MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

8.323: Relativistic Quantum Field Theory I

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PROBLEM SET 5 SOLUTIONS

Problem 1: Subtleties of delta functions *(10 points)*

(a) The Dirac delta function $\delta(t-a)$ is defined by the relation

$$\int \mathrm{d}t\, F(t)\, \delta(t-a) \equiv F(a) \ ,$$

where we should keep in mind that the integral sign in the above definition is not a real integral in the sense of Riemann or Lebesgue, but instead is a symbolic integral, as described in Lecture Notes 4. That is, the integral sign is really just part of the notation for the distribution defined by $\delta(t-a)$, which maps the function $F(t)$ to the number $F(a)$. Given this definition, we have

$$T_{g_1}[\varphi] \equiv \int_{-\infty}^{\infty} \mathrm{d}t\, f(t)\delta(t-a)\varphi(t) = f(a)\varphi(a) \ ,$$

and

$$T_{g_2}[\varphi] \equiv \int_{-\infty}^{\infty} \mathrm{d}t\, f(a)\delta(t-a)\varphi(t) = f(a)\varphi(a) \ .$$

So, the answer is $\boxed{\text{YES,}}$ $g_1(t)$ and $g_2(t)$ describe the same distribution.

(b) As described in Lecture Notes 4, the derivative of a distribution is defined by

$$T'[\varphi] \equiv -T\left[\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right] \ .$$

So,

$$\begin{aligned} T_{h_1}[\varphi] &\equiv \int_{-\infty}^{\infty} \mathrm{d}t\, f(t)\,\varphi(t)\,\delta'(t-a) \\ &= -\int_{-\infty}^{\infty} \frac{\mathrm{d}}{\mathrm{d}t}[f(t)\varphi(t)]\,\delta(t-a) \\ &= -f'(a)\,\varphi(a) - f(a)\,\varphi'(a) \ , \end{aligned}$$

and

$$\begin{aligned} T_{h_2}[\varphi] &\equiv \int_{-\infty}^{\infty} \mathrm{d}t\, f(a)\,\varphi(t)\,\delta'(t-a) \\ &= -\int_{-\infty}^{\infty} \frac{\mathrm{d}}{\mathrm{d}t}[f(a)\varphi(t)]\,\delta(t-a) \\ &= -f(a)\,\varphi'(a) \ . \end{aligned}$$

Thus, $\boxed{h_1(t) \text{ and } h_2(t) \text{ are NOT equal to each other.}}$

(c) The distribution corresponding to $g_1'(t)$ can be written as

$$\begin{aligned} T_{g_1'}[\varphi] &\equiv \int_{-\infty}^{\infty} \mathrm{d}t\, [f'(t)\,\delta(t-a) + f(t)\delta'(t-a)]\,\varphi(t) \\ &= \int_{-\infty}^{\infty} \mathrm{d}t\, f'(t)\,\varphi(t)\,\delta(t-a) - \int_{-\infty}^{\infty} \mathrm{d}t\, \frac{\mathrm{d}}{\mathrm{d}t}[f(t)\,\varphi(t)]\,\delta(t-a) \\ &= f'(a)\,\varphi(a) - f'(a)\,\varphi(a) - f(a)\,\varphi'(a) = -f(a)\varphi'(a)\ . \end{aligned}$$

Alternatively, we could have written

$$\begin{aligned} T_{g_1'}[\varphi] &\equiv T_{g_1}'[\varphi] \equiv -T_{g_1}[\varphi'(t)] \\ &= -\int_{-\infty}^{\infty} \mathrm{d}t\, f(t)\,\delta(t-a)\,\varphi'(t) \\ &= -f(a)\,\varphi'(a)\ , \end{aligned}$$

which results in the same answer. The distribution corresponding to $g_2'(t)$ is then given by

$$\begin{aligned} T_{g_2'}[\varphi] &\equiv \int_{-\infty}^{\infty} \mathrm{d}t\, [f(a)\delta'(t-a)]\,\varphi(t) \\ &= -\int_{-\infty}^{\infty} \mathrm{d}t\, \frac{\mathrm{d}}{\mathrm{d}t}[f(a)\,\varphi(t)] \\ &= -f(a)\,\varphi'(a)\ , \end{aligned}$$

