MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

8.323: Relativistic Quantum Field Theory I

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PROBLEM SET 4 SOLUTIONS

Problem 1: Evaluation of $\langle 0|\phi(x)\phi(y)|0\rangle$ for spacelike separations *(10 points)*

When $x-y$ is spacelike, we can always find a frame in which it is purely spatial. In that case Peskin and Schroeder show, on p. 27, that the two-point function $D(x-y) \equiv \langle 0\,|\phi(x)\,\phi(y)|\,0\rangle$ can be written as

$$D(x-y) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_p}e^{-ip\cdot(x-y)} \tag{1.1a}$$

$$= \frac{-i}{2(2\pi)^2 r}\int_{-\infty}^{\infty}\mathrm{d}p\,\frac{p\,e^{ipr}}{\sqrt{p^2+m^2}}\ , \tag{1.1b}$$

where $(x-y)^2 = -r^2$. This integral, however, is not convergent, so it has no real definition as a function of r. The famous treatise on integrals by Gradshteyn and Ryzhik* explains at the beginning of the section on definite integrals that

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$$

is defined by

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \lim_{\substack{P\to-\infty\\ Q\to\infty}}\int_P^Q f(x)\,\mathrm{d}x\ , \tag{1.2}$$

and is said to converge only if the limit exists for the independent approach of P and Q to $\pm\infty$. If the limit

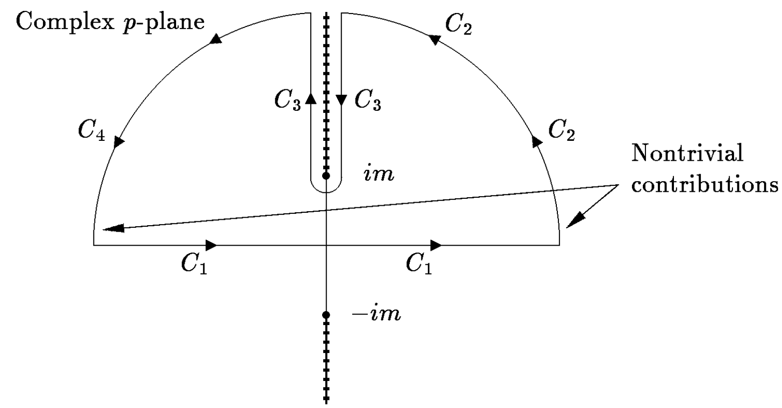
$$\lim_{P\to\infty}\int_{-P}^{+P} f(x)\,\mathrm{d}x \tag{1.3}$$

exists but the previous limit does not, then (1.3) is called the principal value of the improper integral. In this case none of these limits exist, but Eq. (1.1) is nonetheless well-defined as a distribution. Peskin and Schroeder perform a distortion of the integration contour in the complex plane to rewrite it as

$$D(x-y) = \frac{1}{4\pi^2 r}\int_m^{\infty}\mathrm{d}\rho\frac{\rho e^{-\rho r}}{\sqrt{\rho^2-m^2}}\ , \tag{1.4}$$

* *Table of Integrals, Series, and Products,* by I.S. Gradshteyn and I.M. Ryzhik, edited by Alan Jeffrey and Daniel Zwillinger, Academic Press. Seventh Edition, 2007

which is a convergent integral. Of course a divergent integral cannot be legitimately manipulated to give a convergent integral, so P&S's calculation must involve some trickery. The subtlety is that in this case the integral around the circle at infinite radius does not vanish. Instead the integration over the segments C_2 and C_4, in the diagram below, pick up nontrivial contributions from the vicinity of the real axis.



