DATE: Tuesday, May 6, 2008, at 5:00 p.m. in the 8.323 homework box, at the intersection of the 3rd floor of building 8 and the 4th floor of building 16.

REFERENCES: Peskin and Schroeder, Secs. 3.1-3.5. Some class lecture notes will also be posted. On Wigner's Theorem, Problem 5, you might want to look at Steven Weinberg, *The Quantum Theory of Fields, Volume 1: Foundations* (Cambridge University Press, Cambridge, 1995), Section 2.2 and Appendix A of Chapter 2.

Problem 1: The Dirac representation of the Lorentz group *(10 points)*

Show that the defining property of the Dirac matrices,

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \ ,$$

is sufficient to show that the matrices

$$S^{\mu\nu} = \frac{i}{4}\left[\gamma^\mu \, , \, \gamma^\nu\right]$$

have the commutation relations of the Lorentz group, as specified by Eq. (3.17) of Peskin and Shroeder:

$$\left[J^{\mu\nu} \, , \, J^{\rho\sigma}\right] = i\left(g^{\nu\rho} J^{\mu\sigma} - g^{\mu\rho} J^{\nu\sigma} - g^{\nu\sigma} J^{\mu\rho} + g^{\mu\sigma} J^{\nu\rho}\right) \ .$$

The notation for antisymmetrization introduced in Problem 1 of Problem Set 6 may prove useful.

Show also that

$$\left[\gamma^\mu \, , \, S^{\rho\sigma}\right] = (\mathcal{J}^{\rho\sigma})^\mu{}_\nu \, \gamma^\nu \ ,$$

where $(\mathcal{J}^{\mu\nu})_{\alpha\beta}$ is defined by Eq. (3.18) of Peskin and Schroeder,

$$(\mathcal{J}^{\mu\nu})_{\alpha\beta} \equiv i\left(\delta^\mu_\alpha \, \delta^\nu_\beta - \delta^\mu_\beta \, \delta^\nu_\alpha\right) \ .$$

---

† This version replaces the April 28 version. The notation has been improved in Problem 2, using $\Lambda_{\frac{1}{2}}\big(B_3(\eta)\big)$ instead of $B_3(\eta)$, and in Problem 4 the conventions for the definition of $S^{\mu\nu}$ were changed to agree with our standard conventions — this in turn resulted in a change in identity (vi).

Problem 2: Explicit tranformation matrices *(10 points)*

Evaluate explicitly the 4×4 matrix used to represent a boost along the positive z-axis,

$$\Lambda_{\frac{1}{2}}\big(B_3(\eta)\big) \equiv e^{-i\eta K^3} = e^{-i\eta S^{03}} \ .$$

Use Peskin and Schroeder's conventions for the Dirac matrices. How is η related to the velocity of the boost?

Similarly evaluate the 4×4 matrix used to represent a counterclockwise rotation about the positive z-axis,

$$\Lambda_{\frac{1}{2}}\big(R_3(\theta)\big) \equiv e^{-i\theta J^3} = e^{-i\theta S^{12}} \ .$$

Problem 3: Wigner rotations and the transformation of helicity *(15 points)*

The Lorentz transformation properties of spin-$\frac{1}{2}$ particles are actually completely dictated by the properties of the Lorentz group, even if we don't know anything about the Dirac equation.

Consider for example an electron in an eigenstate of momentum $\vec{p}$ with eigenvalue $\vec{p} = 0$; i.e., an electron at rest. We know from nonrelativistic quantum mechanics that the electron will have two possible spin states, which we can label as spin-up and spin-down along the z-axis. If we denote these states by $|\vec{p} = 0, \pm\rangle$, then

$$J^z \, |\vec{p} = 0, \pm\rangle = \pm\frac{1}{2} \, |\vec{p} = 0, \pm\rangle \ . \tag{3.1}$$

If we were to perform a rotation on such a state, the momentum would remain zero, and so the two-state system would transform under the spin-$\frac{1}{2}$ representation of the rotation group, as in nonrelativistic quantum theory. The nonrelativistic theory must apply, because the transformation properties in the nonrelativistic theory were dictated completely by properties of the rotation group, and the rotation group is a subgroup of the Lorentz group.