which means that $\boxed{g_1'(t) \text{ and } g_2'(t) \text{ are equal.}}$

Alternatively, we could have evaluated $g_2'(t)$ by using

$$\begin{aligned} T_{g_2'}[\varphi] &\equiv T_{g_2}'[\varphi] \equiv -T_{g_2}[\varphi'(t)] \\ &= -\int_{-\infty}^{\infty} \mathrm{d}t\, f(a)\,\delta(t-a)\,\varphi'(t) \\ &= -f(a)\,\varphi'(a)\ , \end{aligned}$$

which confirms the previous result.

(d) Starting with Eq. (1.10) from the statement of the problem, we can write

$$\begin{aligned} \int_{-\infty}^{\infty} \mathrm{d}t\, \theta'(t)\,\varphi(t) &\equiv T_\theta'[\varphi] \equiv -T_\theta\left[\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right] \\ &= -\int_{-\infty}^{\infty} \mathrm{d}t\, \theta(t)\varphi'(t) = -\int_0^{\infty} \mathrm{d}t\, \varphi'(t) \\ &= \varphi(0) - \varphi(\infty) = \varphi(0)\ , \end{aligned}$$

where we used the fact that allowed test functions $\varphi(t)$ approach zero as $t \to \infty$. Thus,

$$\int_{-\infty}^{\infty} \mathrm{d}t\, \theta'(t)\, \varphi(t) = \int_{-\infty}^{\infty} \mathrm{d}t\, \varphi(t)\, \delta(t) \ ,$$

so

$$\boxed{\theta'(t) = \delta(t)} \ .$$

Problem 2: $\left(\Box_x + m^2\right) D_F(x-y) = -i\delta^{(4)}(x-y)$ *(10 points)*

Given the definition of the Feynman propagator,

$$D_F(x-y) = \theta(x^0 - y^0)\,\langle 0\,|\phi(x)\,\phi(y)|\,0\rangle + \theta(y^0 - x^0)\,\langle 0\,|\phi(y)\,\phi(x)|\,0\rangle \ ,$$

and using the abbreviation $\partial_0 \equiv \partial/\partial x^0$, we can calculate

$$\begin{aligned} \partial_0 D_F(x-y) = \ & \delta(x^0 - y^0)\,\langle 0\,|\phi(x)\,\phi(y)|\,0\rangle + \theta(x^0 - y^0)\,\langle 0\,|\partial_0\phi(x)\,\phi(y)|\,0\rangle \\ & - \delta(y^0 - x^0)\,\langle 0\,|\phi(y)\,\phi(x)|\,0\rangle + \theta(y^0 - x^0)\,\langle 0\,|\phi(y)\,\partial_0\phi(x)|\,0\rangle \\ = \ & \delta(x^0 - y^0)\,\langle 0\,|[\phi(x)\,,\,\phi(y)]\,|\,0\rangle + \\ & + \theta(x^0 - y^0)\,\langle 0\,|\partial_0\phi(x)\,\phi(y)|\,0\rangle + \theta(y^0 - x^0)\,\langle 0\,|\phi(y)\,\partial_0\phi(x)|\,0\rangle \ . \end{aligned}$$

But

$$\begin{aligned} \delta(x^0 - y^0)\,\langle 0\,|[\phi(x)\,,\,\phi(y)]\,|\,0\rangle &= \delta(x^0 - y^0)\,\left\langle 0\,\middle|\,\left[\phi(\vec{x}, y^0)\,,\,\phi(\vec{y}, y^0)\right]\,\middle|\,0\right\rangle \\ &= 0 \ , \end{aligned}$$

where the second line is a consequence of the canonical commutation relations. Thus,

$$\partial_0 D_F(x-y) = \theta(x^0 - y^0)\,\langle 0\,|\partial_0\phi(x)\,\phi(y)|\,0\rangle + \theta(y^0 - x^0)\,\langle 0\,|\phi(y)\,\partial_0\phi(x)|\,0\rangle \ .$$

