Problem 1: The energy-momentum tensor for source-free electrodynamics

(a) We have the following action

\[ S = \int d^4x \mathcal{L} = \int d^4x \left( \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right), \quad F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}. \]  

(1.1)

Before deriving the equations of motions from it, let us note that \( F_{\mu\nu} \) is antisymmetric: \( F_{\mu\nu} = -F_{\nu\mu} \), and

\[ \frac{\partial F_{\rho\sigma}}{\partial (\partial_{\mu} A_{\nu})} = \delta^{\mu}_{\rho} \delta^{\nu}_{\sigma} - \delta^{\mu}_{\sigma} \delta^{\nu}_{\rho}. \]  

(1.2)

The Euler-Lagrange equations then become

\[ 0 = \partial_{\mu} \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} A_{\nu})} - \frac{\partial \mathcal{L}}{\partial A_{\nu}} = \partial_{\mu} \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} A_{\nu})} \]

\[ = \partial_{\mu} \left( \frac{\partial \mathcal{L}}{\partial F_{\rho\sigma}} \frac{\partial F_{\rho\sigma}}{\partial (\partial_{\mu} A_{\nu})} \right) \]

\[ = \partial_{\mu} \left( -\frac{1}{2} F_{\mu\sigma\rho} (\delta^{\mu}_{\rho} \delta^{\nu}_{\sigma} - \delta^{\mu}_{\sigma} \delta^{\nu}_{\rho}) \right) \]

\[ = -\partial_{\mu} F_{\mu\nu}. \]

(1.3)

We thus get \( \partial_{\mu} F_{\mu\nu} = 0 \), which is nothing other than the inhomogeneous Maxwell equations with no source. If we now set \( \nu = 0 \) in Eq. (1.3), we get \( 0 = \partial_{i} F^{i0} = \partial_{i} E^{i} \), where \( i = 1, 2, 3 \). Thus

\[ \nabla \cdot \vec{E} = 0. \]  

(1.4)

And if we set \( \nu = j \) in Eq. (1.3), we have \( 0 = \partial_{0} F^{0j} + \partial_{j} F^{ij} = -\partial_{i} E^{j} - \partial_{j} e^{ik} B^{k} = -\partial_{j} E^{j} + \left( \nabla \times \vec{B} \right) \). Thus

\[ \nabla \times \vec{B} - \partial_{t} \vec{E} = 0. \]  

(1.5)

(b) Under an infinitesimal translation \( x^{\mu} \rightarrow x^{\mu} - a^{\mu} \), we have

\[ A^{\mu}(x) \rightarrow A^{\mu}(x) = A^{\mu}(x + a) = A^{\mu}(x) + a^{\nu} \partial_{\nu} A^{\mu}(x) \]

\[ \mathcal{L}(x) \rightarrow \mathcal{L}(x + a) = \mathcal{L}(x) + a^{\nu} \partial_{\nu} \mathcal{L}(x) = \mathcal{L}(x) + a^{\nu} \partial_{\nu} (\delta^{\rho\sigma} \mathcal{L}(x)). \]  

(1.10)

From Eq. (1.9), we have

\[ \Delta \mathcal{L} \equiv \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} A_{\lambda})} \Delta (\partial_{\mu} A_{\lambda}) = -F_{\mu\lambda} a^{\nu} \partial_{\nu} \partial_{\lambda} A_{\lambda} = a^{\nu} \partial_{\nu} (F_{\mu\lambda} \partial_{\lambda} A_{\lambda}), \]  

(1.11)

where we used the EOM for \( F_{\mu\nu} \). Comparing Eqs. (1.10) and (1.11), we see that \( \partial_{\mu} (F_{\mu\lambda} \partial_{\lambda} A_{\lambda} - \delta^{\mu}_{\nu} \mathcal{L}) = 0 \), and the energy-momentum tensor is thus

\[ T_{\nu}^{\mu} = -F_{\mu\lambda} \partial_{\nu} A_{\lambda} - \delta^{\mu}_{\nu} \mathcal{L}. \]  