In the relativisitic theory, there must be a unitary operator $U(\Lambda)$ corresponding the each Λ in the Lorentz group. We can use the operators representing boosts to construct a state of nonzero momentum along the z-axis with a definite helicity h:

$$\left| p\hat{z}, h = \pm\frac{1}{2} \right\rangle = U\big(B_z(\eta(p))\big) \ |\vec{p} = 0, \pm\rangle \ , \tag{3.2}$$

where $\eta(p)$ is the boost parameter (rapidity) that brings a rest vector to $p\hat{z}$. Note that $\eta(p)$ will depend on the mass m of the electron, so we assume that it has been specified. Note also that J^z commutes with K^z, so the state described by the equation above is still an eigenstate of J^z.

We can also define states of definite helicity in any other direction. Let

$$\left|\vec{p}, h = \pm\frac{1}{2}\right\rangle \equiv U(B_{\vec{p}})\,|\vec{p}=0,\pm\rangle \ , \tag{3.3}$$

where

$$B_{\vec{p}} = R(\hat{p})B_z\big(\eta(|\vec{p}|)\big) \ , \tag{3.4}$$

where $R(\hat{p})$ is the rotation that rotates the positive z axis into the direction of $\vec{p}$. These states give a complete basis for the Hilbert space of free one-particle electron states.

(a) Consider the state

$$\left|p\hat{z}, h = \pm\frac{1}{2}\right\rangle \ , \tag{3.5}$$

and imagine boosting it in the positive x-direction, by a velocity β

$$\begin{aligned} |\psi\rangle &= U\big(B_x(\eta(\beta))\big)\left|p\hat{z}, h = \pm\frac{1}{2}\right\rangle \\ &= U\big(B_x(\eta(\beta))B_z(\eta(p))\big)\,|\vec{p}=0,\pm\rangle \ . \end{aligned} \tag{3.6}$$

Compute the Lorentz transformation

$$B_x(\eta(\beta))B_z(\eta(p)) \ , \tag{3.7}$$

expressing your answer in the form of a 4×4 Lorentz matrix $\Lambda^\mu{}_\nu$. What is the momentum $\vec{p}\,'$ of the state $|\psi\rangle$?

(b) To express $|\psi\rangle$ in terms of the original basis vectors, we need the inner products

$$\langle\vec{p}\,', h'\,|\psi\,\rangle = \left\langle\vec{p}=0, h'\,\middle|U^\dagger(B_{\vec{p}\,'})\,U\big(B_x(\eta(\beta))B_z(\eta(p))\big)\middle|\,\vec{p}=0,\pm\right\rangle \ , \tag{3.8}$$

where $B_{\vec{p}\,'}$ is defined analogously to Eq. (3.4). Since $U(\Lambda)$ is a unitary representation of the group,

$$U^\dagger(B_{\vec{p}\,'})\,U\big(B_x(\eta(\beta))B_z(\eta(p))\big) = U\big(B_{\vec{p}\,'}^{-1}\,B_x(\eta(\beta))B_z(\eta(p))\big) \ . \tag{3.9}$$

Note, however, that

$$R_W \equiv B_{\vec{p}\,'}^{-1}\,B_x(\eta(\beta))B_z(\eta(p)) \tag{3.10}$$

brings a momentum vector at rest back to a momentum vector at rest, and hence it is a pure rotation. It is called the Wigner rotation. Since the matrix elements of U for rotations are already known, the matrix element needed here is known. Compute the Wigner rotation for this case, describing it first as a Lorentz matrix Λ. What is the axis of rotation? What is the angle of the rotation?

(c) Now consider the $m \to 0$ limit, keeping p and η fixed. This would be the appropriate limit to describe a massless particle with momentum of magnitude p. Show that the Wigner rotation angle approaches zero in this limit, and hence that the helicity of a massless particle is Lorentz invariant.

Problem 4: Useful tricks with Dirac matrices *(10 points)**

Using just the algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ (i.e. without resorting to a particular representation), and defining $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$, $\not{p} \equiv p_\mu \gamma^\mu$, $S^{\mu\nu} \equiv \frac{i}{4}[\gamma^\mu, \gamma^\nu]$, and $\epsilon_{\mu\nu\rho\sigma}$ as the fully antisymmetric tensor with $\epsilon_{0123} = 1$, prove the following results: *(Some useful tricks include the cyclicity of the trace, and inserting* $(\gamma^5)^2 = 1$ *into a trace).*

i. $\mathrm{Tr}\,\gamma^\mu = 0$

ii. $\mathrm{Tr}(\gamma^\mu\,\gamma^\nu) = 4g^{\mu\nu}$

iii. $\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho) = 0$

iv. $(\gamma^5)^2 = 1$

v. $\mathrm{Tr}\,\gamma^5 = 0$

vi. $\not{p}\,\not{q} = 2p \cdot q - \not{q}\,\not{p} = p \cdot q - 2iS^{\mu\nu}\,p_\mu\,q_\nu$