I (Alan Guth) spent some time trying to show that these contributions vanish in the sense of distributions, but that turned out not to be quite true. The contributions from C_2 and C_4 actually give a term proportional to $\delta(r)$. The delta-function can be seen simply by remembering that

$$\int_{-\infty}^{\infty}\mathrm{d}p e^{ipr} = 2\pi\delta(r)\ , \tag{1.5}$$

as was shown in Lecture Notes 4, *Dirac Delta Function as a Distribution.* Thus we can write

$$\int_{-\infty}^{\infty}\mathrm{d}p\frac{pe^{ipr}}{\sqrt{p^2+m^2}} = \int_{-\infty}^{\infty}\mathrm{d}p\frac{\left(p-\sqrt{p^2+m^2}\right)e^{ipr}}{\sqrt{p^2+m^2}} + 2\pi\delta(r)\quad . \tag{1.6}$$

The integrand of the integral on the right falls off as $1/p^2$ for large p, which guarantees that there will be no contribution from C_2 or C_4. The singularity structure and the discontinuity across the branch cut are the same as the integral on the left, so we recover the answer of P&S, modulo the δ-function term:

$$\int_{-\infty}^{\infty}\mathrm{d}p\frac{pe^{ipr}}{\sqrt{p^2+m^2}} = 2i\int_m^{\infty}\mathrm{d}\rho\frac{\rho e^{-\rho r}}{\sqrt{\rho^2-m^2}} + 2\pi\delta(r)\ . \tag{1.7}$$

We will drop the δ-function term on the grounds that the question asks about spacelike separations, which means that $r>0$. The case $r=0$ is a lightlike separation, which is a different regime in which Eq. (1.1) is not valid, since for lightlike separations we cannot find a frame for which $x^0 = y^0$. In the language of distributions, the restriction to $r > 0$ is described as a restriction to test functions $\phi(r)$ which are nonzero only for $r > 0$.

The integral on the right-hand side of Eq. (1.7) is a well-defined integral which can be integrated numerically, or one can type it into a program like *Mathematica*, or look it up in a book such as Gradshteyn and Ryzhik (G&R). G&R tabulate it under *Definite Integrals of Elementary Functions: Combinations of exponentials and algebraic functions*, where it appears (at least in the 4th edition) as Eq. 3.365.2:

$$\int_u^\infty \frac{x e^{-\mu x}\,\mathrm{d}x}{\sqrt{x^2 - u^2}} = u\,K_1(u\mu) \quad [u > 0, \mathrm{Re}\,\mu > 0]. \tag{1.8}$$

Here $K_\nu(z)$ is described in words as a Bessel function of imaginary argument, and can be defined by its integral representation

$$K_\nu(z) = \int_u^\infty e^{-z\cosh t}\,\cosh(\nu t)\,\mathrm{d}t \quad \left[|\arg z| < \frac{\pi}{2} \text{ or } \mathrm{Re}\,z = 0 \text{ and } \nu = 0\right] , \tag{1.9}$$

which appears in G&R as Eq. 8.432.1, or in Abramowitz and Stegun* as Eq. 9.6.24 (p. 376). Note, by the way, that Eq. (1.8) can be connected to Eq. (1.9) by the simple substitution $x = u\cosh t$.

In our notation Eq. (1.8) becomes

$$\int_m^\infty \mathrm{d}\rho \frac{\rho e^{-\rho r}}{\sqrt{\rho^2 - m^2}} = m K_1(mr)\ . \tag{1.10}$$