(1.12)

This is manifestly not symmetric in \( \mu, \nu \), but we can nevertheless construct a symmetric energy-momentum tensor \( \bar{T}_{\nu}^{\mu} = T_{\nu}^{\mu} + \partial_{\lambda} K_{\nu\lambda}^{\mu} \), where \( K_{\nu\lambda}^{\mu} \) is antisymmetric in its first two indices, so that \( \partial_{\mu} K_{\nu\lambda}^{\mu} = 0 \). Let us choose \( K_{\nu\lambda}^{\mu} = F_{\nu\lambda} A^{\mu} \) so that

\[ \bar{T}_{\nu}^{\mu} = -F_{\mu\lambda} \partial_{\nu} A_{\lambda} - g^{\mu\nu} \mathcal{L} + F_{\nu\lambda} A_{\lambda} \]

\[ = F_{\nu\lambda} A_{\lambda} - g^{\mu\nu} \mathcal{L}. \]  

(1.13)

This is manifestly symmetric.
Written in terms of electric and magnetic fields, this becomes

\[ \mathcal{E} \equiv \hat{T}^{00} = F^{0\lambda} F_\lambda^0 - \mathcal{S} = E^2 - \frac{1}{2}(E^2 - B^2) = \frac{1}{2}(E^2 + B^2) \] (1.14)

\[ S_i \equiv \hat{T}^{0i} = F^{0j} F_j^i = -E^i(-\epsilon^{ijk} B^k) = (\vec{E} \times \vec{B})_i \]. (1.15)

(c) The transformation,

\[
A^\mu(x) \rightarrow A'^\mu(x) = A^\mu(x) + a^\nu F^\mu_{\nu}(x)
= A^\mu(x) + a^\nu \partial_\nu A^\mu(x) - \partial^\mu(a^\nu A_\nu(x))
\] (1.16)

is equivalent to a coordinate transformation as before, and a gauge transformation,

\[
A^\mu(x) \rightarrow \tilde{A}^\mu(x) = A^\mu(x) + a^\nu \partial_\nu A^\mu(x)
\]

\[
\tilde{A}^\mu \rightarrow A'^\mu(x) = \tilde{A}^\mu(x) + \partial^\mu \phi,
\] (1.17)

where \( \phi(x) = -a^\nu A_\nu(x) \).

As \( \mathcal{S} \) is gauge invariant, \( \mathcal{S} \) transforms as before in Eq. (1.10). Now apply Noether’s Theorem,

\[
j^\mu = a_\nu T^{\mu\nu} = \frac{\partial \mathcal{S}}{\partial \partial_\mu A_\lambda} a_\nu F^{\nu}_{\lambda} - a^\mu \mathcal{S}
\] (1.19)

\[
T^{\mu\nu} = -F^{\mu\lambda} F^{\nu}_{\lambda} - g^{\mu\nu} \mathcal{S}
\] (1.20)

Problem 2: Waves on a string

\[
L = \int_0^a dx \left[ \frac{\kappa}{2} \left( \frac{\partial y}{\partial t} \right)^2 - \frac{T}{2} \left( \frac{\partial y}{\partial x} \right)^2 \right]
\]

First we will get the general solution

\[
\partial_\mu \left( \frac{\partial y}{\partial (\partial x)} \right) = \frac{\partial y}{\partial y}
\]

\[
\partial_t \left( \frac{\partial y}{\partial (\partial x)} \right) + \partial_x \left( \frac{\partial y}{\partial (\partial x)} \right) = 0
\]

\[
\frac{\partial^2 y}{\partial t^2} - T \frac{\partial^2 y}{\partial x^2} = 0
\]

For fixed ends let \( y = \sqrt{\frac{\kappa}{a}} \sum_{n=1}^\infty \sin \left( \frac{n\pi y}{a} \right) q_n \) \( k \)