Differentiating again,

$$\begin{aligned} \partial_0^2 D_F(x-y) = \ & \delta(x^0 - y^0)\,\langle 0\,|\partial_0\phi(x)\,\phi(y)|\,0\rangle + \theta(x^0 - y^0)\,\left\langle 0\,\middle|\partial_0^2\phi(x)\,\phi(y)\middle|\,0\right\rangle \\ & - \delta(y^0 - x^0)\,\langle 0\,|\phi(y)\,\partial_0\phi(x)|\,0\rangle + \theta(y^0 - x^0)\,\left\langle 0\,\middle|\phi(y)\,\partial_0^2\phi(x)\middle|\,0\right\rangle \\ = \ & \delta(x^0 - y^0)\,\langle 0\,|[\partial_0\phi(x)\,,\,\phi(y)]\,|\,0\rangle + \\ & + \theta(x^0 - y^0)\,\left\langle 0\,\middle|\partial_0^2\phi(x)\,\phi(y)\middle|\,0\right\rangle + \theta(y^0 - x^0)\,\left\langle 0\,\middle|\phi(y)\,\partial_0^2\phi(x)\middle|\,0\right\rangle \ . \end{aligned}$$

This time we find

$$\begin{aligned}\delta(x^0-y^0)\,\langle 0\,|[\partial_0\phi(x)\,,\,\phi(y)]\,|\,0\rangle &= \delta(x^0-y^0)\,\big\langle 0\,\big|\,[\partial_0\phi(\vec{x},y^0)\,,\,\phi(\vec{y},y^0)]\,\big|\,0\big\rangle\\ &= -i\delta(x^0-y^0)\,\delta^{(3)}(\vec{x}-\vec{y})\\ &= -i\delta^{(4)}(x-y)\ ,\end{aligned}$$

where again we used the canonical commutation relations. So

$$\begin{aligned}\partial_0^2 D_F(x-y) &= -i\delta^{(4)}(x-y)+\\ &\quad+\theta(x^0-y^0)\,\big\langle 0\,\big|\partial_0^2\phi(x)\,\phi(y)\big|\,0\big\rangle+\theta(y^0-x^0)\,\big\langle 0\,\big|\phi(y)\,\partial_0^2\phi(x)\big|\,0\big\rangle\ .\end{aligned}$$

Now we just need to add in the other terms, recognizing that the spatial derivatives can be taken through the theta functions, since the arguments of the theta functions are time variables. So,

$$\begin{aligned}\left(\Box_x+m^2\right)D_F(x-y) &= -i\delta^{(4)}(x-y)+\\ &\quad+\theta(x^0-y^0)\left\langle 0\,\middle|(\partial_0^2-\vec{\nabla}^2+m^2)\phi(x)\,\phi(y)\middle|\,0\right\rangle+\\ &\quad+\theta(y^0-x^0)\left\langle 0\,\middle|\phi(y)\,(\partial_0^2-\vec{\nabla}^2+m^2)\phi(x)\middle|\,0\right\rangle\\ &= \boxed{-i\delta^{(4)}(x-y)\ .}\end{aligned}$$

In the last step we used the fact that the Heisenberg field $\phi(x)$ obeys the Klein-Gordon equation.

Problem 3: Coherent states *(15 points)*

(a) From $a^\dagger_{\text{out}}(\vec{p})=S^{-1}\,a^\dagger_{\text{in}}(\vec{p})\,S$, we find

$$\begin{aligned}e^F &= \exp\left(\int\frac{d^3p}{(2\pi)^3}\,\frac{1}{\sqrt{2E_p}}\tilde{\jmath}(p)a^\dagger_{\text{in}}(\vec{p})\right)\\ &= \sum_n\frac{1}{n!}\left(\int\frac{d^3p}{(2\pi)^3}\,\frac{1}{\sqrt{2E_p}}\tilde{\jmath}(p)a^\dagger_{\text{in}}(\vec{p})\right)^n\\ &= S\sum_n\frac{1}{n!}\left(S^{-1}\int\frac{d^3p}{(2\pi)^3}\,\frac{1}{\sqrt{2E_p}}\tilde{\jmath}(p)a^\dagger_{\text{in}}(\vec{p})\,S\right)^n S^{-1}\\ &= S\sum_n\frac{1}{n!}\left(\int\frac{d^3p}{(2\pi)^3}\,\frac{1}{\sqrt{2E_p}}\tilde{\jmath}(p)a^\dagger_{\text{out}}(\vec{p})\right)^n S^{-1}\\ &= S\,e^{F'}S^{-1}.\end{aligned}$$

