MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

8.323: Relativistic Quantum Field Theory I

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PROBLEM SET 3 SOLUTIONS

Problem 1: "Smeared" fields and their variance *(10 points)*

(a) Since $\langle\phi(\vec{x},t)\rangle = 0$, we get $\langle\tilde{\phi}_a(\vec{x},t)\rangle = 0$, and thus

$$\begin{aligned}\sigma^2 =&\langle 0|\tilde{\phi}_a(\vec{x},t)^2|0\rangle\\ =&\langle 0|\frac{1}{\pi^3 a^6}\int d^3y\, d^3z\,\phi(\vec{y},t)\phi(\vec{z},t)e^{-(|\vec{y}-\vec{x}|^2+|\vec{z}-\vec{x}|^2)/a^2}|0\rangle\\ =&\langle 0|\frac{1}{\pi^3 a^6}\int d^3y\, d^3z\,\phi(\vec{y}+\vec{x},t)\phi(\vec{z}+\vec{x},t)e^{-(|\vec{y}|^2+|\vec{z}|^2)/a^2}|0\rangle\\ =&\frac{1}{\pi^3 a^6}\int d^3y\, d^3z\int\frac{d^3p}{(2\pi)^3}\frac{1}{E_{\vec{p}}}\,e^{i\vec{p}\cdot(\vec{y}-\vec{z})}e^{-(|\vec{y}|^2+|\vec{z}|^2)/a^2}\end{aligned}$$

where in the last line we have used Peskin and Scroeder (2.50). To proceed we complete the square

$$i\vec{p}\cdot(\vec{y}-\vec{z})-(|\vec{y}|^2+|\vec{z}|^2)/a^2 = -(\vec{y}-ia^2\vec{p}/2)^2/a^2-(\vec{z}+ia^2\vec{p}/2)^2/a^2-|\vec{p}|^2a^2/2,$$

and use the formula for Gaussian integrals

$$\int d^3y\, e^{-(\vec{y}-ia^2\vec{p}/2)^2/a^2} = a^3\pi^{3/2}.$$

Whence,

$$\begin{aligned}\sigma^2 =&\int\frac{d^3p}{(2\pi)^3}\frac{1}{E_{\vec{p}}}a^6\pi^3e^{-|\vec{p}|^2a^2/2}\\ =&\frac{1}{4\pi^2}\int_0^\infty dp\,\frac{p^2e^{-p^2a^2/2}}{\sqrt{p^2+m^2}}.\end{aligned}$$

(b) To make the integral dimensionless we let $u = ap$ to obtain

$$\sigma^2 = \frac{1}{4\pi^2a^2}\int_0^\infty du\,\frac{u^2e^{-u^2/2}}{\sqrt{u^2+m^2a^2}}$$

In the small a limit

$$\begin{aligned}\sigma^2 &\approx \frac{1}{4\pi^2a^2}\int_0^\infty du\,ue^{-u^2/2}\\ &= \frac{1}{4\pi^2a^2}\\ &= \alpha a^\beta,\end{aligned}$$

where $\alpha = \frac{1}{4\pi^2}$ and $\beta = -2$. Note that σ^2 diverges as a goes to zero; the field is dominated by fluctuations in this limit.

In the large a limit

$$\begin{aligned}\sigma^2 &= \frac{1}{4\pi^2a^3}\int_0^\infty du\,\frac{u^2e^{-u^2/2}}{\sqrt{u^2/a^2+m^2}}\\ &\approx \frac{1}{4\pi^2a^3m}\int_0^\infty du\,u^2e^{-u^2/2}\\ &= \frac{1}{4\pi^2a^3m}\sqrt{\frac{\pi}{2}}\\ &= \alpha a^\beta,\end{aligned}$$

where $\alpha = \frac{1}{4\sqrt{2}\pi^{3/2}m}$ and $\beta = -3$. Note that σ^2 vanishes as a goes to infinity; the field behaves like a classical variable.

