MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

8.323: Relativistic Quantum Field Theory I

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PROBLEM SET 1 SOLUTIONS

Problem 1: The energy-momentum tensor for source-free electrodynamics

(a) We have the following action

$$S = \int d^4x\, \mathcal{L} = \int d^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right), \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ . \tag{1.1}$$

Before deriving the equations of motions from it, let us note that $F_{\mu\nu}$ is antisymmetric: $F_{\mu\nu} = -F_{\nu\mu}$, and

$$\frac{\partial F_{\rho\sigma}}{\partial(\partial_\mu A_\nu)} = \delta^\mu_\rho \delta^\nu_\sigma - \delta^\nu_\rho \delta^\mu_\sigma \ . \tag{1.2}$$

The Euler-Lagrange equations then become

$$\begin{aligned} 0 =& \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} - \frac{\partial \mathcal{L}}{\partial A_\nu} = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \\ =& \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial F_{\rho\sigma}} \frac{\partial F_{\rho\sigma}}{\partial(\partial_\mu A_\nu)}\right) \\ =& \partial_\mu \left(-\frac{1}{2}F^{\rho\sigma}(\delta^\mu_\rho \delta^\nu_\sigma - \delta^\nu_\rho \delta^\mu_\sigma)\right) \\ =& -\partial_\mu F^{\mu\nu} \ . \end{aligned} \tag{1.3}$$

We thus get $\partial_\mu F^{\mu\nu} = 0$, which is nothing other than the inhomogeneous Maxwell equations with no source. If we now set $\nu = 0$ in Eq. (1.3), we get $0 = \partial_i F^{i0} = \partial_i E^i$, where $i = 1, 2, 3$. Thus

$$\vec{\nabla} \cdot \vec{E} = 0 \ . \tag{1.4}$$

And if we set $\nu = j$ in Eq. (1.3), we have $0 = \partial_0 F^{0j} + \partial_i F^{ij} = -\partial_t E^j - \partial_i \epsilon^{ijk} B^k = -\partial_t E^j + \left(\vec{\nabla} \times \vec{B}\right)^j$. Thus

$$\vec{\nabla} \times \vec{B} - \partial_t \vec{E} = 0 \ . \tag{1.5}$$

Note added: To find the homogeneous Maxwell equations, one can use the **dual** field tensor ${}^*F^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$. Using the definition of $F_{\mu\nu}$, Eq. (1.1), we find that $\partial_\mu {}^*F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\partial_\mu F_{\rho\sigma} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\partial_\mu(\partial_\rho A_\sigma - \partial_\sigma A_\rho) = 0$, due to the antisymmetry of $\epsilon^{\mu\nu\rho\sigma}$. Therefore, we get the Bianchi identity

$$\partial_\mu {}^*F^{\mu\nu} = 0 \ . \tag{1.6}$$

Moreover, we have ${}^*F^{0i} = \frac{1}{2}\epsilon^{0i\rho\sigma}F_{\rho\sigma} = \frac{1}{2}\epsilon^{ijk}F_{jk} = -\frac{1}{2}\epsilon^{ijk}\epsilon^{jk\ell}B^\ell = -B^i$, and ${}^*F^{ij} = \epsilon^{ijk0}F_{k0} = -\epsilon^{ijk0}E_k = \epsilon^{ijk}E_k$. In other words, ${}^*F^{\mu\nu}$ is obtained from $F^{\mu\nu}$ by the transformation $\vec{E} \to \vec{B}$ and $\vec{B} \to -\vec{E}$. Using Eq. (1.6) and repeating the steps that led to Eq. (1.4) and Eq. (1.5), we get the homogeneous Maxwell equations:

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{1.7}$$

$$\vec{\nabla} \times \vec{E} + \partial_t \vec{B} = 0 \ . \tag{1.8}$$

(b) Under an infinitesimal translation $x^\mu \to x^\mu - a^\mu$, we have

$$A^\mu(x) \to A'^\mu(x) = A^\mu(x+a) = A^\mu(x) + a^\nu \partial_\nu A^\mu(x) \tag{1.9}$$

$$\mathcal{L}(x) \to \mathcal{L}(x) + a^\mu \partial_\mu \mathcal{L}(x) = \mathcal{L}(x) + a^\nu \partial_\mu \left(\delta^\mu_\nu \mathcal{L}(x)\right) \ . \tag{1.10}$$