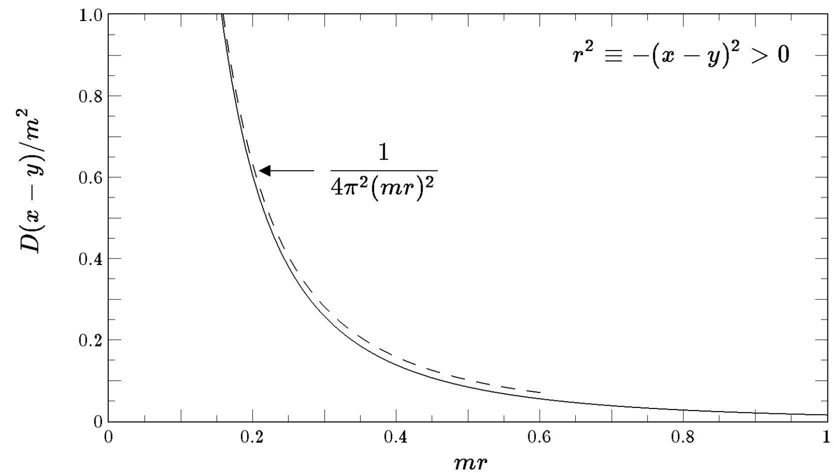
Combining Eqs. (1.1) and (1.7), dropping the δ-function, one obtains the final result

$$\boxed{\frac{D(x-y)}{m^2} = \frac{1}{4\pi^2 mr} K_1(mr).} \tag{1.11}$$

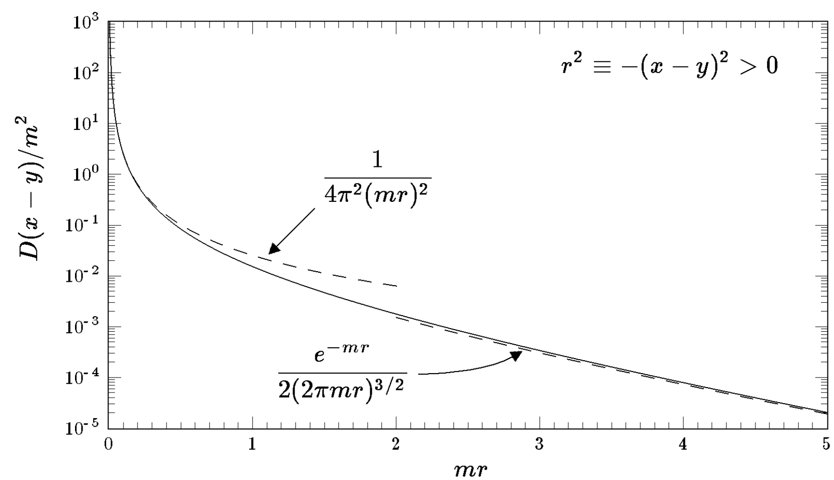
* *Handbook of Mathematical Functions: with Formulas, Graphs, and Mathematical Tables*, edited by Milton Abramowitz and Irene A. Stegun. This document was prepared as a U.S. government publication, so it is not restricted by copyright and is available online at
http://www.convertit.com/Go/Bioresearchonline/Reference/AMS55.ASP
and at
http://www.math.sfu.ca/~cbm/aands/.

For purposes of grading, anyone who found the above answer should get full credit — the extraction of the δ-function described above is by no means required. Alternatively, anyone who constructed graphs equivalent to one of the graphs below should get full credit.

To graph the function, I used Eq. (1.11) with the integral representation (1.9) for $K_1(mr)$. Plotting on a linear scale, it looks like



Note that I plotted also, as a broken line, the asymptotic form for small r that we derived in lecture. A wider range of values can be shown by using a logarithmic scale, where this time I have shown broken lines indicating the asymptotic forms for both small and large r:



Further comments about the $\delta(r)$ term:

The $\delta(r)$ term at first looks very sick, since Eq. (1.1) already contains a factor of $1/r$, so we are finding $\delta(r)/r$, which is not normally well-defined. It is, however, well-defined when r^2 is the square of a Lorentzian 4-vector, as it is here. Note that

$$\frac{\delta(r)}{r} = 2\delta(r^2) \ , \tag{1.12}$$

which is well-defined δ-function contraining the integration to the light cone. I.e.,

$$\int \mathrm{d}^4x\, \delta(x^2)$$

is a well-defined measure of integration, just as

$$\int \mathrm{d}^4p\, \delta(p^2 - m^2)$$

is a well-defined integration over momentum, which we have already encountered.