(c) Let $y = u/(ma)$ and let $s = m^2a^2$.

$$\sigma^2 = \frac{m^2}{4\pi^2}\int_0^\infty dy\,\frac{y^2e^{-sy^2/2}}{\sqrt{1+y^2}}$$

There are many ways to do this. We shall try to put it into the form of the 1st `BesselK` integral representation on the `functions.wolfram.com` website. Let $t = 2y^2+1$,

$$\begin{aligned}\sigma^2 =&\frac{m^2e^{s/4}}{16\pi^2}\int_1^\infty dt\,\left(\frac{t-1}{t+1}\right)^{1/2}e^{-st/4}\\ =&\frac{m^2e^{s/4}}{16\pi^2}\int_1^\infty dt\frac{(t-1)}{\sqrt{(t^2-1)}}e^{-st/4}\\ =&\frac{m^2e^{s/4}}{16\pi^2}\left(-\frac{d}{d(s/4)}K_0(s/4)-K_0(s/4)\right)\end{aligned}$$

where $K_n(z)$ is a modified Bessel function of the second kind. The recursion relation $K_n'(z) = nK_n(z)/z - K_{n+1}(z)$ gives,

$$\sigma^2 = \frac{m^2e^{m^2a^2/4}}{16\pi^2}\left(K_1(m^2a^2/4)-K_0(m^2a^2/4)\right)$$

Problem 2: Casimir Effect in One Dimension

(a) We start with

$$E_0(L) = \frac{1}{2}\sum_{n=1}^{\infty} \omega_n e^{-\omega_n/\omega_c} , \qquad \text{where } \omega_n = n\pi/L.$$

Defining $a \equiv \pi/(L\omega_c)$, this becomes

$$\begin{aligned} E_0(L) &= \frac{\pi}{2L}\sum_{n=1}^{\infty} n e^{-na} = -\frac{\pi}{2L}\frac{\partial}{\partial a}\sum_{n=1}^{\infty} e^{-na} = -\frac{\pi}{2L}\frac{\partial}{\partial a}\left\{e^{-a} + e^{-2a} + \cdots\right\} \\ &= -\frac{\pi}{2L}\frac{\partial}{\partial a}\left[\frac{e^{-a}}{1-e^{-a}}\right] = -\frac{\pi}{2L}\frac{\partial}{\partial a}\left[\frac{1}{e^a - 1}\right] = \frac{\pi}{2L}\frac{e^a}{(e^a-1)^2} \\ &= \frac{\pi}{2L}\frac{1}{(e^{a/2}-e^{-a/2})^2} = \frac{\pi}{2L}\frac{1}{[2\sinh(a/2)]^2} = \boxed{\frac{\pi}{8L\sinh^2\left(\dfrac{\pi}{2L\omega_c}\right)}} . \end{aligned}$$

For small x we have

$$\begin{aligned} \sinh x &\approx x + \frac{1}{6}x^3 + \mathcal{O}(x^5) \\ \sinh^2 x &\approx x^2 + \frac{1}{3}x^4 + \mathcal{O}(x^6) , \end{aligned}$$

so for large ω_c we obtain

$$\begin{aligned} E_0(L) &\approx \frac{\pi}{8L\left(\dfrac{\pi}{2L\omega_c}\right)^2\left[1 + \dfrac{1}{3}\left(\dfrac{\pi}{2L\omega_c}\right)^2 + \mathcal{O}\left(\dfrac{1}{L\omega_c}\right)^4\right]} \\ &= \frac{L\omega_c^2}{2\pi}\left[1 - \frac{1}{3}\left(\frac{\pi}{2L\omega_c}\right)^2 + \mathcal{O}\left(\frac{1}{L\omega_c}\right)^4\right] \\ &= \boxed{\frac{L\omega_c^2}{2\pi} - \frac{\pi}{24L} + \mathcal{O}\left(\frac{1}{L^3\omega_c^2}\right)} . \end{aligned}$$