From Eq. (1.9), we have

$$\Delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\lambda)} \Delta(\partial_\mu A_\lambda) = -F^{\mu\lambda} a^\nu \partial_\mu \partial_\nu A_\lambda = a^\nu \partial_\mu \left(-F^{\mu\lambda}\partial_\nu A_\lambda\right) \ , \tag{1.11}$$

where we used the EOM for $F^{\mu\nu}$. Comparing Eqs. (1.10) and (1.11), we see that $\partial_\mu \left(-F^{\mu\lambda}\partial_\nu A_\lambda - \delta^\mu_\nu \mathcal{L}\right) = 0$, and the energy-momentum tensor is thus

$$T^\mu_{\ \nu} = -F^{\mu\lambda}\partial_\nu A_\lambda - \delta^\mu_{\ \nu}\mathcal{L} \ . \tag{1.12}$$

This is manifestly not symmetric in μ, ν; but we can nevertheless construct a symmetric energy-momentum tensor $\hat{T}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu}$, where $K^{\lambda\mu\nu}$ is antisymmetric in its first two indices, so that $\partial_\mu \partial_\lambda K^{\lambda\mu\nu} = 0$. Let us choose $K^{\lambda\mu\nu} = F^{\mu\lambda}A^\nu$ so that

$$\begin{aligned} \hat{T}^{\mu\nu} &= -F^{\mu\lambda}\partial^\nu A_\lambda - g^{\mu\nu}\mathcal{L} + F^{\mu\lambda}\partial_\lambda A^\nu \\ &= F^{\mu\lambda}F_\lambda^{\ \nu} - g^{\mu\nu}\mathcal{L} \ . \end{aligned} \tag{1.13}$$

This is manifestly symmetric.

Written in terms of electric and magnetic fields, this becomes

$$\mathcal{E} \equiv \hat{T}^{00} = F^{0\lambda} F^0_\lambda - \mathcal{L} = E^2 - \frac{1}{2}(E^2 - B^2) = \frac{1}{2}(E^2 + B^2) \tag{1.14}$$

$$S_i \equiv \hat{T}^{0i} = F^{0j} F_j{}^i = -E^j(-\epsilon^{ijk} B^k) = (\vec{E} \times \vec{B})_i \ . \tag{1.15}$$

(c) The transformation,

$$\begin{aligned} A^\mu(x) \to A'^\mu(x) &= A^\mu(x) + a^\nu F_\nu{}^\mu(x) \\ &= A^\mu(x) + a^\nu \partial_\nu A^\mu(x) - \partial^\mu(a^\nu A_\nu(x)) \end{aligned} \tag{1.16}$$

is equivilent to a coordinate transformation as before, and a gauge transformation,

$$A^\mu(x) \to \tilde{A}^\mu(x) = A^\mu(x) + a^\nu \partial_\nu A^\mu(x) \tag{1.17}$$

$$\tilde{A}^\mu \to A'^\mu(x) = \tilde{A}^\mu(x) + \partial^\mu \phi \ , \tag{1.18}$$

where $\phi(x) = -a^\nu A_\nu(x)$.

As $\mathcal{L}$ is gauge invariant, $\mathcal{L}$ transforms as before in Eq. (1.10). Now apply Noether's Theorem,

$$j^\mu = a_\nu T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\lambda)} a_\nu F^\nu{}_\lambda - a^\mu \mathcal{L} \tag{1.19}$$

$$T^{\mu\nu} = -F^{\mu\lambda} F^\nu{}_\lambda - g^{\mu\nu} \mathcal{L} \tag{1.20}$$

Problem 2: Waves on a string

$$L = \int_0^a dx \left[\frac{\sigma}{2} \left(\frac{\partial y}{\partial t} \right)^2 - \frac{T}{2} \left(\frac{\partial y}{\partial x} \right)^2 \right]$$