The presence of the δ-function term is real, although our calculation does not find the right coefficient, since we assumed that $x - y$ is spacelike ($r^2 > 0$) before we extracted the δ-function. Perhaps the most careful textbook for discussing these coordinate-space functions is *Introduction to the Theory of Quantized Fields,* by N.N. Bogoliubov and D.V. Shirkov, first published in Russian in 1957, and published in English by Wiley-Interscience in 1959. In Appendix I they give the following expression for $D(x-y)$:

$$\begin{aligned} D(x-y) = \frac{-i}{4\pi}\varepsilon(x^0)\delta(\lambda) &+ \frac{m}{8\pi\sqrt{\lambda}}\theta(\lambda)\left[N_1(m\sqrt{\lambda}) + i\varepsilon(x^0)J_1(m\sqrt{\lambda}\right] \\ &+ \frac{m}{4\pi^2\sqrt{-\lambda}}\theta(-\lambda)K_1(m\sqrt{-\lambda}) \ , \end{aligned} \tag{1.13}$$

where

$$\lambda \equiv (x-y)^2 \equiv (x^0 - y^0)^2 - (\vec{x} - \vec{y})^2 \ . \tag{1.14}$$

Actually Bogoliubov and Shirkov seem to give the opposite sign for the last term, if I am understanding their conventions correctly, but I changed it to agree with our sign. Most textbooks discuss the propagator only in a momentum-space representation, as in Eq. (1.1a), so the coordinate space complications are usually bypassed. Bogoliubov and Shirkov give the coordinate-space Feynman propagator as

$$\begin{aligned} D_F(x-y) = \frac{-i}{4\pi}\delta(\lambda) &+ \frac{m}{8\pi\sqrt{\lambda}}\theta(\lambda)\left[N_1(m\sqrt{\lambda}) + iJ_1(m\sqrt{\lambda}\right] \\ &+ \frac{m}{4\pi^2\sqrt{-\lambda}}\theta(-\lambda)K_1(m\sqrt{-\lambda}) \ , \end{aligned} \tag{1.15}$$

which I think is correct.

Problem 2: A tale of three cutoffs *(15 points)*

(a) The Fourier transform of $\varphi_0(\omega)$ is

$$\begin{aligned} \tilde{\varphi}_0(t) &= \int_{-\infty}^{\infty} \mathrm{d}\omega\, e^{i\omega t} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(\omega-\omega_1)^2/(2\sigma^2)} \\ &= e^{i\omega_1 t} e^{-\sigma^2 t^2/2} \ . \end{aligned} \tag{2.1}$$

Then

$$F^{(1)}[\varphi_0] = \tilde{F}_g[\varphi_0] = F_g[\tilde{\varphi}_0] = \int_0^{\infty} \mathrm{d}t\, e^{-i\omega_0 t}\, \tilde{\varphi}(t) = \int_0^{\infty} \mathrm{d}t\, e^{i(\omega_1-\omega_0)t}\, e^{-\sigma^2 t^2/2} \ . \tag{2.2}$$

This can be evaluated in terms of the error function by using

$$\int_x^{\infty} \mathrm{d}t\, e^{-t^2} = \frac{\sqrt{\pi}}{2}\left[1 - \Phi(x)\right] \ , \tag{2.3}$$

and by completing the square,

$$i(\omega_1-\omega_0)t - \frac{\sigma^2 t^2}{2} = -\frac{\sigma^2}{2}\left[t - \frac{i(\omega_1-\omega_0)}{\sigma^2}\right]^2 - \frac{(\omega_1-\omega_0)^2}{2\sigma^2} \ ,$$

resulting in

$$F^{(1)}[\varphi_0] = \tilde{F}_g[\varphi_0] = \frac{1}{\sigma}\sqrt{\frac{\pi}{2}}\, e^{-(\omega_1-\omega_0)^2/(2\sigma^2)}\left[1 - \Phi\left(\frac{i(\omega_0-\omega_1)}{\sqrt{2}\,\sigma}\right)\right] \ . \tag{2.4}$$