(b) The total energy is

$$E^{\text{total}}(a) = E_0(a) + 2E_0\left(\frac{L-a}{2}\right) .$$

Using our result from part (a), we have

$$\begin{aligned} E^{\text{total}}(a) &= \frac{\omega_c^2}{2\pi}\Big[a + (L-a)\Big] - \frac{\pi}{24}\left[\frac{1}{a} + \frac{4}{L-a}\right] \\ &= \frac{\omega_c^2 L}{2\pi} - \frac{\pi}{24}\left[\frac{1}{a} + \frac{4}{L} + \mathcal{O}\left(\frac{a}{L^2}\right)\right] . \end{aligned}$$

Since the energy is lowered as a is decreased, the force is attractive. If a is increased by Δa, each of the two walls moves a distance $\frac{1}{2}\Delta a$, and each is moving against a force of magnitude F. Hence the total work done is $F \cdot \Delta a$, so $F = |\partial E^{\text{total}}(a)/\partial a|$. Then

$$\frac{\partial E^{\text{total}}(a)}{\partial a} = -\frac{\pi}{24}\left[-\frac{1}{a^2} + \mathcal{O}\left(\frac{1}{L^2}\right)\right] ,$$

so

$$\boxed{F = \frac{\pi}{24a^2}} .$$

Problem 3: The Casimir Effect in Electrodynamics

(a) You were not asked to derive the boundary conditions on $\vec{E}$ and $\vec{B}$, but for pedagogical purposes I will begin by reviewing this derivation. The key idea is that the conducting plates are treated as perfect conductors, so $\vec{E}$ must vanish inside the plates because otherwise there would be infinite currents. The rest comes from the *homogenous* Maxwell equations:

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} , \qquad \vec{\nabla} \cdot \vec{B} = 0 . \tag{3.1}$$

Note that the *inhomogeneous* Maxwell equations,

$$\vec{\nabla} \times \vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c}\vec{J} , \qquad \vec{\nabla} \cdot \vec{E} = 4\pi\rho , \tag{3.2}$$

do not constrain the boundary conditions on the fields, but instead determine the surface charges and surface currents. Since $\vec{E}$ must vanish in the conductor, the first of Eqs. (3.1) implies that $\vec{B}$ inside the conductor must be static. Since we are only interested in oscillating fields, we can set $\vec{B} = 0$ inside the conductor. To continue, adopt a coordinate system so that the plate boundary is at $z = 0$, with the conductor filling the region $z < 0$. Then $\vec{\nabla} \cdot \vec{B} = 0$ implies that

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0 . \tag{3.3}$$

But we assume that all the functions are continuous in x and y, so any discontinuity in B_z at the surface would produce a δ-function from the $\partial B_z/\partial z$ term which could not be canceled by the other terms. Thus B_z must vanish just outside the plate, so we write the boundary condition

$$B_\perp = 0 \ . \tag{3.4}$$

Here $\perp$ indicates the component perpendicular to the surface. Similarly the $\vec\nabla \times \vec E$ equation implies that

$$\begin{aligned} \frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y} &= -\frac{1}{c}\frac{\partial B_x}{\partial t} \\ \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} &= \frac{1}{c}\frac{\partial B_y}{\partial t} \ , \end{aligned} \tag{3.5}$$

which by similar arguments implies that E_x and E_y must be continuous at the boundary. Thus, the boundary condition on $\vec E$ is that

$$\vec E_\parallel = 0 \ , \tag{3.6}$$

where $\parallel$ indicates the components parallel to the surface.

The boundary conditions for $\vec A$ can be found by using the relation between $\vec A$ and the fields $\vec E$ and $\vec B$:

$$\vec E = -\vec\nabla\phi - \frac{1}{c}\frac{\partial \vec A}{\partial t} \ , \qquad \vec B = \vec\nabla \times \vec A \ , \tag{3.7}$$

where $\phi(x) \equiv A^0(x)$. In Coulomb gauge $\vec\nabla\cdot\vec A = 0$ and $\phi = 0$. Then $\vec E = -(1/c)\partial\vec A/\partial t$, so for sinusoidally varying fields,

$$\vec E_\parallel = 0 \quad \Longrightarrow \quad \boxed{\vec A_\parallel = 0 \ .} \tag{3.8}$$

Then the gauge condition

$$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 0 \tag{3.9}$$

implies the boundary condition

$$\frac{\partial A_z}{\partial z} = \boxed{\frac{\partial}{\partial n} A_\perp = 0 \ ,} \tag{3.10}$$

since Eq. (3.8) implies that A_x and A_y vanish at the boundary.