First we will get the general solution

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu y)} \right) = \frac{\partial \mathcal{L}}{\partial y}$$

$$\partial_t \left(\frac{\partial \mathcal{L}}{\partial(\partial_t y)} \right) + \partial_x \left(\frac{\partial \mathcal{L}}{\partial(\partial_x y)} \right) = 0$$

$$\sigma \frac{\partial^2 y}{\partial t^2} - T \frac{\partial^2 y}{\partial x^2} = 0$$

For fixed ends let $y = \sqrt{\frac{2}{a}} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{a}\right) q_n(t)$

Plug into eq. of. motion

$$\sigma \sum_n \sin\left(\frac{n\pi x}{a}\right) \ddot{q}_n + T \sum_n \sin\left(\frac{n\pi x}{a}\right) \frac{n^2\pi^2}{a^2} q_n = 0$$

so $\sigma \ddot{q}_n + T \frac{n^2\pi^2}{a^2} q_n = 0$

Alternately Plug into L

$$L = \frac{\sigma}{2}\int_0^a dx\, \frac{2}{a} \sum_{n,m} \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi x}{a}\right) \dot{q}_n \dot{q}_m$$

$$- \frac{T}{2}\int_0^a dx\, \frac{2}{a} \sum_{n,m} \cos\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi x}{a}\right)\left(\frac{n\pi}{a}\right)\left(\frac{m\pi}{a}\right) q_n q_m$$

Use orthonomality of $\sin\left(\frac{n\pi x}{a}\right)$ and of $\cos\left(\frac{n\pi x}{a}\right)$

$$L = \sum_n \left[\frac{\sigma}{2}\dot{q}_n^2 - \frac{T}{2}\left(\frac{n\pi}{a}\right)^2 q_n^2\right]$$

equation of motion $\sigma \ddot{q}_n + T\left(\frac{n\pi}{a}\right)^2 q_n = 0$

as before.

Problem 3: Fields with SO(3) symmetry

Noether's Theorem for multiple fields ϕ_K

$$\phi_K \to \phi_K + \alpha^b \Delta_{b,K}\phi$$

Symmetry if $\mathcal{L} \to \mathcal{L} + \alpha^b \partial_\mu \mathcal{J}^\mu_b$

Then
$$j^\mu_b = \sum_K \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_K)} \Delta_{b,K}\phi - \mathcal{J}^\mu_b$$

In the problem
$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi_a \partial^\mu \phi_a - \frac{1}{2} m^2 \phi_a \phi_a$$

$$\phi_a \to R_{ab}\phi_b \qquad RR^T = 1$$

For 3 fields, R has 3 parameters

$$\phi_a' = \phi_a + \delta\theta\, \varepsilon_{abc} n_b \phi_c$$

$n_b n_b = 1$ unit vector in 3 dim, 2 parameters
Rotation angle θ is 3rd.

$\mathcal{L}$ is invariant $\quad \Delta_{b,a}\phi = \varepsilon_{abc}\phi_c$

$$j^\mu_b = \sum_a \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \varepsilon_{abc}\phi_c$$

$$j^\mu_b = \partial^\mu \phi_a \varepsilon_{abc} \phi_c$$

or if you prefer $\quad j^\mu_a = \varepsilon_{abc} \partial^\mu \phi_b \phi_c$

Check conservation

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi_a \partial^\mu \phi_a - V(\phi_a \phi_a)$$

euler-lagrange $\quad \partial_\mu \partial^\mu \phi_a = -2V' \cdot \phi_a$

Verify $\quad \partial_\mu j^\mu_a = 0$

$$\partial_\mu j^\mu_a = \varepsilon_{abc} (\partial_\mu \partial^\mu \phi_b) \phi_c + \varepsilon_{abc} \partial^\mu \phi_b \partial_\mu \phi_c$$

$$= \varepsilon_{abc} (-2V') \phi_b \phi_c$$

$$+ \varepsilon_{abc} (\partial_t \phi_b \partial_t \phi_c - \partial_x \phi_b \partial_x \phi_c - \dots)$$