vii. $\mathrm{Tr}(\not{p}\,\not{q}) = 4p \cdot q$

viii. $\mathrm{Tr}(\not{p}_1 \cdots \not{p}_n) = 0$ if n is odd

ix. $\mathrm{Tr}(\not{p}_1\,\not{p}_2\,\not{p}_3\,\not{p}_4) = 4[(p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) - (p_1 \cdot p_3)(p_2 \cdot p_4)]$

x. $\mathrm{Tr}(\gamma^5\not{p}_1\not{p}_2) = 0$

xi. $\gamma_\mu\not{p}\gamma^\mu = -2\not{p}$

xii. $\gamma_\mu\not{p}_1\not{p}_2\gamma^\mu = 4p_1 \cdot p_2$

xiii. $\gamma_\mu\not{p}_1\not{p}_2\not{p}_3\gamma^\mu = -2\not{p}_3\not{p}_2\not{p}_1$

xiv. $\mathrm{Tr}(\gamma^5\not{p}_1\not{p}_2\not{p}_3\not{p}_4) = 4i\epsilon_{\mu\nu\rho\sigma}p_1^\mu p_2^\nu p_3^\rho p_4^\sigma$

Problem 5 (Extra Credit): Wigner's Symmetry Representation Theorem *(10 points extra credit)*

This problem will be a guided exercise in which a proof of Wigner's theorem* will be constructed. The proof that you will construct is a modified version of the proof given in Steven Weinberg's textbook, the complete reference for which was given at the start of the problem set. This version is in my opinion simpler, and

---

* Problem taken from David Tong's *Lectures on Quantum Field Theory,* http://www.damtp.cam.ac.uk/user/dt281/qft.html.

* The theorem was originally proven in *Gruppentheorie und ihre Anwendung auf die Quanten-mechanik der Atomspektren* (Braunschweig, 1931), pp. 251–3, by Eugene P. Wigner. An English translation was published by Academic Press in 1959.

it also avoids what I believe is a minor flaw† in Weinberg's argument. Weinberg in turn claims to have remedied a flaw in Wigner's original proof, so historical precedent seems to suggest that any proof of Wigner's theorem is flawed. If you find any flaws in this one, you will get extra credit.

First, we need some definitions that will be used in the statement of the theorem. Consider a quantum theory formulated on a Hilbert space $\mathcal{H}$. A physical state corresponds to a *ray* $\mathcal{R}$ in the Hilbert space, where a ray is defined as a set of normalized vectors ($\langle \Psi | \Psi \rangle = 1$), where $|\Psi\rangle$ and $|\Psi'\rangle$ belong to the same ray if they are equal up to a phase (i.e., if $|\Psi'\rangle = e^{i\theta} |\Psi\rangle$ for some real θ). I will use the notation $|\Psi\rangle \in \mathcal{R}$ or $\mathcal{R} \ni |\Psi\rangle$ to indicate that $|\Psi\rangle$ belongs to the ray $\mathcal{R}$, and I will define $\mathcal{R}(\Psi)$ to denote the ray that contains the vector $|\Psi\rangle$. We will consider a transformation T defined on physical states, so T maps one ray onto another. I will sometimes use the abbreviation $T(\Psi)$ to denote $T\big(\mathcal{R}(\Psi)\big)$, the image under T of the ray that contains the vector $|\Psi\rangle$. T will be said to be probability-preserving if

$$|\langle \psi_2' | \psi_1' \rangle| = |\langle \psi_2 | \psi_1 \rangle| \tag{5.1}$$

whenever

$$|\psi_1'\rangle \in T(\psi_1) \quad \text{and} \quad |\psi_2'\rangle \in T(\psi_2) \ . \tag{5.2}$$

If U is an operator on the Hilbert space $\mathcal{H}$, then T is said to be represented by U if

$$|\Psi\rangle \in \mathcal{R} \quad \text{implies} \quad U|\Psi\rangle \in T(\mathcal{R}) \ . \tag{5.3}$$

An operator U on $\mathcal{H}$ is said to be *linear* if

$$U\big(\alpha |\psi_1\rangle + \beta |\psi_2\rangle\big) = \alpha\, U |\psi_1\rangle + \beta\, U |\psi_2\rangle \ , \tag{5.4}$$

and it is said to be *antilinear* if

$$U\big(\alpha |\psi_1\rangle + \beta |\psi_2\rangle\big) = \alpha^*\, U |\psi_1\rangle + \beta^*\, U |\psi_2\rangle \ . \tag{5.5}$$

