MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

8.323: Relativistic Quantum Field Theory I

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PROBLEM SET 2 SOLUTIONS

Problem 1: Complex scalar fields

$$\mathcal{L} = (\partial_\mu \phi^*)\,(\partial^\mu \phi) - m^2 \phi^* \phi \tag{1.1}$$

(a) Treat ϕ and ϕ^* "as if they were independent," which is done by defining

$$\frac{\partial}{\partial \phi} \equiv \frac{1}{2}\left(\frac{\partial}{\partial \operatorname{Re}\phi} - i\frac{\partial}{\partial \operatorname{Im}\phi}\right) \quad \text{and} \quad \frac{\partial}{\partial \phi^*} \equiv \frac{1}{2}\left(\frac{\partial}{\partial \operatorname{Re}\phi} + i\frac{\partial}{\partial \operatorname{Im}\phi}\right)\ , \tag{1.2}$$

with analogous definitions for $\partial/\partial(\partial_0\phi)$ and $\partial/\partial(\partial_0\phi^*)$. The conjugate momenta are

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0\phi)} = \partial_0\phi^* \qquad \text{and} \qquad \pi^* = \frac{\partial \mathcal{L}}{\partial(\partial_0\phi^*)} = \partial_0\phi\ . \tag{1.3}$$

And the canonical commutation relations are thus

$$\begin{aligned}
&[\phi(\vec{x},t),\, \partial_0\phi^*(\vec{y},t)] = i\delta^{(3)}(\vec{x}-\vec{y}) \\
&[\phi^*(\vec{x},t),\, \partial_0\phi(\vec{y},t)] = i\delta^{(3)}(\vec{x}-\vec{y}) \\
&\text{others} = 0\,.
\end{aligned} \tag{1.4}$$

And the Hamiltonian is

$$\begin{aligned}
H &= \int d^3x\,\left(\pi\partial_0\phi + \pi^*\partial_0\phi^* - \mathcal{L}\right) \\
&= \int d^3x\,\left(\pi\pi^* + \pi^*\pi - \left(\pi\pi^* - \vec{\nabla}\phi^*\cdot\vec{\nabla}\phi - m^2\phi^*\phi\right)\right) \\
&= \int d^3x\,\left(\pi^*\pi + \vec{\nabla}\phi^*\cdot\vec{\nabla}\phi + m^2\phi^*\phi\right).
\end{aligned} \tag{1.5}$$

Heisenberg EOM: $\partial_0\phi(x) = -i[\phi(x),\, H]$. From the commutation relations, one can calculate that $-i[\phi(x),\, H] = \pi^*(x)$, and therefore $\partial_0\phi(x) = \pi^*(x)$. Now let us do the same with π^*.

$$\begin{aligned}
\partial_0^2\phi(x) =& \partial_0\pi^*(x) = -i[\pi^*(x),\, H] \\
=& -i\int d^3y\,\left[\pi^*(x),\, \left(\vec{\nabla}\phi^*(y)\cdot\vec{\nabla}\phi(y) + m^2\phi^*(y)\phi(y)\right)\right] \\
=& \nabla^2\phi(x) - m^2\phi(x)\,,
\end{aligned} \tag{1.6}$$

by using the canonical commutation relations and integrating by part. We see that Eq. (1.6) implies the Klein-Gordon equation for ϕ

$$\partial_\mu \partial^\mu \phi + m^2 \phi = 0 \,. \tag{1.7}$$

(b) We know from the equations of motion (1.7) that ϕ and

$$\begin{aligned} \phi(x) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}} e^{-ip\cdot x} + b^\dagger_{\vec{p}} e^{ip\cdot x} \right) \\ \phi^*(x) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(b_{\vec{p}} e^{-ip\cdot x} + a^\dagger_{\vec{p}} e^{ip\cdot x} \right) . \end{aligned} \tag{1.8}$$

Now ϕ is not real, so $a_{\vec{p}}$ and $b_{\vec{p}}$ are independent; the choice of using the names $a_{\vec{p}}$ and $b^\dagger_{\vec{p}}$ for the operators in the expansion of $\phi(x)$ can be justified by working out their commutation relations from Eq. (1.4). Indeed one finds:

$$\begin{aligned} &[a_{\vec{p}}, a^\dagger_{\vec{q}}] = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}) \\ &[b_{\vec{p}}, b^\dagger_{\vec{q}}] = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}) \\ &\text{others} = 0 \,. \end{aligned} \tag{1.9}$$

This shows that $a^\dagger_{\vec{q}}$ and $b^\dagger_{\vec{q}}$ are creation operators, and we have thus two kinds of particles: the one created by $a^\dagger_{\vec{q}}$ and the one created by $b^\dagger_{\vec{q}}$. Using Eq. (1.8), we can express the Hamiltonian in terms of a, $a^\dagger$, b and $b^\dagger$:

$$\begin{aligned} H = \int d^3x \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \Bigg(& \frac{1}{2} \sqrt{E_{\vec{p}} E_{\vec{q}}} (a_{\vec{p}} e^{-ip\cdot x} - b^\dagger_{\vec{p}} e^{ip\cdot x})(a^\dagger_{\vec{q}} e^{iq\cdot x} - b_{\vec{q}} e^{-iq\cdot x}) + \\ & + \frac{1}{2\sqrt{E_{\vec{p}} E_{\vec{q}}}} (\vec{p} \cdot \vec{q})(b_{\vec{p}} e^{-ip\cdot x} - a^\dagger_{\vec{p}} e^{ip\cdot x})(b^\dagger_{\vec{q}} e^{iq\cdot x} - a_{\vec{q}} e^{-iq\cdot x}) + \\ & + \frac{m^2}{2\sqrt{E_{\vec{p}} E_{\vec{q}}}} (b_{\vec{p}} e^{-ip\cdot x} + a^\dagger_{\vec{p}} e^{ip\cdot x})(b^\dagger_{\vec{q}} e^{iq\cdot x} + a_{\vec{q}} e^{-iq\cdot x}) \Bigg) . \end{aligned} \tag{1.10}$$

Since $\dot{H} = 0$, we can evaluate Eq. (1.10) at $t = 0$; we then use $\int d^3x e^{i\vec{k}\cdot\vec{x}} = (2\pi)^3 \delta^{(3)}(\vec{k})$, and integrate over d^3q to get

$$\begin{aligned} H = \int \frac{d^3p}{(2\pi)^3} \Bigg(& \frac{E_{\vec{p}}}{2} (a_{\vec{p}} a^\dagger_{\vec{p}} - a_{\vec{p}} b_{-\vec{p}} - b^\dagger_{\vec{p}} a^\dagger_{-\vec{p}} + b^\dagger_{\vec{p}} b_{\vec{p}}) + \\ & + \frac{\vec{p}^2}{2E_{\vec{p}}} (b_{\vec{p}} b^\dagger_{\vec{p}} + b_{\vec{p}} a_{-\vec{p}} + a^\dagger_{\vec{p}} b^\dagger_{-\vec{p}} + a^\dagger_{\vec{p}} a_{\vec{p}}) + \\ & + \frac{m^2}{2E_{\vec{p}}} (b_{\vec{p}} b^\dagger_{\vec{p}} + b_{\vec{p}} a_{-\vec{p}} + a^\dagger_{\vec{p}} b^\dagger_{-\vec{p}} + a^\dagger_{\vec{p}} a_{\vec{p}}) \Bigg) . \end{aligned} \tag{1.11}$$

But $p^2 + m^2 = E_{\vec{p}}^2$. thus

$$H = \int \frac{d^2p}{(2\pi)^3} E_{\vec{p}} \left(a_{\vec{p}}^\dagger a_{\vec{p}} + b_{\vec{p}}^\dagger b_{\vec{p}} + \frac{1}{2}[a_{\vec{p}},\, a_{\vec{p}}^\dagger] + \frac{1}{2}[b_{\vec{p}},\, b_{\vec{p}}^\dagger] \right). \tag{1.12}$$

The last two terms correspond to the zero-point energy and can be discarded. Note, by the way, that we would have gotten the same answer if we had evaluated Eq. (1.10) at an arbitrary time: the surviving terms were time-independent, and the terms that cancelled would have still cancelled if the time-dependence was included.