Plug into eq. of motion

\[
\kappa \sum_n \sin \left( \frac{n\pi y}{a} \right) \ddot{q}_n + T \sum_n \sin \left( \frac{n\pi y}{a} \right) \frac{n^2 \pi^2}{a^2} q_n = 0
\]

\[
\kappa \ddot{q}_n + T \frac{n^2 \pi^2}{a^2} q_n = 0
\]

Alternatively plug into \( L \)
Problem 3: Fields with SO(3) symmetry

Noether's Theorem for multiple fields $\phi_k$

$\phi_k \rightarrow \phi_k + \lambda^b \Delta b_k \phi_k$

Symmetry if $L \rightarrow L + \lambda^b \partial \phi^b$

Then

$J^b = \sum \frac{1}{k} \frac{\partial}{\partial (\theta^\mu \phi^k)} \Delta b_k \phi - \partial \frac{\partial}{\partial (\partial \phi^k)}$

Int the problem

$L = \frac{1}{2} \partial \mu \phi_a \partial^\mu \phi_a - \frac{1}{2} m^2 \phi_a \phi_a$

$\phi_a \rightarrow R_{ab} \phi_b$

$R R^T = 1$

For 3 fields, $R_3$ has 3 parameters

$\phi_a' = \phi_a + \delta \theta \varepsilon_{abc} N_b \phi_c$

$N_b N_b = 1$

Unit vector in 3 dim, 2 parameters

Rot: theta angle $\theta$, 3rd.

$L$ is invariant

$\Delta b_a \phi = \varepsilon_{abc} \phi_c$

$J^b = \sum \frac{\partial}{\partial (\theta^\mu \phi^k)} \varepsilon_{abc} \phi_c$

Equation of motion

$\ddot{q}_n + \frac{m^2}{2} (\frac{\pi}{2})^2 q_n = 0$

as before.
Problem 4: Lorentz transformations and Noether's theorem for scalar fields

(a) We are given
\[ x'^\lambda = x^\lambda - \Sigma^\lambda_\sigma x^\sigma , \]
so lowering the index gives
\[ x'^\lambda = x^\lambda - \Sigma^\lambda_\rho x^\rho . \]
Then to first order in \( \Sigma \),
\[ x'^\lambda x'^\lambda = (x^\lambda - \Sigma^\lambda_\sigma x^\sigma) (x_\lambda - \Sigma_\lambda_\rho x^\rho) \]
\[ = x^\lambda x_\lambda - \Sigma^\lambda_\sigma x^\sigma x_\lambda - \Sigma_\lambda_\rho x^\lambda x^\rho \]
\[ = x^\lambda x_\lambda - \Sigma_\lambda_\sigma x^\sigma x_\lambda - \Sigma_\lambda_\rho x^\lambda x^\rho . \]
But the two terms in \( \Sigma \) each vanish due to the antisymmetry of \( \Sigma^\lambda_\sigma \), so we have
\[ x'^\lambda x'^\lambda = x^\lambda x_\lambda, \]
as expected.