$$= 0 \quad \text{since}$$

$$\varepsilon_{abc} V^a V^b = 0$$

Problem 4: Lorentz transformations and Noether's theorem for scalar fields

(a) We are given
$$x'^{\lambda} = x^{\lambda} - \Sigma^{\lambda}{}_{\sigma} x^{\sigma} \ , \tag{4.1}$$
so lowering the index gives
$$x'_{\lambda} = x_{\lambda} - \Sigma_{\lambda\rho} x^{\rho} \ . \tag{4.2}$$
Then to first order in Σ,
$$\begin{aligned} x'^{\lambda}\, x'_{\lambda} &= \left(x^{\lambda} - \Sigma^{\lambda}{}_{\sigma} x^{\sigma}\right)\left(x_{\lambda} - \Sigma_{\lambda\rho} x^{\rho}\right) \\ &= x^{\lambda}\, x_{\lambda} - \Sigma^{\lambda}{}_{\sigma} x^{\sigma} x_{\lambda} - \Sigma_{\lambda\rho} x^{\lambda} x^{\rho} \\ &= x^{\lambda}\, x_{\lambda} - \Sigma_{\lambda\sigma} x^{\sigma} x^{\lambda} - \Sigma_{\lambda\rho} x^{\lambda} x^{\rho} \ . \end{aligned} \tag{4.3}$$
But the two terms in Σ each vanish due to the antisymmetry of $\Sigma_{\lambda\sigma}$, so we have $x'^{\lambda}\, x'_{\lambda} = x^{\lambda}\, x_{\lambda}$, as expected.

(b) According to Noether's theorem, if a field theory possesses a symmetry
$$\phi(x) \longrightarrow \phi'(x) = \phi(x) + \alpha^b \Delta\phi_b(x) \tag{4.4}$$
under which the Lagrangian density $\mathscr{L}$ is transformed by the addition of a total derivative,
$$\mathscr{L}(x) \longrightarrow \mathscr{L}'(x) = \mathscr{L}(x) + \alpha^b \partial_\mu \mathscr{J}^{\mu}_b(x) \ , \tag{4.5}$$
where α^b represents a set of infinitesimal constants, then the currents
$$j^{\mu}_b(x) = \frac{\partial \mathscr{L}}{\partial\left(\partial_\mu \phi\right)} \Delta_b \phi - \mathscr{J}^{\mu}_b \tag{4.6}$$
are conserved:
$$\partial_\mu j^{\mu}_b = 0 \ , \quad \text{for each } b. \tag{4.7}$$
The Lorentz-invariance of the scalar field Lagrangian can be stated in this form, with $\alpha^b \leftrightarrow \Sigma^{\lambda\sigma}$, and $\Delta\phi_b(x) \leftrightarrow x_\sigma \partial_\lambda \phi(x)$. The Lagrangian density is a Lorentz scalar, so the tranformation acts only on the argument x of $\mathscr{L}(x)$:
$$\mathscr{L}'(x') = \mathscr{L}(x) \ , \tag{4.8}$$
which implies that
$$\mathscr{L}'(x) = \mathscr{L}(x) + \Sigma^{\lambda\sigma} x_\sigma \partial_\lambda \mathscr{L}(x) \ , \tag{4.9}$$

exactly like the scalar field. We can make contact with Noether's theorem by writing the second term above as

$$\Sigma^{\lambda\sigma} x_\sigma \partial_\lambda \mathcal{L}(x) = \Sigma^{\lambda\sigma} \partial_\mu \left(x_\sigma \delta^\mu_\lambda \mathcal{L}(x) \right) \ . \tag{4.10}$$

Thus

$$\alpha^b j^\mu_b = \Sigma^{\lambda\sigma} \left\{ \partial^\mu \phi x_\sigma \partial_\lambda \phi(x) - x_\sigma \delta^\mu_\lambda \mathcal{L} \right\} \ . \tag{4.11}$$

Since $\Sigma^{\lambda\sigma}$ is antisymmetric, it is only the part of the above expression that is antisymmetric in λ and σ that is required to obey the conservation equation. Thus, raising the λ and σ indices, a conserved current $j_1^{\mu\lambda\sigma}$ can be written as

$$j_1^{\mu\lambda\sigma} = x^\sigma \partial^\mu \phi \partial^\lambda \phi - x^\lambda \partial^\mu \phi \partial^\sigma \phi - (x^\sigma \eta^{\mu\lambda} - x^\lambda \eta^{\mu\sigma}) \mathcal{L} \ . \tag{4.12}$$