(b) Now consider a conducting box of size $L_x \times L_y \times L_z$, where I have changed the notation from the original $a \times b \times c$ to avoid confusion between the length of the box and the speed of light. Let one corner be at the origin, so the box includes the region

$$\begin{aligned} 0 \le x &\le L_x \\ 0 \le y &\le L_y \\ 0 \le z &\le L_z \ . \end{aligned} \tag{3.11}$$

In Coulomb gauge, the free space Maxwell equations become

$$\nabla^2 \vec A - \frac{1}{c^2}\frac{\partial^2 \vec A}{\partial t^2} = 0 \ , \tag{3.12}$$

so plane wave solutions have the form

$$\vec A(\vec x, t) = \vec\epsilon\, e^{i(\vec k\cdot\vec x - \omega t)} \ , \tag{3.13}$$

where the equations of motion (3.12) imply

$$\omega = c|\vec k| \ , \tag{3.14}$$

and the gauge condition $\vec\nabla\cdot\vec A$ implies that

$$\vec\epsilon\cdot\vec k = 0. \tag{3.15}$$

Fourier analysis guarantees us that the plane wave solutions are complete.

We are interested in standing wave modes, which can always be regarded as superpositions of modes of the form (3.13) with wave vectors $\vec k$ and $-\vec k$. Consider first the behavior of $A_x(x,y,z,t)$. The boundary conditions imply that A_x must vanish at $y = 0$ and $y = L_y$, and at $z = 0$ and $z = L_z$, and that its x-derivative must vanish at $x = 0$ and $x = L_x$. The boundary condition at $x = 0$ can be satisfied by using only cosine factors in the expansion, while the boundary conditions at $y = 0$ and $z = 0$ can be satisfied by using only sine functions. So, a general term in the Fourier sum will have the form

$$A_x = \epsilon_x(\vec k)\cos(k_x x)\sin(k_y y)\sin(k_z z) \ . \tag{3.16}$$

To also match the boundary conditions at $x = L_x$, $y = L_y$, and $z = L_z$, we must insist that

$$k_x = \frac{\pi n_x}{L_x} \ , \quad k_y = \frac{\pi n_y}{L_y} \ , \quad k_z = \frac{\pi n_z}{L_z} \ , \quad \text{where } n_i = 0, 1, 2, \ldots \ . \tag{3.17}$$

The completeness of Fourier sums* then guarantees that an arbitrary function of x, y, and z which obeys boundary conditions specified for A_x can always be written as

$$A_x(x,y,z) = \sum_{n_x=0}^{\infty}\sum_{n_y=0}^{\infty}\sum_{n_z=0}^{\infty} \epsilon_x(\vec{k}\,)\cos(k_x x)\sin(k_y y)\sin(k_z z) \ , \tag{3.18a}$$

where k_i and n_i are related by Eq. (3.17). A_y and A_z can be expanded similarly, giving

$$A_y(x,y,z) = \sum_{n_x=0}^{\infty}\sum_{n_y=0}^{\infty}\sum_{n_z=0}^{\infty} \epsilon_y(\vec{k}\,)\sin(k_x x)\cos(k_y y)\sin(k_z z) \ , \tag{3.18b}$$

$$A_z(x,y,z) = \sum_{n_x=0}^{\infty}\sum_{n_y=0}^{\infty}\sum_{n_z=0}^{\infty} \epsilon_z(\vec{k}\,)\sin(k_x x)\sin(k_y y)\cos(k_z z) \ . \tag{3.18c}$$

The gauge condition is again given by Eq. (3.15). If all three n_i's are nonzero, the gauge condition reduces the number of independent solutions from 3 to 2. That is, only 2 of the coefficients $\epsilon_i(\vec{k}\,)$ can be chosen independently. To see what happens if only two n_i's are nonzero, let $n_z = 0$, with n_x and n_y nonzero. Then $A_x = A_y = 0$, since both are proportional to $\sin(k_z z)$, so there is only one independent solution, proportional to $\epsilon_z(\vec{k}\,)$. If only one n_i is nonzero, then there are no solutions, as A_x, A_y, and A_z all vanish identically.