(b) Evaluating the integral of Eq. (1.7) of the Problem Set,

$$\tilde{g}_\epsilon(\omega) = \frac{i}{\omega - \omega_0 + i\epsilon} \ . \tag{2.5}$$

This expression, however, will not be needed for the method that we will pursue. Eqs. (1.8) and (1.4) of the Problem Set imply that

$$F^{(2)}[\varphi_0] = \lim_{\epsilon\to 0} \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \mathrm{d}\omega\, \tilde{g}_\epsilon(\omega)\, e^{-(\omega-\omega_1)^2/(2\sigma^2)} \ , \tag{2.6}$$

which can be rewritten by replacing $\tilde{g}_\epsilon(\omega)$ by its definition, Eq. (1.7) of the Problem Set:

$$F^{(2)}[\varphi_0] = \lim_{\epsilon\to 0} \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \mathrm{d}\omega\, \left[\int_0^{\infty} \mathrm{d}t\, e^{i(\omega-\omega_0+i\epsilon)t}\right] e^{-(\omega-\omega_1)^2/(2\sigma^2)} \ . \tag{2.7}$$

Since the integral is absolutely convergent, one can interchange the order of integration, giving

$$F^{(2)}[\varphi_0] = \lim_{\epsilon\to 0} \frac{1}{\sqrt{2\pi}\sigma} \int_0^\infty \mathrm{d}t \int_{-\infty}^\infty \mathrm{d}\omega\, e^{i\,(\omega-\omega_0+i\epsilon)\,t}\, e^{-(\omega-\omega_1)^2/(2\sigma^2)} \tag{2.8a}$$

$$= \lim_{\epsilon\to 0} \int_0^\infty \mathrm{d}t\, e^{i\,(\omega_1-\omega_0+i\epsilon)\,t}\, e^{-\frac{1}{2}\sigma^2 t^2} \,, \tag{2.8b}$$

where the integral over ω was carried out by completion of the square. Completing the square in t, the integral can be brought to the Fresnel form, giving

$$F^{(2)}[\varphi_0] = \lim_{\epsilon\to 0} \frac{1}{\sigma}\sqrt{\frac{\pi}{2}}\, e^{-(\omega_1-\omega_0+i\epsilon)^2/(2\sigma^2)} \left[1 - \Phi\left(\frac{i(\omega_0-\omega_1-i\epsilon)}{\sqrt{2}\,\sigma}\right)\right] \,. \tag{2.9}$$

This function is continuous in ϵ in the vicinity of $\epsilon = 0$, so the answer is the same as Eq. (2.4). Without explicitly evaluating the integral in Eq. (2.8b), one could instead have used Lebesgue's Dominated Convergence Theorem to show the limit could be taken inside the integral, and therefore the result is identical to the right-hand expression in Eq. (2.2).

(c) The Fourier transform of $g_\epsilon^{(3)}(t)$ is given by

$$\tilde{g}_\epsilon^{(3)} = \frac{i}{\omega-\omega_0}\left[1 - e^{i\Lambda(\omega-\omega_0)}\right] \,, \tag{2.10}$$

where $\Lambda \equiv 1/\epsilon$, but again we will not actually use this expression. Instead, we will manipulate $F^{(3)}[\varphi_0]$ by a procedure analogous to that used to derive Eq. (2.8b), obtaining in this case

$$F^{(3)}[\varphi_0] = \lim_{\epsilon\to 0} \int_0^{1/\epsilon} \mathrm{d}t\, e^{i(\omega_1-\omega_0)t}\, e^{-\sigma^2 t^2/2} \,. \tag{2.11}$$