(c) Again, since Q is conserved, we can evaluate it at $t = 0$:

$$\begin{aligned} Q =& \frac{i}{2} \int d^3x (\phi^* \pi^* - \pi\phi) = \frac{i}{2} \int d^3x \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{1}{2\sqrt{E_{\vec{p}} E_{\vec{q}}}} \times \\ & \times \left(-iE_{\vec{q}} (b_{-\vec{p}} + a_{\vec{p}}^\dagger)(a_{\vec{q}} - b_{-\vec{q}}^\dagger) e^{i\vec{x}\cdot(\vec{p}-\vec{q})} - iE_{\vec{p}}(-b_{-\vec{p}} + a_{\vec{p}}^\dagger)(a_{\vec{q}} + b_{-\vec{q}}^\dagger) e^{i\vec{x}\cdot(\vec{p}-\vec{q})} \right) \\ =& \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \left(a_{\vec{p}}^\dagger a_{\vec{p}} - b_{\vec{p}}^\dagger b_{\vec{p}} - [b_{\vec{p}},\, b_{\vec{p}}^\dagger] \right), \end{aligned} \tag{1.13}$$

where we have made the substitution $\vec{p} \to -\vec{p}$ and $\vec{q} \to -\vec{q}$ in some of the terms to factorize the oscillating part. We discard the last term (infinite zero point contribution), and write

$$Q = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \left(a_{\vec{p}}^\dagger a_{\vec{p}} - b_{\vec{p}}^\dagger b_{\vec{p}} \right). \tag{1.14}$$

We have then

$$[Q, a_{\vec{p}}^\dagger] = \frac{1}{2} a_{\vec{p}}^\dagger, \qquad [Q, b_{\vec{p}}^\dagger] = -\frac{1}{2} b_{\vec{p}}^\dagger, \tag{1.15}$$

which means that the particle created by $a_{\vec{p}}^\dagger$ has charge $+\frac{1}{2}$, whereas the particle created by $b_{\vec{p}}^\dagger$ has charge $-\frac{1}{2}$.

(d)

$$\mathcal{L} = \partial_\mu \phi^* \cdot \partial^\mu \phi - m^2 \phi^* \cdot \phi, \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \tag{1.16}$$

$\mathcal{L}$ is manifestly invariant under $\phi \to g\phi$ for any $g \in SU(2)$. An element of $SU(2)$ can be written $g = \exp\left(-\frac{i}{2}\sigma^i \epsilon_i\right)$, where the σ^i $(i = 1, 2, 3)$ are the Pauli matrices (defined in page xx of Peskin & Schroeder), and the ϵ_i are real parameters. For g close to the identity, $\epsilon_i \ll 1$ and we can expand: $g = 1 - \frac{i}{2}\sigma^i \epsilon_i + \mathcal{O}(\epsilon^2)$.

If now we define $(\delta\phi)_a^i \equiv -\frac{i}{2}(\sigma^i)_{ab}\phi_b$, we find three currents

$$J^{i\mu} = \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi_a\right)} (\delta\phi)_a^i + \text{h.c.} = -\frac{i}{2} \left(\partial^\mu \phi_a^* (\sigma^i)_{ab} \phi_b - \phi_a^* (\sigma^i)_{ab} \partial^\mu \phi_b \right), \tag{1.17}$$

where h.c. stands for the hermitian conjugate. And the corresponding conserved charges are thus

$$Q^i = \int d^3x J^0 = \frac{i}{2}\int d^3x \left(\phi_a^*(\sigma^i)_{ab}\pi_b^* - \pi_a(\sigma^i)_{ab}\phi_b\right) . \tag{1.18}$$

Let us now calculate the commutator

$$\begin{aligned}
[Q^i, Q^j] =& -\frac{1}{4}\int d^3x d^3y \left([\phi_a^*(\sigma^i)_{ab}\pi_b^* \,,\, \phi_c^*(\sigma^j)_{cd}\pi_d^*] + [\pi_a(\sigma^i)_{ab}\phi_b \,,\, \pi_c(\sigma^j)_{cd}\phi_d]\right)\\
=& -\frac{1}{4}\int d^3x d^3y \left(\phi_a^*(\sigma^i)_{ab}[\pi_b^* \,,\, \phi_c^*(\sigma^j)_{cd}\pi_d^*] + (\sigma^i)_{ab}[\phi_a^* \,,\, \phi_c^*(\sigma^j)_{cd}\pi_d^*]\pi_b^*\right) - \text{h.c.}\\
=& -\frac{1}{4}\int d^3x \left((-i)\phi_a^*(\sigma^i)_{ab}(\sigma^j)_{bd}\pi_d^* + i\phi_c^*(\sigma^j)_{cd}(\sigma^i)_{db}\pi_b^*\right) - \text{h.c.}\\
=& -\frac{1}{4}\int d^3x \left((-i)\phi_a^*[\sigma^i, \sigma^j]_{ab}\pi_b^* + i\pi_a[\sigma^i, \sigma^j]_{ab}\phi_b\right)\\
=& -\frac{1}{4}\int d^3x 2i\epsilon^{ijk}(-i)\left(\phi_a^*(\sigma^k)_{ab}\pi_b^* - \pi_a(\sigma^k)_{ab}\phi_b\right)\\
=& i\epsilon^{ijk}Q^k \, .
\end{aligned} \tag{1.19}$$

In the second line, we used that $[A\,, B]^\dagger = -[A^\dagger\ , B^\dagger]$, and in the fifth line we used $[\sigma^i, \sigma^j] = 2i\epsilon^{ijk}\sigma^k$. Looking at the result, we see that the charges Q^i generate an $SU(2)$ algebra.

Note that we still have the $U(1)$ symmetry $\phi_a \to e^{i\theta}\phi_a$, which was studied in (c). Together with $SU(2)$, it forms a total symmetry group $SU(2) \times U(1) = U(2)$.

In general, the conserved charges generate an algebra isomorphic to the Lie algebra of the symmetry group: For N complex scalars,

$$\mathcal{L} = \partial_\mu\phi^* \cdot \partial^\mu\phi - m^2\phi^*\cdot\phi\, , \quad \phi = \begin{pmatrix}\phi_1\\ \vdots \\ \phi_2\end{pmatrix} , \tag{1.20}$$

is invariant under $\phi \to g\phi$, where $g \in SU(N)$. And we have N^2-1 charges Q^i , $i = 1, \ldots, N^2-1$, such that $[Q^i, Q^j] = if^{ijk}Q^k$. And the f^{ijk} are the structure constants of $SU(N)$.

As for the missing currents, we can see where they come from if we re-express the Lagrangian as:

$$\begin{aligned}
\mathcal{L} &= \sum_a \left[(\partial_\mu \phi_a^*)(\partial^\mu \phi) - m^2 \phi^* \phi\right] \\
&= \sum_a \left[(\partial_\mu \text{Re } \phi_a)^2 + (\partial_\mu \text{Im } \phi_a)^2 - m^2 (\text{Re } \phi_a)^2 - m^2 (\text{Im } \phi_a)^2\right] \\
&= \sum_a \left[(\partial_\mu \Phi)^T (\partial_\mu \Phi) - m^2 \Phi^T \Phi\right] \ ,
\end{aligned} \tag{1.21}$$

where

$$\Phi = \begin{pmatrix} \text{Re } \Phi_1 \\ \text{Im } \Phi_1 \\ \vdots \\ \text{Re } \Phi_n \\ \text{Im } \Phi_n \end{pmatrix} \tag{1.22}$$

is a vector in a $2N$–dimensional Euclidean space. The Lagrangian is clearly invariant if $\Phi'^T \Phi' \equiv (O\Phi)^T (O\Phi) = \Phi^T O^T O \Phi = \Phi^T \Phi$, i.e. $O^T O = I$ (identity), which is fulfilled by any orthogonal $2N$–dimensional matrix. The theory therefore has a more general $O(2N)$ symmetry, and if we let γ^i be its independent generators, the Noether current will be

$$j_i^\mu \propto \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \gamma^i \Phi = (\partial_\mu \Phi)^T \gamma^i \Phi \ , \tag{1.23}$$

with an associated charge

$$Q^i = \int d^3x \ (\partial_0 \Phi)^T \gamma^i \Phi = \int d^3x \ \Pi^T \gamma^i \Phi \ . \tag{1.24}$$

The number of conserved charges is equal to the number of independent generators of $O(2N)$, which is $N(2N-1)$. We can see this works for $N = 2$, where the number of generators is indeed 6.