An operator is said to be *unitary* if

$$\langle U\psi_2 | U\psi_1 \rangle = \langle \psi_2 | \psi_1 \rangle \ , \tag{5.6}$$

and it is said to be *antiunitary* if

$$\langle U\psi_2 | U\psi_1 \rangle = \langle \psi_2 | \psi_1 \rangle^* \ . \tag{5.7}$$

---

† On pp. 92 and 93 of Weinberg's text, he uses a number of equations in which C_1 or C_1' appears in the denominator, where C_1 and C_1' are expansion coefficients of an arbitrary state in a particular basis. The argument is therefore inapplicable to states for which these particular coefficients vanish. The gap can be filled, but doing so makes the proof more cumbersome.

Now Wigner's theorem can be stated:

> Given any probability-preserving invertible transformation T on the rays of a Hilbert space $\mathcal{H}$, then one and only one of the following two statements is true:
>
> (a) We can construct an operator U on the Hilbert space $\mathcal{H}$ which represents T and which is linear and unitary.
>
> (b) We can construct an operator U on the Hilbert space $\mathcal{H}$ which represents T and which is antilinear and antiunitary.
>
> In either case, the operator U is uniquely defined, up to an overall phase.

To prove the theorem, we begin by proving some properties that T must have if it is probability-preserving and invertible. Let $|\psi_1\rangle$, $|\psi_2\rangle$, ... be a complete orthonormal set of vectors in $\mathcal{H}$. For each $k = 1, 2, \ldots$, choose some particular vector

$$|\tilde{\psi}_k\rangle \in T(\psi_k) \; . \tag{5.8}$$

(a) Show that the vectors $|\tilde{\psi}_1\rangle$, $|\tilde{\psi}_2\rangle$, ... also form a complete orthonormal set of vectors in $\mathcal{H}$.

(b) Now consider the vectors

$$|\phi_k\rangle \equiv \frac{1}{\sqrt{2}}\big(\,|\psi_1\rangle + |\psi_k\rangle\big) \; , \tag{5.9}$$

for $k = 2, 3, \ldots$. Show that for each k,

$$T(\phi_k) \ni \frac{1}{\sqrt{2}}\big(\,|\tilde{\psi}_1\rangle + e^{i\theta_k}\,|\tilde{\psi}_k\rangle\big) \tag{5.10}$$

for some real θ_k.

Now define

$$\begin{aligned} |\psi'_1\rangle &= |\tilde{\psi}_1\rangle \\ |\psi'_k\rangle &= e^{i\theta_k}\,|\tilde{\psi}_k\rangle \;\text{ for } k = 2, 3, \ldots \, , \end{aligned} \tag{5.11}$$

so

$$T(\phi_k) \ni |\phi'_k\rangle \, , \quad \text{where} \quad |\phi'_k\rangle = \frac{1}{\sqrt{2}}\big(\,|\psi'_1\rangle + |\psi'_k\rangle\big) \; . \tag{5.12}$$

(c) Now consider the vectors

$$|\Phi(\theta)\rangle \equiv \frac{1}{\sqrt{2}}\big(\,|\psi_1\rangle + e^{i\theta}\,|\psi_2\rangle\big) \; , \tag{5.13}$$

where θ is a real number. By considering the inner product of these vectors with the $|\psi_k\rangle$ and with $|\phi_2\rangle$, show that either

$$T\big(\Phi(\theta)\big) \ni \big|\Phi'_+(\theta)\big\rangle\ , \quad \text{where } \big|\Phi'_+(\theta)\big\rangle = \frac{1}{\sqrt{2}}\big(\,|\psi'_1\rangle + e^{i\theta}\,|\psi'_2\rangle\big) \qquad \text{(case A)} \tag{5.14a}$$

or

$$T\big(\Phi(\theta)\big) \ni \big|\Phi'_-(\theta)\big\rangle\ , \quad \text{where } \big|\Phi'_-(\theta)\big\rangle = \frac{1}{\sqrt{2}}\big(\,|\psi'_1\rangle + e^{-i\theta}\,|\psi'_2\rangle\big) \qquad \text{(case B)}\ . \tag{5.14b}$$

If $\theta = n\pi$, where n is an integer, then these two cases are identical. Otherwise $\big|\Phi'_+(\theta)\big\rangle$ and $\big|\Phi'_-(\theta)\big\rangle$ belong to different rays, so only one of the two cases can apply. The choice between case A and case B is not our choice, but is determined by the properties of T, which defines the ray $T\big(\Phi(\theta)\big)$.