(b) According to Noether's theorem, if a field theory possesses a symmetry
\[ \phi(x) \longrightarrow \phi'(x) = \phi(x) + \alpha^b \Delta \phi_b(x) \]
under which the Lagrangian density \( \mathcal{L} \) is transformed by the addition of a total derivative,
\[ \mathcal{L}(x) \longrightarrow \mathcal{L}'(x) = \mathcal{L}(x) + \alpha^b \partial_\mu \mathcal{J}^\mu_b(x) , \]
where \( \alpha^b \) represents a set of infinitesimal constants, then the currents
\[ j^\mu_b(x) = \partial_\mu \mathcal{J}^\mu_b(x) \]
are conserved:
\[ \partial_\mu j^\mu_b = 0 , \quad \text{for each } b. \]
The Lorentz-invariance of the scalar field Lagrangian can be stated in this form, with \( \alpha^b \leftrightarrow \Sigma^\lambda_\sigma \), and \( \Delta \phi_b(x) \leftrightarrow x^\lambda \partial_\lambda \phi(x) \). The Lagrangian density is a Lorentz scalar, so the transformation acts only on the argument \( x \) of \( \mathcal{L}(x) \):
\[ \mathcal{L}'(x') = \mathcal{L}(x) , \]
which implies that
\[ \mathcal{L}'(x') = \mathcal{L}(x) + \Sigma^\lambda_\sigma x^\sigma \partial_\lambda \mathcal{L}(x) , \]
exactly like the scalar field. We can make contact with Noether’s theorem by writing the second term above as
\[ \Sigma^{\lambda \sigma} x_\sigma \partial_\lambda \xi \phi (x) = \Sigma^{\lambda \sigma} \partial_\mu \left( x_\sigma \delta_\lambda^\mu \xi \phi (x) \right) . \] (4.10)

Thus
\[ \alpha^b_j \mu = \Sigma^{\lambda \sigma} \left\{ \partial^\nu \phi x_\sigma \partial_\lambda \phi (x) - x_\sigma \delta_\lambda^\mu \xi \phi \right\} . \] (4.11)

Since \( \Sigma^{\lambda \sigma} \) is antisymmetric, it is only the part of the above expression that is antisymmetric in \( \lambda \) and \( \sigma \) that is required to obey the conservation equation. Thus, raising the \( \lambda \) and \( \sigma \) indices, a conserved current \( j_1^{\mu \lambda \sigma} \) can be written as
\[ j_1^{\mu \lambda \sigma} = x^\nu \partial^\mu \phi \partial^\nu \phi - x^\lambda \partial^\mu \phi \partial^\nu \phi - (x^\sigma \eta^{\mu \lambda} - x^\lambda \eta^{\mu \sigma}) \xi \phi \] (4.12)

Recalling that the energy-momentum tensor can be written as
\[ T^{\mu \nu} = \partial^\mu \phi \partial^\nu \phi - \eta^{\mu \nu} \xi \phi , \] (4.13)

the conserved current can then be rewritten as
\[ j_1^{\mu \lambda \sigma} = x^\nu T^{\mu \nu \lambda} - x^\lambda T^{\mu \nu \sigma} . \] (4.14)

This current differs from the one defined in the problem set by an overall sign, but of course any fixed multiple of a conserved current is also a conserved current. Hence, Noether’s theorem implies also that
\[ j^{\mu \lambda \sigma} \equiv - j_1^{\mu \lambda \sigma} = x^\lambda T^{\mu \nu \sigma} - x^\sigma T^{\mu \nu \lambda} \] (4.15)

is conserved.

To verify that the equations of motion imply that the current in the box above is conserved, one can first check that \( T^{\mu \nu} \) is conserved. The equations of motion are
\[ \Box \phi = \partial_\mu \partial^\mu \phi = -m^2 \phi , \] (4.16)

and
\[ T^{\mu \nu} = \partial^\mu \phi \partial^\nu \phi - \frac{1}{2} \eta^{\mu \nu} \left[ \partial_\lambda \phi \partial^\lambda \phi - m^2 \phi \right] . \] (4.17)

Then
\[ \partial_\mu T^{\mu \nu} = \Box \partial^\nu \phi + \partial^\mu \phi \partial_\mu \partial^\nu \phi - \partial^\lambda \phi \partial^\nu \partial_\lambda \phi + m^2 \phi \partial^\nu \phi \]
\[ = -m^2 \partial^\nu \phi + \partial^\mu \phi \partial_\mu \partial^\nu \phi - \partial^\lambda \phi \partial^\nu \partial_\lambda \phi + m^2 \phi \partial^\nu \phi \] (4.18)

\[ = 0 . \]