Problem 2: Lorentz transformations and Noether's theorem for scalar fields (continued)

The scalar field can be expanded in creation and annihilation operators as

$$\phi(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ a(\vec{p}) e^{-ip\cdot x} + a^\dagger(\vec{p}) e^{ip\cdot x} \right\} \ . \tag{2.1}$$

We will need only the expressions for $t = 0$, so we can simplify a bit by writing

$$\begin{aligned}\phi(\vec{x},0) &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left\{a(\vec{p})e^{i\vec{p}\cdot\vec{x}} + a^\dagger(\vec{p})e^{-i\vec{p}\cdot\vec{x}}\right\} \\ &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}e^{i\vec{p}\cdot\vec{x}}\left\{a(\vec{p}) + a^\dagger(-\vec{p})\right\} \ ,\end{aligned} \tag{2.2}$$

The first spatial derivative is then

$$\partial_j\phi(\vec{x},0) = i\int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{p^j}{\sqrt{2E_p}}e^{i\vec{p}\cdot\vec{x}}\left\{a(\vec{p}) + a^\dagger(-\vec{p})\right\} \ . \tag{2.3}$$

To find the first time derivative we have to start with Eq. (2.1) and then set $t = 0$ after differentiation:

$$\partial_0\phi(\vec{x},0) = -i\int \frac{\mathrm{d}^3 p}{(2\pi)^3}\sqrt{\frac{E_p}{2}}e^{i\vec{p}\cdot\vec{x}}\left\{a(\vec{p}) - a^\dagger(-\vec{p})\right\}. \tag{2.4}$$

The conserved quantity is the spatial integral of the time-component of the conserved current, or

$$M^{\lambda\sigma} \equiv \int \mathrm{d}^3 x\, j^{0\lambda\sigma} \ . \tag{2.5}$$

I will separately treat the jk and $0k$ components of this tensor. For the jk components, one can define

$$J^i = \frac{1}{2}\epsilon_{ijk}M^{jk}, \tag{2.6}$$

where ϵ_{ijk} is the totally antisymmetric tensor with $\epsilon_{123} = 1$, so

$$M^{ij} = \epsilon_{ijk}J^k \ . \tag{2.7}$$

Here I am treating $\vec{J}$ as a vector in a Euclidean 3-space, so there is no need to distinguish between upper and lower indices. Using the expression for the conserved current $j^{\mu\lambda\sigma}$, one has

$$J^i = \epsilon_{ijk}\int \mathrm{d}^3 x\, x^j T^{0k} \ , \tag{2.8}$$

where

$$T^{0k} = -\partial_0\phi\partial_k\phi \ . \tag{2.9}$$

There is a potential ordering problem here, since we derived Noether's theorem classically, but the two factors in Eq. (2.9) do not commute as operators. The commutator is certainly a c-number, however, so at worst it might produce an infinite constant, like the zero-point energy we found when we computed the energy density of the vacuum. In this case, however, we are computing a vector quantity $\vec{J}$, so rotational symmetry guarantees that the vacuum expectation value must vanish. If we want to see explicitly why the commutator of the two factors in Eq. (2.9) cannot contribute, we can write the commutator explicitly as

$$[\partial_0\phi(\vec{x},0)\,,\,\partial_k\phi(\vec{x},0)] = \frac{\partial}{\partial x^k}\left[\dot{\phi}(\vec{y},0)\,,\,\phi(\vec{x},0)\right]\Big|_{\vec{y}=\vec{x}} = -i\,\frac{\partial}{\partial x^k}\delta^3(\vec{x}-\vec{y})\Big|_{\vec{y}=\vec{x}}\ . \tag{2.10}$$

When this quantity is inserted into the integral of Eq. (2.8), the differential operator $\partial/\partial x^k$ is integrated by parts to give $\partial x^j/\partial x^k = \delta^j_k$, which then vanishes when contracted with ϵ_{ijk}.

In the course of evaluating the expansion of $M^{\lambda\sigma}$ in terms of creation and annihilation operators, we will find many occasions when c-number terms will arise from the commutation of a and $a^\dagger$. It is customary and useful to write these expressions in "normal-ordered" form, which means that the annihilation operators are written to the right of the creation operators, so that the vacuum expectation value of the operator expression is always zero. The normal ordering does not necessarily affect the value of the expression, since the c-numbers that arise from the commutators can be kept. The final expression is then the sum of a normal-ordered operator expression and a c-number, where the c-number can then be identified with the vacuum expectation value. Thus we can write

$$M^{\lambda\sigma} = M^{\lambda\sigma}_{\text{normal ordered}} + \left\langle 0\left|M^{\lambda\sigma}\right|0\right\rangle\ . \tag{2.11}$$

If the c-number contribution $\left\langle 0\left|M^{\lambda\sigma}\right|0\right\rangle$ is kept, it would be found to contain a number of ill-defined expressions, an example of which is $\int \mathrm{d}^3p\,\vec{p}\,\delta^3(0)$. While standard mathematics would say that such an expression is undefined, we can argue physically that it must vanish by rotational invariance — there is no preferred direction in which it could point. Since $M^{00} \equiv 0$, all the quantities involved will have one or two spatial indices; thus, if the theory is to be rotationally invariant, all the contributions to $\left\langle 0\left|M^{\lambda\sigma}\right|0\right\rangle$ must sum to zero. One way to be precise about this is to formulate the theory on a finite-sized cubic lattice, and then take the limit as the lattice size approaches infinity and the lattice spacing approaches zero. Although the lattice does not have full rotational symmetry, it is symmetric under rotations by 90° about any axis, and that is enough to prove that the vacuum expectation value of any vector or antisymmetric tensor quantity must vanish. For the rest of this calculation we will freely normal order the operators without keeping track of the c-number contributions, relying on the fact that rotational invariance will require that the sum of the c-number contributions must vanish.

Using Eqs. (2.3), (2.4), and (2.9) to evaluate Eq. (2.8), one has

$$J^i = -\frac{1}{2}\epsilon_{ijk}\int \mathrm{d}^3x\, x^j \int \frac{\mathrm{d}^3p}{(2\pi)^3}\int \frac{\mathrm{d}^3q}{(2\pi)^3}\sqrt{\frac{E_p}{E_q}}q^k e^{i(\vec{p}+\vec{q})\cdot\vec{x}} \\ \times \left\{a(\vec{p}) - a^\dagger(-\vec{p})\right\}\left\{a(\vec{q}) + a^\dagger(-\vec{q})\right\} \ . \tag{2.12}$$

The integral over $\vec{x}$ can then be carried out by using

$$\int \mathrm{d}^3x\, x^j\, e^{i(\vec{p}+\vec{q})\cdot\vec{x}} = -i\frac{\partial}{\partial p^j}\int \mathrm{d}^3x e^{i(\vec{p}+\vec{q})\cdot\vec{x}} = -(2\pi)^3 i\frac{\partial}{\partial p^j}\delta^3\left(\vec{p}+\vec{q}\right) \ . \tag{2.13}$$

Inserting (2.13) into (2.12), one uses the fact that the derivative of a delta function is defined by integration by parts. After integrating $\partial/\partial p^j$ by parts, the expression becomes

$$J^i = -\frac{i}{2}\epsilon_{ijk}\int \frac{\mathrm{d}^3p\,\mathrm{d}^3q}{(2\pi)^3}\delta^3\left(\vec{p}+\vec{q}\right)\frac{\partial}{\partial p^j}\left[\sqrt{\frac{E_p}{E_q}}q^k\left\{a(\vec{p}) - a^\dagger(-\vec{p})\right\}\left\{a(\vec{q}) + a^\dagger(-\vec{q})\right\}\right] \ . \tag{2.14}$$

When $\partial/\partial p^j$ acts on $\sqrt{E_p}$ it produces a term proportional to p^j, but $p^j q^k$ vanishes when the delta function is used and the antisymmetry of ϵ_{ijk} is taken into account. So the only terms that survive arise from the derivatives of the creation and annihilation operators:

$$J^i = \frac{i}{2}\epsilon_{ijk}\int \frac{\mathrm{d}^3p}{(2\pi)^3}p^k\left\{\frac{\partial a(\vec{p})}{\partial p^j} - \frac{\partial a^\dagger(-\vec{p})}{\partial p^j}\right\}\left\{a(-\vec{p}) + a^\dagger(\vec{p})\right\} \ . \tag{2.15}$$

By expanding the product of the two expressions in curly brackets, we will get four terms. To discuss them one at a time, I will label them as $J^i_{\pm\pm}$, where the first subscript indicates whether the first factor is a creation $(+)$ or annihilation $(-)$ operator, and the second subscript indicates whether the second factor is a creation or annihilation operator. Then

$$J^i_{--} = \frac{i}{2}\epsilon_{ijk}\int \frac{\mathrm{d}^3p}{(2\pi)^3}p^k\frac{\partial a(\vec{p})}{\partial p^j}a(-\vec{p}) \ . \tag{2.16}$$