Recalling that the energy-momentum tensor can be written as

$$T^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi - \eta^{\mu\nu} \mathcal{L} \ , \tag{4.13}$$

the conserved current can then be rewritten as

$$j_1^{\mu\lambda\sigma} = x^\sigma T^{\mu\lambda} - x^\lambda T^{\mu\sigma} \ . \tag{4.14}$$

This current differs from the one defined in the problem set by an overall sign, but of course any fixed multiple of a conserved current is also a conserved current. Hence, Noether's theorem implies also that

$$\boxed{j^{\mu\lambda\sigma} \equiv -j_1^{\mu\lambda\sigma} = x^\lambda T^{\mu\sigma} - x^\sigma T^{\mu\lambda}} \tag{4.15}$$

is conserved.

To verify that the equations of motion imply that the current in the box above is conserved, one can first check that $T^{\mu\nu}$ is conserved. The equations of motion are

$$\Box \phi \equiv \partial_\mu \partial^\mu \phi = -m^2 \phi \ , \tag{4.16}$$

and

$$T^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi - \frac{1}{2} \eta^{\mu\nu} \left[\partial_\lambda \phi \partial^\lambda \phi - m^2 \phi^2 \right] \ . \tag{4.17}$$

Then

$$\begin{aligned} \partial_\mu T^{\mu\nu} &= \Box \phi \partial^\nu \phi + \partial^\mu \phi \partial_\mu \partial^\nu \phi - \partial^\lambda \phi \partial^\nu \partial_\lambda \phi + m^2 \phi \partial^\nu \phi \\ &= -m^2 \partial^\nu \phi + \partial^\mu \phi \partial_\mu \partial^\nu \phi - \partial^\lambda \phi \partial^\nu \partial_\lambda \phi + m^2 \phi \partial^\nu \phi \\ &= 0 \ . \end{aligned} \tag{4.18}$$

It then follows that

$$\begin{aligned} \partial_\mu j^{\mu\lambda\sigma} &= \delta^\lambda_\mu T^{\mu\sigma} + x^\lambda \partial_\mu T^{\mu\sigma} - \delta^\sigma_\mu T^{\mu\lambda} - x^\sigma \partial_\mu T^{\mu\lambda} \\ &= T^{\lambda\sigma} - T^{\sigma\lambda} \\ &= 0 \ . \end{aligned} \tag{4.19}$$

That is, $j^{\mu\lambda\sigma}$ is conserved as long as $T^{\mu\nu}$ is both symmetric and conserved.

(c) The conservation of $j^{\mu\lambda\sigma}$ implies that the quantity

$$K^i \equiv \int \mathrm{d}^3x \, j^{00i}(\vec{x}) \tag{4.20}$$

is conserved. For clarity we can replace T^{00} by $\mathcal{H}$, the energy density, and T^{0i} by $\mathfrak{p}^i$, the momentum density. Then

$$\vec{K} = \int \mathrm{d}^3x \left[\vec{\mathfrak{p}}\, t - \mathcal{H} \vec{x} \right] \ . \tag{4.21}$$

If we let M be the total energy (or mass, since $c = 1$), then we can define the center of mass position as

$$\vec{x}_{\mathrm{cm}} = \frac{1}{M} \int \mathrm{d}^3x \, \vec{x} \, \mathcal{H}(\vec{x}, t) \ , \tag{4.22}$$

and we know that the total momentum $\vec{P}$ can be written as

$$\vec{P} = \int \mathrm{d}^3x \, \vec{\mathfrak{p}}(\vec{x}, t) \ . \tag{4.23}$$

Then

$$\vec{K} = -M \left[\vec{x}_{\mathrm{cm}} - \frac{\vec{P}}{M} t \right] \ , \tag{4.24}$$

so this (explicitly time-dependent) conservation law implies that the position of the center of mass moves precisely at velocity $\vec{P}/M$.