(d) Show that for a given transformation T, the same case in Eqs. (5.14a) and (14b) applies to all values of θ. (*Hint:* Suppose that case A applies for $\theta = \theta_A$ and case B applies for $\theta = \theta_B$, where $\theta_A \neq n\pi$ and $\theta_B \neq n\pi$. Consider the inner product $\langle\Phi(\theta_B)\,|\Phi(\theta_A)\,\rangle$.)

(e) Now consider the vectors

$$|\Psi_N(\alpha_2,\alpha_3,\ldots,\alpha_N)\rangle = \frac{1}{\sqrt{N}}\big(\,|\psi_1\rangle + e^{i\alpha_2}\,|\psi_2\rangle + e^{i\alpha_3}\,|\psi_3\rangle + \ldots + e^{i\alpha_N}\,|\psi_N\rangle\big), \tag{5.15}$$

where α_2, α_3, ... , α_N are real numbers. For case A, show that

$$T\big(\Psi_N(\alpha_2,\ldots,\alpha_N)\big) \ni \big|\Psi'_{N,+}(\alpha_2,\ldots,\alpha_N)\big\rangle\ , \quad \text{where}$$

$$\big|\Psi'_{N,+}(\alpha_2,\ldots,\alpha_N)\big\rangle = \frac{1}{\sqrt{N}}\big(\,|\psi'_1\rangle + e^{i\alpha_2}\,|\psi'_2\rangle + e^{i\alpha_3}\,|\psi'_3\rangle + \ldots + e^{i\alpha_N}\,|\psi'_N\rangle\big)\ , \tag{5.16a}$$

and for case B, show that

$$T\big(\Psi_N(\alpha_2,\ldots,\alpha_N)\big) \ni \big|\Psi'_{N,-}(\alpha_2,\ldots,\alpha_N)\big\rangle\ , \quad \text{where}$$

$$\big|\Psi'_{N,-}(\alpha_2,\ldots,\alpha_N)\big\rangle = \frac{1}{\sqrt{N}}\big(\,|\psi'_1\rangle + e^{-i\alpha_2}\,|\psi'_2\rangle + e^{-i\alpha_3}\,|\psi'_3\rangle + \ldots + e^{-i\alpha_N}\,|\psi'_N\rangle\big)\ . \tag{5.16b}$$

(*Hint:* Note that for $N = 1$ and $N = 2$, this statement has already been proven. See if you can construct an argument using induction on N which demonstrates the result for all N.)

(f) Now we are ready to consider an arbitrary vector, which can be expanded in the complete orthonormal basis as

$$|\Psi\rangle = \sum_{k=1}^{\infty} C_k\,|\psi_k\rangle \quad . \tag{5.17}$$

Show that for case A,

$$T(\Psi) \ni \left|\Psi'_+\right\rangle \quad \text{where} \quad \left|\Psi'_+\right\rangle = \sum_{k=1}^{\infty} C_k \left|\psi'_k\right\rangle \tag{5.18a}$$

and that for case B,

$$T(\Psi) \ni \left|\Psi'_-\right\rangle \quad \text{where} \quad \left|\Psi'_-\right\rangle = \sum_{k=1}^{\infty} C_k^* \left|\psi'_k\right\rangle \ . \tag{5.18b}$$

(g) For case A, define

$$U\left|\Psi\right\rangle = \left|\Psi'_+\right\rangle = \sum_{k=1}^{\infty} C_k \left|\psi'_k\right\rangle \ , \tag{5.19a}$$

and for case B define

$$U\left|\Psi\right\rangle = \left|\Psi'_-\right\rangle = \sum_{k=1}^{\infty} C_k^* \left|\psi'_k\right\rangle \ , \tag{5.19b}$$

where $\left|\Psi\right\rangle$ is the state defined in Eq. (5.17). From part (f), U is clearly a representation of T, as defined by Eq. (5.3). Show for case A that U is linear and unitary, and for case B that it is antilinear and antiunitary.

(h) Finally, prove that U is unique up to an overall phase. (*Hint:* Assume that U_1 and U_2 both satisfy all the properties described in the theorem. Consider the product $U_2^{-1}U_1$, which in either case A or B is a linear transformation which maps each ray onto itself. Show that such a map is necessarily an overall phase times the identity operator.)