It is helpful at this point to remember that we are expressing a conserved quantity, so we can check for consistency by putting in the time dependence that would be seen in the Heisenberg picture:

$$e^{iHt}a(\pm\vec{p})e^{-iHt} = a(\pm\vec{p})e^{-iE_pt} \ . \tag{2.17}$$

Thus J^i_{--} is proportional to $e^{-2iE_p t}$. Its time dependence will be of the form

$$J^i_{--} = [c_1 + c_2 t] e^{-2iE_p t} \ , \tag{2.18}$$

where the $c_2 t$ term arises when the derivative with respect to p^j in Eq. (2.16) acts on the time dependence of $a(\vec{p})e^{-iE_p t}$. By contrast, $J^i_{++} \propto e^{2iE_p t}$, while J^i_{+-} and J^i_{-+} will have no exponential time dependence. If this sum is to be constant, J^i_{--} and J^i_{++} must both vanish. To see how this happens explicitly, change variables of integration in Eq. (2.16) by substituting $\vec{q} = -\vec{p}$. Of course p^k becomes $-q^k$, and

$$\frac{\partial a(\vec{p})}{\partial p^j} = \frac{\partial q^\ell}{\partial p^j} \frac{\partial a(-\vec{q})}{\partial q^\ell} = -\delta^\ell_j \frac{\partial a(-\vec{q})}{\partial q^\ell} = -\frac{\partial a(-\vec{q})}{\partial q^j} \ . \tag{2.19}$$

So

$$J^i_{--} = \frac{i}{2} \epsilon_{ijk} \int \frac{\mathrm{d}^3 q}{(2\pi)^3} q^k \frac{\partial a(-\vec{q})}{\partial q^j} a(\vec{q}) \ . \tag{2.20}$$

Now if one integrates by parts, recognizing that the annihilation operators commute with each other and that the term proportional to $\partial q^k / \partial q^j = \delta^k_j$ vanishes due to the antisymmetry of ϵ_{ijk}, one finds an expression that is equal to the negative of the one that we started with, except that $\vec{p}$ has been replaced by $\vec{q}$. Thus the expression must vanish. The argument that $J^i_{++} = 0$ is completely analogous.

Eq. (2.17) implies that J^i_{+-} is time-independent, so presumably it makes a nontrivial contribution to the conserved angular momentum. From Eq. (2.15) one extracts

$$J^i_{+-} = -\frac{i}{2} \epsilon_{ijk} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} p^k \frac{\partial a^\dagger(-\vec{p})}{\partial p^j} a(-\vec{p}) \ . \tag{2.21}$$

This term could be left as it is, but it can be made to look a little simpler by making the following changes:

(1) Change variables of integration by $\vec{p} \to -\vec{p}$, so the arguments of the creation and annihilation operators become simply $\vec{p}$. This results in two canceling sign changes, since p^k and $\partial a^\dagger(-\vec{p})/\partial p^j$ (see Eq. (2.19)) change sign.

(2) Integrate by parts, so the derivative acts on the annihilation operator. Again there is a term proportional to δ^k_j which vanishes when contracted with ϵ_{ijk}. This step results in a change of sign.

(3) Interchange j and k in the integrand, just to restore alphabetical order. This results in a change of sign, canceling the change in step (2).

The result is

$$J^i_{+-} = -\frac{i}{2} \epsilon_{ijk} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} p^j a^\dagger(\vec{p}) \frac{\partial a(\vec{p})}{\partial p^k} \ . \tag{2.22}$$

Finally we look back at Eq. (2.15) and extract

$$J^i_{-+} = \frac{i}{2}\epsilon_{ijk}\int \frac{\mathrm{d}^3p}{(2\pi)^3} p^k \frac{\partial a(\vec{p})}{\partial p^j} a^\dagger(\vec{p}) \ . \tag{2.23}$$

Following the earlier discussion about normal ordering, we can interchange the order of the creation operator and the derivative of the annihilation operator. If we then also interchange j and k in the integrand, resulting in a change of sign, and the final result is identical to the expression for J^i_{+-}. The final answer is then twice J^i_{+-}, or

$$\boxed{J^i = -i\epsilon_{ijk}\int \frac{\mathrm{d}^3p}{(2\pi)^3} p^j a^\dagger(\vec{p}) \frac{\partial a(\vec{p})}{\partial p^k} \ .} \tag{2.24}$$

It is worth checking that Eq. (2.24) agrees with our recollections of nonrelativistic quantum mechanics in situations where the momenta are nonrelativistic. Recall that

$$\left[a(\vec{p})\, , \, a^\dagger(\vec{q})\right] = (2\pi)^3\delta^3(\vec{p}-\vec{q}) \ , \tag{2.25}$$

so one can construct normalized nonrelativistic momentum eigenstates by

$$|\vec{p}_{\mathrm{NR}}\rangle = \frac{1}{(2\pi)^{3/2}} a^\dagger(\vec{p})\, |0\rangle \ \ , \tag{2.26}$$

with

$$\langle \vec{q}_{\mathrm{NR}}\, |\vec{p}_{\mathrm{NR}}\, \rangle = \delta^3(\vec{q}-\vec{p}) \ . \tag{2.27}$$

For a one-particle nonrelativistic state $|\psi\rangle$, one can make contact with conventional quantum theory by defining

$$\psi(\vec{p}) \equiv \langle \vec{p}_{\mathrm{NR}}\, |\psi\, \rangle \ \ . \tag{2.28}$$

One can then see that

$$\langle 0\, | a(\vec{p}) |\, \psi\rangle = (2\pi)^{3/2}\, \langle \vec{p}_{\mathrm{NR}}\, |\psi\, \rangle = (2\pi)^{3/2}\psi(\vec{p}) \ , \tag{2.29}$$

and that for two nonrelativistic one-particle states $|\psi_1\rangle$ and $|\psi_2\rangle$,

$$\left\langle \psi_2 \left| a^\dagger(\vec{p}\,')a(\vec{p}) \right| \psi_1 \right\rangle = (2\pi)^3 \psi_2^*(\vec{p}\,')\psi_1(\vec{p}) \ . \tag{2.30}$$

Using Eq. (2.30) with Eq. (2.24), one has

$$\left\langle \psi_2 \left| J^i \right| \psi_1 \right\rangle = -i\epsilon_{ijk} \int \mathrm{d}^3p\, \psi_2^*(\vec{p})\, p^j\, \frac{\partial}{\partial p^k}\, \psi_1(\vec{p}) \ . \tag{2.31}$$

Remembering that the position operator in momentum space can be written as

$$\vec{r} = i\nabla_{\vec{p}} \ , \tag{2.32}$$

one can rewrite Eq. (2.31) as

$$\left\langle \psi_2 \middle| \vec{J} \middle| \psi_1 \right\rangle = \int \mathrm{d}^3 p \, \psi_2^*(\vec{p}) \, \vec{r} \times \vec{p} \, \psi_1(\vec{p}) \ , \tag{2.33}$$

which is exactly what we expect for an angular momentum operator.

Going back to Eq. (2.5), we will now express M^{0k} in terms of creation and annihilation operators. Using

$$j^{\mu\lambda\sigma} = x^\lambda T^{\mu\sigma} - x^\sigma T^{\mu\lambda} \tag{2.34}$$

from part (b), Eq. (2.5) implies that

$$M^{0k} \equiv K^k = \int \mathrm{d}^3 x \, j^{00k} = \int \mathrm{d}^3 x \, \left[T^{0k} t - \mathcal{H} \vec{x} \right] \ . \tag{2.35}$$

The first term is easy, since t factors out of the integral, which then becomes the integral for the total momentum. Using $T^{0k} = -\partial_0 \phi \partial_k \phi$, Eqs. (2.3) and (2.4) can be used to show that

$$\begin{aligned} P^k &\equiv \int \mathrm{d}^3 x \, T^{0k} \\ &= -\frac{1}{2} \int \mathrm{d}^3 x \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{\mathrm{d}^3 q}{(2\pi)^3} \sqrt{\frac{E_p}{E_q}} q^k e^{i(\vec{p}+\vec{q})\cdot\vec{x}} \\ &\qquad \times \left\{ a(\vec{p}) - a^\dagger(-\vec{p}) \right\} \left\{ a(\vec{q}) + a^\dagger(-\vec{q}) \right\} \\ &= \frac{1}{2} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} p^k \left\{ a(\vec{p}) - a^\dagger(-\vec{p}) \right\} \left\{ a(-\vec{p}) + a^\dagger(\vec{p}) \right\} \ . \end{aligned} \tag{2.36}$$

When the two factors on the right are expanded, the terms in either two annihilation operators or two creation operators lead to integrals which become the negative of their original form under a change of variable $\vec{p} \to -\vec{p}$, so these integrals vanish. We are left with

$$P^k = \frac{1}{2} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} p^k \left\{ a(\vec{p}) a^\dagger(\vec{p}) - a^\dagger(-\vec{p}) a(-\vec{p}) \right\} \ . \tag{2.37}$$

The second term can be rewritten by a change of integration variable $\vec{p} \to -\vec{p}$, which makes it equal to the first term up to ordering. (Note that the sign change

$p^k \to -p^k$ cancels the explicit minus sign in the equation.) The normal-ordering prescription allows us to change the order, so the two terms are equal. We therefore obtain

$$\boxed{P^k = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} p^k a^\dagger(\vec{p}) a(\vec{p}) \ ,} \tag{2.38}$$

as we would expect. When acting on a basis state $a^\dagger(\vec{p}_1)\, a^\dagger(\vec{p}_2)\, \ldots\, a^\dagger(\vec{p}_n)\, |0\rangle$, the integral in Eq. (2.38) adds up the momenta of each of the particles.