The most general time-dependent solution consistent with the equations of motion (3.12) is then given by

$$\vec{A}(\vec{x},t) = \mathrm{Re}\left\{ \sum_{n_x=0}^{\infty}\sum_{n_y=0}^{\infty}\sum_{n_z=0}^{\infty} \vec{\epsilon}(\vec{k}\,)\cdot\vec{W}(\vec{k},\vec{x}\,)\,e^{-i\omega(\vec{k}\,)t} \right\} \ , \tag{3.19}$$

where $\vec{\epsilon}(\vec{k}\,)$ is a complex vector satisfying the gauge condition (3.15), $\omega(\vec{k}\,) = c|\vec{k}\,|$, and

$$\begin{aligned} W_x(\vec{k},\vec{x}\,) &\equiv \cos(k_x x)\sin(k_y y)\sin(k_z z) \ , \\ W_y(\vec{k},\vec{x}\,) &\equiv \sin(k_x x)\cos(k_y y)\sin(k_z z) \ , \\ W_z(\vec{k},\vec{x}\,) &\equiv \sin(k_x x)\sin(k_y y)\cos(k_z z) \ . \end{aligned} \tag{3.20}$$

* If you are familiar with the completeness of Fourier series for a single variable, note that the completeness generalizes very easily to functions of several variables. Start by considering y and z to be fixed, and Fourier expand the resulting function of x. Since y and z were treated as fixed parameters, each Fourier coefficient will be a function of y and z. Then consider z alone to be fixed, and expand each of these Fourier coefficients as a function of y. Each Fourier coefficient in this sum will then be a function of z, which in turn can be Fourier-expanded.

We can now express the zero-point energy of the system by assigning an energy $\frac{1}{2}\hbar\omega$ to each harmonic oscillator in the system described by Eq. (3.19). For each triplet $\{n_i\}$ for which all three are nonzero, we found that the contribution to the static configuration of Eqs. (3.18) is specified by two real numbers. For the time-dependent solution of Eq. (3.19), however, the contribution is specified by two complex numbers, or equivalently four real numbers. So, does it count as two harmonic oscillators, or four? The answer is two, since the time-dependent solution for a single harmonic oscillator is specified by two real numbers: for example, its initial coordinate and its initial time derivative. Thus there are two harmonic oscillator modes for each triplet $\{n_i\}$ for which all three n_i are nonzero, and one for each triplet for which only two are nonzero.

The sum over modes will be divergent, so we put in a convergence factor $F(|\vec{k}\,|)$, which is 1 for $|\vec{k}|$ below some cutoff and goes to zero rapidly for large k. Although the sum over modes is divergent, we will find that the *change* in the energy when the plates are moved is finite.

Using the convergence factor $F(|\vec{k}\,|)$, the sum over zero-point energies can be written

$$\begin{aligned} E_0(L_x,L_y,L_z) = \frac{\hbar c}{2}\Bigg\{ &\sum_{n_x,n_y=1}^{\infty} \sqrt{k_x^2+k_y^2}\,F\left(\sqrt{k_x^2+k_y^2}\right) \\ &+ \sum_{n_y,n_z=1}^{\infty} \sqrt{k_y^2+k_z^2}\,F\left(\sqrt{k_y^2+k_z^2}\right) + \sum_{n_x,n_z=1}^{\infty} \sqrt{k_x^2+k_z^2}\,F\left(\sqrt{k_x^2+k_z^2}\right) \\ &+ 2\sum_{n_x,n_y,n_z=1}^{\infty} \sqrt{k_x^2+k_y^2+k_z^2}\,F\left(\sqrt{k_x^2+k_y^2+k_z^2}\right) \Bigg\} \ , \end{aligned} \tag{3.21}$$

where k_i is related to n_i by Eq. (3.17).