The integral can be expressed as

$$F^{(3)}[\varphi_0] = \lim_{\epsilon\to 0} \frac{1}{\sigma}\sqrt{\frac{\pi}{2}}\, e^{-(\omega_1-\omega_0)^2/(2\sigma^2)} \left\{ \Phi\left[\frac{\sigma}{\sqrt{2}}\left(\frac{1}{\epsilon} - \frac{i(\omega_1-\omega_0)}{\sigma^2}\right)\right] - \Phi\left[\frac{i(\omega_0-\omega_1)}{\sqrt{2}\,\sigma}\right] \right\} \,. \tag{2.12}$$

One can carry out the limit by arguing that

$$\lim_{\epsilon\to 0} \Phi\left[\frac{\sigma}{\sqrt{2}}\left(\frac{1}{\epsilon} - \frac{i(\omega_1-\omega_0)}{\sigma^2}\right)\right] = 1 \,, \tag{2.13}$$

since analyticity allows one to distort the contour of integration so that

$$\Phi\left[\frac{\sigma}{\sqrt{2}}\left(\Lambda - \frac{i(\omega_1-\omega_0)}{\sigma^2}\right)\right] = \frac{2}{\sqrt{\pi}} \int_0^{\frac{\sigma}{\sqrt{2}}\Lambda} \mathrm{d}t\, e^{-t^2} + \frac{2}{\sqrt{\pi}} \int_{\frac{\sigma}{\sqrt{2}}\Lambda}^{\frac{\sigma}{\sqrt{2}}\left[\Lambda - \frac{i(\omega_1-\omega_0)}{\sigma^2}\right]} \mathrm{d}t\, e^{-t^2} \,. \tag{2.14}$$

In the limit the first term becomes

$$\frac{2}{\sqrt{\pi}} \int_0^\infty \mathrm{d}t\, e^{-t^2} = 1 \,, \tag{2.15}$$

and the second term vanishes because the real part of the exponent becomes arbitrarily negative. Alternatively, one could have used Lebesgue's Dominated Convergence Theorem to take the limit of Eq. (2.11) inside the integral sign, reproducing the result of Eq. (2.2). To apply the Dominated Convergence theorem to Eq. (2.11), one would first rewrite it as

$$F^{(3)}[\varphi_0] = \lim_{\epsilon\to 0} \int_0^\infty \mathrm{d}t\, \theta\left(\frac{1}{\epsilon} - t\right) e^{i(\omega_1-\omega_0)t}\, e^{-\sigma^2 t^2/2} \,. \tag{2.16}$$

(d) The kind of construction that can cause trouble is something like

$$g_\epsilon(t) = \begin{cases} 0 & \text{if } t < 0 \\ e^{-i\omega_0 t} & \text{if } 0 \le t \le \frac{1}{\epsilon} \\ e^{\frac{1}{2}\alpha t^2} & \text{if } 1/\epsilon < t < 1 + \frac{1}{\epsilon} \\ 0 & \text{if } t > 1 + \frac{1}{\epsilon} \,. \end{cases} \tag{2.17}$$

In this case the analogue of Eq. (2.8b) becomes

$$F[\varphi_0] = \lim_{\epsilon\to 0} \int_0^{1/\epsilon} \mathrm{d}t\, e^{i(\omega_1-\omega_0)t}\, e^{-\sigma^2 t^2/2} + \lim_{\epsilon\to 0} \int_{\frac{1}{\epsilon}}^{1+\frac{1}{\epsilon}} \mathrm{d}t\, e^{\frac{1}{2}\alpha^2 t^2}\, e^{i\omega_1 t}\, e^{-\frac{1}{2}\sigma^2 t^2} \,. \tag{2.18}$$

If $\alpha < \sigma$ the extra term vanishes in the limit, but if $\alpha > \sigma$ it is divergent. But a valid distribution has to be defined for any Schwartz function, so Eq. (2.17) does not lead to a valid distribution.