Returning to the second term of M^{0k}, as seen in Eq. (2.35), we begin by naming it G^k, so

$$\begin{aligned} \vec{G} &\equiv \int \mathrm{d}^3 x\, \vec{x}\, \mathcal{H} = \int \mathrm{d}^3 x\, \vec{x}\, T^{00} \\ &= \frac{1}{2} \int \mathrm{d}^3 x\, \vec{x} \left[\dot{\phi}^2 + \left(\vec{\nabla}\phi\right)^2 + m^2 \phi^2 \right] \ , \end{aligned} \tag{2.39}$$

where

$$M^{0k} = K^k = \vec{P}\, t - \vec{G} \ . \tag{2.40}$$

Using again the expansions of the scalar field from Eqs. (2.2)–(2.4) one has

$$\begin{aligned} G^k = \frac{1}{4} \int \mathrm{d}^3 x\, x^k \int & \frac{d^3 p}{(2\pi)^3} \frac{d^3 q}{(2\pi)^3} \frac{1}{\sqrt{E_p E_q}} e^{i(\vec{p}+\vec{q})\cdot\vec{x}} \\ & \times \left\{ -E_p E_q \left[a(\vec{p}) - a^\dagger(-\vec{p})\right] \left[a(\vec{q}) - a^\dagger(-\vec{q})\right] \right. \\ & \left. + \left(m^2 - \vec{p}\cdot\vec{q}\right) \left[a(\vec{p}) + a^\dagger(-\vec{p})\right] \left[a(\vec{q}) + a^\dagger(-\vec{q})\right] \right\} \ . \end{aligned} \tag{2.41}$$

Now use

$$\int \mathrm{d}^3 x\, x^k e^{i(\vec{p}+\vec{q})\cdot\vec{x}} = -i \frac{\partial}{\partial p^k} \int \mathrm{d}^3 x\, e^{i(\vec{p}+\vec{q})\cdot\vec{x}} = -(2\pi)^3 i \frac{\partial}{\partial p^k} \delta^3(\vec{p}+\vec{q}) \ . \tag{2.42}$$

Inserting this expression into Eq. (2.41) and then integrating by parts, the resulting expression can be conveniently divided into three pieces, $G^k = G^k_{(1)} + G^k_{(2)} + G^k_{(3)}$, depending on where the derivative acts:

$$\begin{aligned} G^k_{(1)} = \frac{i}{4} \int & \frac{d^3 p}{(2\pi)^3} \frac{d^3 q}{(2\pi)^3} \frac{\partial}{\partial p^k} \left(\frac{1}{\sqrt{E_p E_q}} \right) (2\pi)^3 \delta^3(\vec{p}+\vec{q}) \\ & \times \left\{ -E_p E_q \left[a(\vec{p}) - a^\dagger(-\vec{p})\right] \left[a(\vec{q}) - a^\dagger(-\vec{q})\right] \right. \\ & \left. + \left(m^2 - \vec{p}\cdot\vec{q}\right) \left[a(\vec{p}) + a^\dagger(-\vec{p})\right] \left[a(\vec{q}) + a^\dagger(-\vec{q})\right] \right\} \ , \end{aligned} \tag{2.43}$$

$$
\begin{aligned}
G^k_{(2)} = \frac{i}{4}\int \frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{E_pE_q}}(2\pi)^3\delta^3(\vec{p}+\vec{q}) \\
\times\left\{-\frac{\partial E_p}{\partial p^k}E_q\left[a(\vec{p})-a^\dagger(-\vec{p})\right]\left[a(\vec{q})-a^\dagger(-\vec{q})\right]\right. \\
\left.-q^k\left[a(\vec{p})+a^\dagger(-\vec{p})\right]\left[a(\vec{q})+a^\dagger(-\vec{q})\right]\right\} ,
\end{aligned}
\tag{2.44}
$$

and

$$
\begin{aligned}
G^k_{(3)} = \frac{i}{4}\int \frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{E_pE_q}}(2\pi)^3\delta^3(\vec{p}+\vec{q}) \\
\times\left\{-E_pE_q\left[\frac{\partial a(\vec{p})}{\partial p^k}-\frac{\partial a^\dagger(-\vec{p})}{\partial p^k}\right]\left[a(\vec{q})-a^\dagger(-\vec{q})\right]\right. \\
\left.+\left(m^2-\vec{p}\cdot\vec{q}\right)\left[\frac{\partial a(\vec{p})}{\partial p^k}+\frac{\partial a^\dagger(-\vec{p})}{\partial p^k}\right]\left[a(\vec{q})+a^\dagger(-\vec{q})\right]\right\} ,
\end{aligned}
\tag{2.45}
$$

Recalling that $\partial E_p/\partial p^k = p^k/E_p$, one can simplify $G^k_{(1)}$ as follows:

$$
\begin{aligned}
G^k_{(1)} &= -\frac{i}{8}\int \frac{d^3p}{(2\pi)^3}\frac{p^k}{E_p^3}\left\{-E_p^2\left[a(\vec{p})-a^\dagger(-\vec{p})\right]\left[a(-\vec{p})-a^\dagger(\vec{p})\right]\right. \\
&\quad \left.+E_p^2\left[a(\vec{p})+a^\dagger(-\vec{p})\right]\left[a(-\vec{p})+a^\dagger(\vec{p})\right]\right\} \\
&= -\frac{i}{4}\int \frac{d^3p}{(2\pi)^3}\frac{p^k}{E_p}\left[a^\dagger(\vec{p})a(\vec{p})+a^\dagger(-\vec{p})a(-\vec{p})\right] ,
\end{aligned}
\tag{2.46}
$$

where we have used the normal-ordering prescription to re-order the first term. By a change of integration variable $\vec{p}\to-\vec{p}$, the second term becomes the negative of the first term, and the result vanishes: $G^k_{(1)} = 0$.

One finds a similar behavior for $G^k_{(2)}$:

$$
\begin{aligned}
G^k_{(2)} &= \frac{i}{4}\int \frac{d^3p}{(2\pi)^3}\frac{1}{E_p}\left\{-p^k\left[a(\vec{p})-a^\dagger(-\vec{p})\right]\left[a(-\vec{p})-a^\dagger(\vec{p})\right]\right. \\
&\quad \left.+p^k\left[a(\vec{p})+a^\dagger(-\vec{p})\right]\left[a(-\vec{p})+a^\dagger(\vec{p})\right]\right\} \\
&= \frac{i}{2}\int \frac{d^3p}{(2\pi)^3}\frac{p^k}{E_p}\left[a^\dagger(\vec{p})a(\vec{p})+a^\dagger(-\vec{p})a(-\vec{p})\right] \\
&= 0 .
\end{aligned}
\tag{2.47}
$$

Finally, the answer must be given entirely by $G^k_{(3)}$. Starting from Eq. (2.45),

$$\begin{aligned}
G^k = G^k_{(3)} &= \frac{i}{4}\int \frac{\mathrm{d}^3p}{(2\pi)^3} E_p \left\{ - \left[\frac{\partial a(\vec{p})}{\partial p^k} - \frac{\partial a^\dagger(-\vec{p})}{\partial p^k}\right]\left[a(-\vec{p}) - a^\dagger(\vec{p})\right] \right. \\
&\qquad \left. + \left[\frac{\partial a(\vec{p})}{\partial p^k} + \frac{\partial a^\dagger(-\vec{p})}{\partial p^k}\right]\left[a(-\vec{p}) + a^\dagger(\vec{p})\right] \right\} \\
&= \frac{i}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3} E_p \left[\frac{\partial a(\vec{p})}{\partial p^k} a^\dagger(\vec{p}) + \frac{\partial a^\dagger(-\vec{p})}{\partial p^k} a(-\vec{p})\right] \\
&= \boxed{\frac{i}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3} E_p \left[a^\dagger(\vec{p})\frac{\partial a(\vec{p})}{\partial p^k} - \frac{\partial a^\dagger(\vec{p})}{\partial p^k} a(\vec{p})\right]\ ,}
\end{aligned} \tag{2.48}$$

where we have used the normal-ordering prescription in the last step.