(c) We approximate as follows:

$$\sum_{n_x=1}^{\infty} \approx \int_0^{\infty} \mathrm{d}n_x = \frac{L_x}{\pi}\int_0^{\infty}\mathrm{d}k_x \ , \qquad \sum_{n_y=1}^{\infty} \approx \int_0^{\infty} \mathrm{d}n_y = \frac{L_y}{\pi}\int_0^{\infty}\mathrm{d}k_y \ . \tag{3.22}$$

Then we can rewrite Eq. (3.21) as

$$E_0(L_x,L_y,L_z) = \big(T_1+T_2+T_3+T_4\big)\hbar c \ , \tag{3.23}$$

where

$$T_1 = \frac{L_x L_y}{2\pi^2} \int_0^\infty \mathrm{d}k_x \int_0^\infty dk_y \sqrt{k_x^2 + k_y^2}\, F\left(\sqrt{k_x^2 + k_y^2}\right)$$

$$T_2 = \frac{L_y}{2\pi} \sum_{n_z=1}^{\infty} \int_0^\infty \mathrm{d}k_y \sqrt{k_y^2 + \left(\frac{n_z \pi}{L_z}\right)^2}\, F\left(\sqrt{k_y^2 + \left(\frac{n_z \pi}{L_z}\right)^2}\right)$$

$$T_3 = \frac{L_x}{2\pi} \sum_{n_z=1}^{\infty} \int_0^\infty \mathrm{d}k_x \sqrt{k_x^2 + \left(\frac{n_z \pi}{L_z}\right)^2}\, F\left(\sqrt{k_x^2 + \left(\frac{n_z \pi}{L_z}\right)^2}\right)$$

$$T_4 = \frac{L_x L_y}{\pi^2} \sum_{n_z=1}^{\infty} \int_0^\infty \mathrm{d}k_x \int_0^\infty \mathrm{d}k_y \sqrt{k_x^2 + k_y^2 + \left(\frac{n_z \pi}{L_z}\right)^2}\, F\left(\sqrt{k_x^2 + k_y^2 + \left(\frac{n_z \pi}{L_z}\right)^2}\right) . \tag{3.24}$$

From now on we will set $L_x = L_y = L$, and $L_z = a$, as specified in the problem.

To convert to polar coordinates in the first integral, we first notice that the integrand is even in k_x and k_y, so we may extend the domain of both integrals to cover the range $-\infty$ to ∞ and compensate by dividing by 4. Converting the resulting integral to polar coordinates, we obtain

$$T_1 = \frac{L^2}{4\pi} \int_0^\infty \mathrm{d}k\, k^2 F(k) . \tag{3.25}$$

The terms T_2 and T_3 are each proportional to only one power of L, compared to two powers for T_1 and T_4, so they can be neglected in the limit of large L. Converting to polar coordinates in the fourth term as before, we obtain

$$T_4 = \frac{L^2}{2\pi} \sum_{n_z=1}^{\infty} \int_0^\infty \mathrm{d}k\, k \sqrt{k^2 + \left(\frac{n_z \pi}{a}\right)^2}\, F\left(\sqrt{k^2 + \left(\frac{n_z \pi}{a}\right)^2}\right) . \tag{3.26}$$

Changing variables of integration to

$$u = \left(\frac{ak}{\pi}\right)^2 + n_z^2 , \tag{3.27}$$

so $k\, \mathrm{d}k = (\pi^2/2L_z^2)\, du$, we find

$$T_4 = \frac{\pi^2 L^2}{4a^3} \sum_{n_z=1}^{\infty} \int_{n_z^2}^\infty \mathrm{d}u\, \sqrt{u}\, F\left(\frac{\pi}{a}\sqrt{u}\right) . \tag{3.28}$$

Putting the terms together in Eq. (3.23), one has

$$E_0(L, L, a) = \frac{\hbar c L^2}{4\pi} \int_0^\infty \mathrm{d}k\, k^2 F(k) + \frac{\hbar c \pi^2 L^2}{4a^3} \sum_{n_z=1}^{\infty} \int_{n_z^2}^\infty \mathrm{d}u\, \sqrt{u}\, F\left(\frac{\pi}{a}\sqrt{u}\right) . \tag{3.29}$$