Pulling together the results of Eqs. (2.38), (2.40), and (2.48), one has

$$\boxed{M^{0k} = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\left\{ tp^k a^\dagger(\vec{p})\, a(\vec{p}) - \frac{iE_p}{2}\left[a^\dagger(\vec{p})\frac{\partial a(\vec{p})}{\partial p^k} - \frac{\partial a^\dagger(\vec{p})}{\partial p^k} a(\vec{p})\right]\right\}\ .} \tag{2.49}$$

It is useful to check that when the full Heisenberg time-dependence of the operators is included, this operator is actually conserved. Using Eq. (2.17), one has

$$\begin{aligned}
M^{0k}(t) = \int \frac{\mathrm{d}^3p}{(2\pi)^3} &\left\{ tp^k a^\dagger(\vec{p})\, a(\vec{p}) \right. \\
&\left. - \frac{iE_p}{2}\left[a^\dagger(\vec{p})e^{iE_pt}\frac{\partial}{\partial p^k}\left(a(\vec{p})e^{-iE_pt}\right) - \frac{\partial}{\partial p^k}\left(a^\dagger(\vec{p})e^{iE_pt}\right) a(\vec{p})e^{-iE_pt}\right]\right\}\ .
\end{aligned} \tag{2.50}$$

Using $\partial E_p/\partial p^k = p^k/E_p$, the terms proportional to t cancel, leaving only the manifestly time-independent expression

$$M^{0k}(t) = -\frac{i}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\left[a^\dagger(\vec{p})\frac{\partial a(\vec{p})}{\partial p^k} - \frac{\partial a^\dagger(\vec{p})}{\partial p^k} a(\vec{p})\right]\ . \tag{2.51}$$

Problem 3: Lorentz transformations and Noether's theorem for the electromagetic potential $A_\mu(x)$

(a) We begin by stating Noether's theorem for the case of multiple fields:

The generic symmetry transformation for Noether's theorem can be written as

$$\phi_k(x) \longrightarrow \phi'_k(x) = \phi_k(x) + \alpha^b \Delta_b \phi_k(x) \;, \tag{3.1}$$

where the α^b represent a set of infinitesimal parameters. Here $\Delta_b \phi_k(x)$ is a complicated notation for a single quantity that represents the change in $\phi_k(x)$ induced by the transformation b. (Note: in lecture I used the symbol $\Delta_{b,k}\phi(x)$ for this quantity, but I think the notation that I am using here is clearer. Remember, however, that $\Delta_b \phi_k(x)$ can depend not only on ϕ_k, but on all of the fields and any derivativates of them.) If this transformation causes the Lagrangian density $\mathcal{L}$ to change by nothing more than the addition of a total derivative,

$$\mathcal{L}(x) \longrightarrow \mathcal{L}'(x) = \mathcal{L}(x) + \alpha^b \partial_\mu \mathcal{J}^\mu_b(x) \;, \tag{3.2}$$

then we say that the Lagrangian possesses a symmetry, and the current $\alpha^b j^\mu_b$, where

$$j^\mu_b(x) \equiv \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi_k\right)} \Delta_b \phi_k - \mathcal{J}^\mu_b \;, \tag{3.3}$$

is conserved:

$$\partial_\mu \left[\alpha^b j^\mu_b\right] = 0 \;. \tag{3.4}$$

If the α^b are linearly independent, then the j^μ_b currents are individually conserved:

$$\partial_\mu j^\mu_b = 0 \;, \quad \text{for each } b. \tag{3.5}$$

For this problem we replace ϕ_k by A_ν and α^b by $\Sigma^{\lambda\sigma}$. The transformation properties of $A_\nu(x)$ were stated on the problem set as Eq. (7), but I will restate them here, making for convenience a different choice of when to write indices as upper or lower:

$$A'_\nu(x) = A_\nu(x) + \Sigma^{\lambda\sigma}\left\{x_\sigma\, \partial_\lambda A_\nu(x) - \eta_{\nu\lambda}\, A_\sigma(x)\right\} \;. \tag{3.6}$$

To match Eq. (3.1), we rewrite this as

$$A'_\nu(x) = A_\nu(x) + \Sigma^{\lambda\sigma} \Delta_{\lambda\sigma} A_\nu(x) \;, \tag{3.7}$$

where

$$\Delta_{\lambda\sigma} A_\nu(x) = x_\sigma\, \partial_\lambda A_\nu(x) - \eta_{\nu\lambda}\, A_\sigma(x) \;. \tag{3.8}$$

The Lagrangian is a Lorentz scalar, so the only change it experiences in the transformation comes from the change of the argument:

$$\mathcal{L}'(x) = \mathcal{L}(x) + \Sigma^{\lambda\sigma} x_\sigma \partial_\lambda \mathcal{L}(x) = \mathcal{L}(x) + \Sigma^{\lambda\sigma} \partial_\mu \mathcal{J}^\mu_{\lambda\sigma} \;, \tag{3.9}$$

where

$$\mathscr{J}^{\mu}_{\lambda\sigma} = x_\sigma \delta^\mu_\lambda \mathscr{L}(x) \ . \tag{3.10}$$

You can work out the transformation of $\mathscr{L}$ by using the transformation (3.6) directly, and you should get the same answer. From $\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, one has

$$\frac{\partial \mathscr{L}}{\partial(\partial_\mu A_\nu)} = -F^{\mu\nu} \ , \tag{3.11}$$

so Noether's theorem guarantees that the following current (from Eqs. (3.3) and (3.4)) is conserved:

$$\begin{aligned} \alpha^b j^\mu_b &= \Sigma^{\lambda\sigma}\left\{\frac{\partial \mathscr{L}}{\partial(\partial_\mu A_\nu)}\Delta_{\lambda\sigma}A_\nu - \mathscr{J}^\mu_{\lambda\sigma}\right\} \\ &= \Sigma^{\lambda\sigma}\left\{-F^{\mu\nu}\left[x_\sigma\,\partial_\lambda A_\nu - \eta_{\nu\lambda}\,A_\sigma\right] - x_\sigma\,\delta^\mu_\lambda\,\mathscr{L}\right\} \ . \end{aligned} \tag{3.12}$$

Since $\Sigma^{\lambda\sigma}$ is antisymmetric, only the part of the quantity in curly brackets that is antisymmetric in λ and σ has to be conserved. Changing an overall sign to match the convention used in the problem set, we have derived the conservation of the current

$$\boxed{j^\mu{}_{\lambda\sigma} = \left\{x_\sigma\left[F^{\mu\nu}\,\partial_\lambda A_\nu + \delta^\mu_\lambda\,\mathscr{L}\right] - F^\mu{}_\lambda\,A_\sigma\right\} - \{\lambda \leftrightarrow \sigma\} \ .} \tag{3.13}$$

(b) Calculating the divergence of the current in Eq. (3.13),

$$\partial_\mu\, j^\mu{}_{\lambda\sigma} = \left\{F_{\sigma\nu}\,\partial_\lambda A^\nu + \eta_{\sigma\lambda}\,\mathscr{L} + x_\sigma\left[F^{\mu\nu}\,\partial_\lambda\partial_\mu A_\nu + \partial_\lambda\mathscr{L}\right] - F^\mu{}_\lambda\partial_\mu A_\sigma\right\} - \{\lambda \leftrightarrow \sigma\} \ . \tag{3.14}$$

where we used the equations of motion, $\partial_\mu F^{\mu\nu} = 0$. Note that the term in $\eta_{\sigma\lambda}$ vanishes when antisymmetrized. Looking at the sum of the first and last terms in curly brackets,

$$\begin{aligned} &\{F_{\sigma\nu}\,\partial_\lambda A^\nu - F^\mu{}_\lambda\partial_\mu A_\sigma\} - \{\lambda \leftrightarrow \sigma\} \\ &\qquad = \{-F_{\lambda\mu}\,\partial_\sigma A^\mu - F^\mu{}_\lambda\partial_\mu A_\sigma\} - \{\lambda \leftrightarrow \sigma\} \\ &\qquad = \{F^\mu{}_\lambda\,\partial_\sigma A_\mu - F^\mu{}_\lambda\partial_\mu A_\sigma\} - \{\lambda \leftrightarrow \sigma\} \\ &\qquad = \{F^\mu{}_\lambda\,F_{\sigma\mu}\} - \{\lambda \leftrightarrow \sigma\} = 0 \ , \end{aligned} \tag{3.15}$$

since $F^\mu{}_\lambda\,F_{\sigma\mu}$ is symmetric under $\lambda \leftrightarrow \sigma$. Finally, looking at the terms in square brackets in Eq. (3.14),

$$\begin{aligned} &F^{\mu\nu}\,\partial_\lambda\partial_\mu A_\nu + \partial_\lambda\mathscr{L} \\ &\qquad = \frac{1}{2}F^{\mu\nu}\,\partial_\lambda F_{\mu\nu} + \partial_\lambda\mathscr{L} \\ &\qquad = \frac{1}{4}\partial_\lambda\left(F^{\mu\nu}F_{\mu\nu}\right) + \partial_\lambda\mathscr{L} = 0 \ . \end{aligned} \tag{3.16}$$

Finally,

$$\boxed{\partial_\mu \, j^\mu{}_{\lambda\sigma} = 0 \ ,} \tag{3.17}$$

as expected.