If we define

$$G(n) \equiv \int_{n^2}^\infty \mathrm{d}u\, \sqrt{u}\, F\left(\frac{\pi}{a}\sqrt{u}\right) , \tag{3.30}$$

then Eq. (3.29) can be rewritten as

$$\boxed{E_0(L, L, a) = \frac{\hbar c L^2}{4\pi} \int_0^\infty \mathrm{d}k\, k^2 F(k) + \frac{\hbar c \pi^2 L^2}{4a^3} \sum_{n=1}^{\infty} G(n) ,} \tag{3.31}$$

or even more compactly as

$$\boxed{E_0(L, L, a) = \frac{\hbar c \pi^2 L^2}{4a^3} \left\{ \frac{1}{2} G(0) + \sum_{n=1}^{\infty} G(n) \right\} .} \tag{3.32}$$

Note that it was apparent from Eqs. (3.24) that T_1 is equal to half the $n_z = 0$ term from T_4, so the simplification in Eq. (3.32) is not surprising.

(d) To use the Euler-Maclaurin formula, we need to evaluate the terms on the right-hand side. From Eq. (3.30), we see that

$$G'(n) = -2n^2 F\left(\frac{n\pi}{a}\right) . \tag{3.33}$$

To calculate further derivatives near $n = 0$ we observe that $F(n) \equiv 1$ in the vicinity of $n = 0$, so we may set this factor equal to one and compute

$$\begin{aligned} G''(n \approx 0) &= -4n \\ G'''(n \approx 0) &= -4 . \end{aligned} \tag{3.34}$$

Hence, at $n = 0$, we have

$$G' = G'' = 0 , \qquad G''' = -4 , \tag{3.35}$$

with all higher derivatives vanishing.

To compute $\int_0^\infty \mathrm{d}nG(n)$, it is easiest to rewrite $G(n)$ of Eq. (3.30) by undoing the change of variables (3.27), and undoing the change to polar coordinates introduced in Eq. (3.26):

$$\begin{aligned} G(n) &= \frac{2a^3}{\pi^3}\int_0^\infty \mathrm{d}k\, k\sqrt{k^2+\left(\frac{n\pi}{a}\right)^2}\, F\left(\sqrt{k^2+\left(\frac{n\pi}{a}\right)^2}\right) \\ &= \frac{a^3}{\pi^4}\int_{-\infty}^\infty \mathrm{d}k_x \int_{-\infty}^\infty \mathrm{d}k_y \sqrt{k_x^2+k_y^2+\left(\frac{n\pi}{a}\right)^2}\, F\left(\sqrt{k_x^2+k_y^2+\left(\frac{n\pi}{a}\right)^2}\right) . \end{aligned} \tag{3.36}$$

Then using the substitution $k_z = n\pi/a$,

$$\begin{aligned} \int_0^\infty \mathrm{d}n\, G(n) &= \frac{a}{\pi}\int_0^\infty \mathrm{d}k_z\, G(ak_z/\pi) \\ &= \frac{a^4}{\pi^5}\int_0^\infty \mathrm{d}k_z \int_{-\infty}^\infty \mathrm{d}k_x \int_{-\infty}^\infty \mathrm{d}k_y \sqrt{k_x^2+k_y^2+k_z^2}\, F\left(\sqrt{k_x^2+k_y^2+k_z^2}\right) \\ &= \frac{a^4}{2\pi^5}\int_{-\infty}^\infty \mathrm{d}k_z \int_{-\infty}^\infty \mathrm{d}k_x \int_{-\infty}^\infty \mathrm{d}k_y\, |\vec{k}|\, F\left(|\vec{k}|\right) \\ &= \frac{2a^4}{\pi^4}\int_0^\infty \mathrm{d}k\, k^3\, F(k) \quad . \end{aligned} \tag{3.37}$$

(It is not necessary to simplify $\int_0^\infty \mathrm{d}nG(n)$ this much, but it is necessary to determine that it is proportional to a^4.)