(c) Given the equations from the problem set,

$$\hat{\jmath}^{\mu\lambda\sigma} = j^{\mu\lambda\sigma} + \partial_\kappa N^{\kappa\mu\lambda\sigma} \tag{3.18}$$

and

$$N^{\kappa\mu\lambda\sigma} = x^\lambda \, F^{\mu\kappa} \, A^\sigma - x^\sigma \, F^{\mu\kappa} \, A^\lambda \ , \tag{3.19}$$

we can begin by calculating the derivative needed in Eq. (3.18):

$$\partial_k \left[x^\lambda \, F^{\mu\kappa} \, A^\sigma - x^\sigma \, F^{\mu\kappa} \, A^\lambda \right] = \left\{ F^{\mu\lambda} \, A^\sigma + x^\lambda \, F^{\mu\kappa} \, \partial_\kappa A^\sigma \right\} - \{ \lambda \leftrightarrow \sigma \} \ , \tag{3.20}$$

where I have used the equations of motion, $\partial_\mu F^{\mu\nu} = 0$. Then using Eqs. (3.13) and (3.18),

$$\begin{aligned} \hat{\jmath}^\mu{}_{\lambda\sigma} = \Big\{ & x_\sigma \left[F^{\mu\nu} \, \partial_\lambda A_\nu + \delta^\mu_\lambda \, \mathcal{L} \right] - F^\mu{}_\lambda \, A_\sigma \\ & + F^\mu{}_\lambda \, A_\sigma + x_\lambda \, F^{\mu\kappa} \, \partial_\kappa A_\sigma \Big\} - \{ \lambda \leftrightarrow \sigma \} \ . \end{aligned} \tag{3.21}$$

The antisymmetry in λ and σ allows us to rewrite

$$x_\lambda \, F^{\mu\kappa} \, \partial_\kappa A_\sigma$$

as

$$-x_\sigma \, F^{\mu\nu} \, \partial_\nu A_\lambda \ ,$$

so Eq. (3.21) simplifies to

$$\hat{\jmath}^\mu{}_{\lambda\sigma} = \left\{ x_\sigma \left[F^{\mu\nu} \, F_{\lambda\nu} + \delta^\mu_\lambda \, \mathcal{L} \right] \right\} - \{ \lambda \leftrightarrow \sigma \} \ . \tag{3.22}$$

From the solutions to Problem 1(b) of Problem Set 1, the symmetric energy-momentum tensor for $A_\mu(x)$ is given by

$$\hat{T}^{\mu\nu} = F^{\mu\lambda} \, F_\lambda{}^\nu - g^{\mu\nu} \, \mathcal{L} \ . \tag{3.23}$$

So finally

$$\boxed{\hat{\jmath}^{\mu\lambda\sigma} = x^\lambda \, \hat{T}^{\mu\sigma} - x^\sigma \, \hat{T}^{\mu\lambda} \ ,} \tag{3.24}$$

which matches the result for scalar fields found in Problem Set 1, Problem 4.

Problem 4: Non-uniqueness of the harmonic oscillator quantization

(a) By definition:

$$x_2 = \frac{1}{\omega_0}(a^2 + a^{\dagger 2}) \ . \tag{4.1}$$

In the Heisenberg picture,

$$x_2(t) = e^{iHt} x_2 e^{-iHt} = \frac{1}{\omega_0}\left[e^{iHt} a e^{-iHt} e^{iHt} a e^{-iHt} + e^{iHt} a^\dagger e^{-iHt} e^{iHt} a^\dagger e^{-iHt}\right] \ . \tag{4.2}$$

Now

$$\begin{aligned} [H, a] &= \left[\omega_0\left(a^\dagger a + \frac{1}{2}\right), a\right] \\ &= \omega_0 [a^\dagger a, a] \\ &= \omega_0 [a^\dagger, a] a \\ &= -\omega_0 a \ . \end{aligned} \tag{4.3}$$

Therefore

$$\begin{aligned} Ha &= a(H - \omega_0) \\ \Rightarrow H^n a &= a(H - \omega_0)^n \\ \Rightarrow e^{iHt} a &= \sum_n \frac{1}{n!}(it)^n H^n a \\ &= a \sum_n \frac{1}{n!}[i(H - \omega_0)t]^n \\ &= a e^{i(H-\omega_0)t} \\ \Rightarrow e^{iHt} a e^{-iHt} &= a e^{-i\omega_0 t} \ . \end{aligned} \tag{4.4}$$

Similarly

$$e^{iHt} a^\dagger e^{-iHt} = a^\dagger e^{i\omega_0 t} \ . \tag{4.5}$$

Thus $x_2(t)$ becomes

$$\begin{aligned} x_2(t) &= \frac{1}{\omega_0}[a e^{-i\omega_0 t} a e^{-i\ \omega_0 t} + a^\dagger e^{i\omega_0 t} a^\dagger e^{i\omega_0 t}] \\ &= \boxed{\frac{1}{\omega_0}[a^2 e^{-2i\omega_0 t} + a^{\dagger 2} e^{2i\omega_0 t}] \ .} \end{aligned} \tag{4.6}$$

Hence,

$$\begin{aligned} \frac{dx_2}{dt} =& \frac{1}{\omega_0} 2i\omega_0[-a^2 e^{-2i\omega_0 t} + a^{\dagger 2} e^{2i\omega_0 t}] \\ \frac{d^2 x_2}{dt^2} =& \frac{1}{\omega_0}(-4\omega_0)^2[a^2 e^{-2i\omega_0 t} + a^{\dagger 2} e^{2i\omega_0 t}] \\ =& -(2\omega_0)^2 x_2 \ , \end{aligned} \tag{4.7}$$

which has the desired form, with $\omega = 2\omega_0$.

(b)

$$\begin{aligned} a^2 =& \left(\sqrt{\frac{\omega_0}{2}} q + \frac{i}{\sqrt{2\omega_0}} p\right)^2 \\ =& \frac{\omega_0}{2} q^2 + \frac{i}{2}(qp + pq) - \frac{1}{2\omega_0} p^2 \ . \\ \Rightarrow a^{\dagger 2} =& (a^2)^\dagger = \frac{\omega_0}{2} q^2 - \frac{i}{2}(qp + pq) - \frac{1}{2\omega_0} p^2 \ . \end{aligned} \tag{4.8}$$

So

$$\begin{aligned} x_2 =& \frac{1}{\omega_0}(a^2 + a^{\dagger 2}) \\ =& \frac{1}{\omega_0}\left(\omega_0 q^2 - \frac{1}{\omega_0} p^2\right) = \boxed{q^2 - \frac{p^2}{\omega_0^2} \ ,} \end{aligned} \tag{4.9}$$

as desired. This same equation clearly holds in the Heisenberg picture as well:

$$\boxed{x_2(t) = q(t)^2 - \frac{p(t)^2}{\omega_0^2} \ .} \tag{4.10}$$

So

$$\begin{aligned} \frac{dx_2(t)}{dt} =& 2q(t)\dot{q}(t) - \frac{2}{\omega_0^2} p(t)\dot{p}(t) \\ \frac{d^2 x_2(t)}{dt^2} =& 2\left[\dot{q}(t)^2 + q(t)\ddot{q}(t) - \frac{1}{\omega_0^2}\dot{p}(t)^2 - \frac{1}{\omega_0^2} p(t)\ddot{p}(t)\right] \ . \end{aligned} \tag{4.11}$$

From the equations of motion:

$$\begin{aligned} \dot{q} =& p \\ \dot{p} =& -\omega_0^2 q \\ \ddot{q} =& \dot{p} = -\omega_0^2 q \\ \ddot{p} =& -\omega_0^2 \dot{q} = -\omega_0^2 p \ . \end{aligned} \tag{4.12}$$

Hence:

$$\begin{aligned}\frac{d^2x_2(t)}{dt^2} =&2\left[p^2 + q(-\omega_0^2 q) - \frac{1}{\omega_0^2}(-\omega_0^2 q)^2 - \frac{1}{\omega_0^2}p(\omega_0^2 p)\right]\\ =&2[2p^2 - 2\omega_0^2 q^2]\\ =&-4\omega_0^2[q(t)^2 - \frac{p(t)^2}{\omega_0^2}\\ =&-4\omega_0^2 x(t)^2\ ,\end{aligned} \tag{4.13}$$

which is the same equation as before.

(c) Let $\boxed{x_3 \equiv a^3 + (a^\dagger)^3.}$ As before,

$$\begin{aligned}x_3(t) =&e^{iHt}x_3 e^{-iHt}\\ =&(ae^{-i\omega_0 t})^3 + (a^\dagger e^{i\omega_0 t})^3\\ =&\boxed{a^3 e^{-3i\omega_0 t} + a^{\dagger 3}e^{3i\omega_0 t}\ .}\\ \Rightarrow \frac{d^2x_3(t)}{dt^2} =&(-3i\omega_0)^2 a^2 e^{-3i\omega_0 t} + (3i\omega_0)^2(a^\dagger)^3 e^{3i\omega_0 t}\\ =&-(3\omega_0)^2(a^3 e^{-3i\omega_0 t} + (a^\dagger)^3 e^{3i\omega_0 t})\\ =&-(3\omega_0)^2 x_3(t)\ ,\end{aligned} \tag{4.14}$$

which has the desired form. Now let us find x_3 in terms of p and q. Let us write for brevity:

$$a =\tilde{q} + i\tilde{p}, \qquad \tilde{q} \equiv \sqrt{\frac{\omega_0}{2}}q, \quad \tilde{p} \equiv \frac{1}{\sqrt{2\omega_0}}p\ . \tag{4.15}$$

Then

$$x_3 = a^3 + (a^\dagger)^3 = (\tilde{q} + i\tilde{p})^3 + (\tilde{q} - i\tilde{p})^3\ . \tag{4.16}$$

Expanding we obtain

$$\begin{aligned}(\tilde{q} + i\tilde{p})^3 =&\tilde{q}^3 + i(\tilde{q}^2\tilde{p} + \tilde{q}\tilde{p}\tilde{q} + \tilde{p}\tilde{q}^2) - (\tilde{q}\tilde{p}^2 - \tilde{p}\tilde{q}\tilde{p} - \tilde{p}^2\tilde{q}) - i\tilde{p}^3\ ,\\ (\tilde{q} - i\tilde{p})^3 =&\tilde{q}^3 - i(\tilde{q}^2\tilde{p} + \tilde{q}\tilde{p}\tilde{q} + \tilde{p}\tilde{q}^2) - (\tilde{q}\tilde{p}^2 - \tilde{p}\tilde{q}\tilde{p} - \tilde{p}^2\tilde{q}) + i\tilde{p}^3\ .\end{aligned} \tag{4.17}$$

Thus

$$x_3 =2\tilde{q}^3 - 2(\tilde{q}\tilde{p}^2 + \tilde{p}\tilde{q}\tilde{p} + \tilde{p}^2\tilde{q})\ . \tag{4.18}$$

Now

$$\begin{aligned}\tilde{p}^2\tilde{q} =& \tilde{p}^2\tilde{q} - \tilde{p}\tilde{q}\tilde{p} + \tilde{p}\tilde{q}\tilde{p}\\ =& \tilde{p}([\tilde{p},\tilde{q}] + \tilde{q}\tilde{p}) \ ,\\ \tilde{q}\tilde{p}^2 =& \tilde{q}\tilde{p}^2 - \tilde{p}\tilde{q}\tilde{p} + \tilde{p}\tilde{q}\tilde{p}\\ =& ([\tilde{q},\tilde{p}] + \tilde{p}\tilde{q})\tilde{p} \ .\end{aligned} \tag{4.19}$$

Therefore

$$\begin{aligned}\tilde{q}\tilde{p}^2 + \tilde{p}\tilde{q}\tilde{p} + \tilde{p}^2\tilde{q} =& 3\tilde{p}\tilde{q}\tilde{p} + \tilde{p}([\tilde{p},\tilde{q}] + [\tilde{q},\tilde{p}])\\ =& 3\tilde{p}\tilde{q}\tilde{p} \ .\end{aligned} \tag{4.20}$$

Finally

$$\begin{aligned}x_3 =& 2\tilde{q}^3 - 6\tilde{p}\tilde{q}\tilde{p}\\ =& \boxed{\frac{\omega_0^{3/2}}{\sqrt{2}}q^3 - \frac{3}{\sqrt{2\omega_0}}pqp \ .}\end{aligned} \tag{4.21}$$

(d) Since N is diagonal in the same basis in which the Hamiltonian is diagonal, it follows that

$$[H\ ,\ N] = 0. \tag{4.22}$$

It follows immediately that

$$\begin{aligned}z_\lambda(t) =& \frac{1}{\sqrt{2\omega_0}}\left[e^{iHt}a(1+\lambda N)e^{-iHt} + e^{iHt}(1+\lambda N)a^\dagger e^{-iHt}\right]\\ =& \frac{1}{\sqrt{2\omega_0}}\left[e^{iHt}ae^{-iHt}e^{iHt}(1+\lambda N)e^{-iHt} + e^{iHt}(1+\lambda N)e^{-iHt}e^{iHt}a^\dagger e^{-iHt}\right]\\ =& \frac{1}{\sqrt{2\omega_0}}\left[e^{iHt}ae^{-iHt}(1+\lambda N) + (1+\lambda N)e^{iHt}a^\dagger e^{-iHt}\right]\\ =& \boxed{\frac{1}{\sqrt{2\omega_0}}\left[a(1+\lambda N)e^{-i\omega t} + (1+\lambda N)a^\dagger e^{i\omega t}\right]\ ,}\end{aligned} \tag{4.23}$$

Differentiating this expression,

$$\boxed{\frac{\mathrm{d}^2 z_\lambda}{\mathrm{d}t^2} = -\omega_0^2\, z_\lambda\ .} \tag{4.24}$$

Problem 5: Space-time translations of a_k

(a) We want to prove that

$$e^A B e^{-A} = B + [A,\, B] + \frac{1}{2}[A,\, [A,\, B]] + \frac{1}{3!}[A,\, [A,\, [A,\, B]]] + ... \tag{5.1}$$

For this, we first adopt a systematic definition of the terms that appear in the above formula. Let

$$C^{(0)}(A,\, B) \equiv B \ , \tag{5.2}$$

and then define inductively the sequence $C^{(n)}(A,\, B)$, by

$$C^{(n)}(A,\, B) \equiv \Big[A\,,\, C^{(n-1)}(A,\, B) \Big] \ . \tag{5.3}$$

We then have to show that

$$e^A B e^{-A} = \sum_{n=0}^{\infty} \frac{1}{n!} C^{(n)}(A,\, B) \ . \tag{5.4}$$

Now consider $F(t) = e^{tA} B e^{-tA}$, remembering that the quantity that we are trying to express is $F(1)$. We can try to write it as a power series in t by noting that

$$\begin{aligned} \frac{\mathrm{d}F}{\mathrm{d}t} &= A e^{tA} B e^{-tA} - e^{tA} B e^{-tA} A \\ &= e^{tA} \, [A\,,\, B] \, e^{-tA} \\ &= e^{tA} \, C^{(1)}(A,\, B) \, e^{-tA} \ . \end{aligned} \tag{5.5}$$

Iterating, one can see that each derivative with respect to t will result in taking the commutator of A with the middle of the three factors on the right, so in general

$$\frac{\mathrm{d}^n F}{\mathrm{d}t^n} = e^{tA} \, C^{(n)}(A,\, B) \, e^{-tA} \ . \tag{5.6}$$

The Taylor series then gives

$$F(1) = \sum_{n=0}^{\infty} \frac{1}{n!} \left. \frac{\mathrm{d}^n F}{\mathrm{d}t^n} \right|_{t=0} = \sum_{n=0}^{\infty} \frac{1}{n!} \, C^{(n)}(A,\, B) \ , \tag{5.7}$$

which completes the proof.

(b) Let us use the following 4-momentum operator:

$$\hat{P}^\mu = \int \frac{d^3 p}{(2\pi)^3} p^\mu a^\dagger_{\vec{p}} a_{\vec{p}} \,, \quad \text{where}\, p^0 = E_{\vec{p}} \,. \tag{5.8}$$

Now note that

$$[i(\hat{P}\cdot x),\, a_{\vec{k}}] = [i\int \frac{d^3p}{(2\pi)^3}(p\cdot x)a^\dagger_{\vec{p}}a_{\vec{p}},\, a_{\vec{k}}] = -i(k\cdot x)a_{\vec{k}}\,, \qquad (5.9)$$

and from this we see easily that

$$C^{(n)}(i(\hat{P}\cdot x),\, a_{\vec{k}}) = (-i(k\cdot x))^n a_{\vec{k}}\,. \qquad (5.10)$$

And therefore

$$\boxed{e^{i\hat{P}\cdot x}a_{\vec{k}}e^{-i\hat{P}\cdot x} = \sum_{n=0}^{\infty}\frac{(-i(k\cdot x))^n}{n!}a_{\vec{k}} = e^{-ik\cdot x}a_{\vec{k}}\ .} \qquad (5.11)$$