Thus, the Euler-Maclaurin summation formula tells us that

$$\frac{1}{2}G(0) + \sum_{n=1}^\infty G(n) = \int_0^\infty \mathrm{d}n\, G(n) - \frac{1}{12}G'(0) + \frac{1}{720}G'''(0) + \ldots \quad , \tag{3.38}$$

which in this case gives

$$\frac{1}{2}G(0) + \sum_{n=1}^\infty G(n) = \frac{2a^4}{\pi^4}\int_0^\infty \mathrm{d}k\, k^3\, F(k) - \frac{1}{180} \qquad \text{(exactly)} , \tag{3.39}$$

where the formula is exact because all higher derivatives of $G(n)$ vanish. Using this expression in Eq. (3.32), one finds

$$\begin{aligned} E_0(L,L,a) &= \frac{\hbar c\pi^2 L^2}{4a^3}\left\{\frac{2a^4}{\pi^4}\int_0^\infty \mathrm{d}k\, k^3\, F(k) - \frac{1}{180}\right\} \\ &= \boxed{L^2\left\{\frac{\hbar ca}{2\pi^2}\int_0^\infty \mathrm{d}k\, k^3\, F(k) - \frac{\hbar c\pi^2}{720a^3}\right\} \ .} \end{aligned} \tag{3.40}$$

This has the desired form, with

$$c_1 = \frac{\hbar c}{2\pi^2}\int_0^\infty \mathrm{d}k\, k^3\, F(k) \ , \qquad c_2 = 0 \ . \tag{3.41}$$

(e) The total energy for the three regions is given by

$$\begin{aligned} E_0^{\rm total}(a) &= E_0(L,L,a) + 2E_0\left(L,L,\frac{L-a}{2}\right) \\ &= -\frac{\hbar c\pi^2 L^2}{720}\left[\frac{1}{a^3} + \frac{16}{(L-a)^3}\right] + \frac{\hbar cL^3}{2\pi^2}\int_0^\infty \mathrm{d}k\, k^3\, F(k) \ . \end{aligned} \tag{3.42}$$

Note that the final, infinite term can be written

$$\frac{\hbar cL^3}{2\pi^2}\int_0^\infty \mathrm{d}k\, k^3\, F(k) = 2L^3\int \frac{\mathrm{d}^3k}{(2\pi)^3}\,\frac{1}{2}\hbar ck\, F(k) \ , \tag{3.43}$$

where the factor of 2 in front is associated with the two polarization states for each photon. In this form one can see immediately the sum over modes, with a contribution of $\frac{1}{2}\hbar\omega$ for each mode. The force is given only by the finite term. It is attractive, since the energy is lowered as a is decreased. As in Problem 1, there are two plates which each move a distance $\frac{1}{2}\,\Delta a$, so $|F| = |\partial E_0^{\rm total}(a)/\partial a|$. Thus

$$P = \frac{|F|}{L^2} = -\frac{\hbar c\pi^2}{720}\frac{\partial}{\partial a}\left[\frac{1}{a^3} + \frac{16}{L^3} + \mathcal{O}\left(\frac{a}{L^4}\right)\right] = \boxed{\frac{\hbar c\pi^2}{240a^4} \ .} \tag{3.44}$$

Inserting numbers,

$$P(a) = \frac{\pi^2\cdot(1.055\cdot 10^{-34}\ \mathrm{kg\ m^2\ s^{-1}})(2.99792\cdot 10^8\ \mathrm{m\ s^{-1}})}{240(10^{-24}\ \mathrm{m}^4\cdot\tilde{a}^4)} \ , \tag{3.45}$$

where

$$\tilde{a} = \frac{a}{1\ \mu\mathrm{m}} \tag{3.46}$$

is the plate separation in μm (10^{-6} m). Then

$$\begin{aligned} P(a) &= \frac{.0013}{\tilde{a}^4}\ \mathrm{kg\ m^{-1}\ s^{-2}} \\ &= \frac{.0013}{\tilde{a}^4}\ \mathrm{N/m^2} \\ &= \boxed{\frac{.013}{\tilde{a}^4}\ \mathrm{dyne/cm^2} \ .} \end{aligned} \tag{3.47}$$