Using $a_{\rm out}(\vec{p}) = S^{-1}\, a_{\rm in}(\vec{p})\, S$ we can similarly obtain $e^G = S\, e^{G'} S^{-1}$. So letting

$$\lambda = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} |\tilde{\jmath}(p)|^2 \ ,$$

we obtain

$$S = e^{-\frac{1}{2}\lambda}\, e^F e^G = e^{-\frac{1}{2}\lambda} S\, e^{F'} e^{G'} S^{-1}.$$

Multiplying each expression by S on the right and S^{-1} on the left gives

$$S = e^{-\frac{1}{2}\lambda}\, e^{F'} e^{G'} \ .$$

Then since $e^{G'}|0_{\rm out}\rangle = |0_{\rm out}\rangle$, we conclude that

$$|0_{\rm in}\rangle = S\,|0_{\rm out}\rangle = e^{-\frac{1}{2}\lambda}\, e^{F'} e^{G'} |0_{\rm out}\rangle = e^{-\frac{1}{2}\lambda}\, e^{F'} |0_{\rm out}\rangle \ .$$

(b) We found in Problem 5(a) of Problem Set 2 that

$$e^A B e^{-A} = B + [A,\, B] + \frac{1}{2}[A,\, [A,\, B]] + \frac{1}{3!}[A,\, [A,\, [A,\, B]]] + \ldots$$

So since $[a^\dagger, a] = -1$, we find $e^{-za^\dagger} a e^{za^\dagger} = a + z$. Thus

$$a|z\rangle = a\, e^{za^\dagger}|0\rangle = e^{za^\dagger} e^{-za^\dagger} a\, e^{za^\dagger}|0\rangle = e^{za^\dagger}(a+z)|0\rangle = z|z\rangle.$$

$|z\rangle$ is an eigenstate of the annihilation operator with eigenvalue z.

(c) Using $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$ and $\langle n|m\rangle = \delta_{nm}$, we find

$$\begin{aligned}
\langle z_2|z_1\rangle &= \langle 0|e^{z_2^* a} e^{z_1 a^\dagger}|0\rangle \\
&= \left(\sum_m \frac{(z_2^*)^m \langle 0|a^m}{m!}\right)\left(\sum_n \frac{z_1^n (a^\dagger)^n |0\rangle}{n!}\right) \\
&= \left(\sum_m \frac{(z_2^*)^m \langle m|}{\sqrt{m!}}\right)\left(\sum_n \frac{z_1^n |n\rangle}{\sqrt{n!}}\right) \\
&= \sum_n \frac{(z_2^* z_1)^n}{n!} \\
&= e^{z_2^* z_1}.
\end{aligned}$$

(d)

$$\begin{aligned}
\langle q\rangle_z &= \frac{\langle z|q|z\rangle}{\langle z|z\rangle} \\
&= \frac{\langle z|\frac{1}{\sqrt{2}}(a^\dagger + a)|z\rangle}{\langle z|z\rangle} \\
&= \frac{\langle z|\frac{1}{\sqrt{2}}(z^* + z)|z\rangle}{\langle z|z\rangle} \\
&= \sqrt{2}\,Re(z),
\end{aligned}$$

and

$$\begin{aligned}
\langle p\rangle_z &= \frac{\langle z|p|z\rangle}{\langle z|z\rangle} \\
&= \frac{\langle z|\frac{i}{\sqrt{2}}(a^\dagger - a)|z\rangle}{\langle z|z\rangle} \\
&= \frac{\langle z|\frac{i}{\sqrt{2}}(z^* - z)|z\rangle}{\langle z|z\rangle} \\
&= \sqrt{2}\,Im(z).
\end{aligned}$$

(e)

$$\begin{aligned}
\Delta q^2 &= \langle q^2\rangle - \langle q\rangle^2 \\
&= \frac{1}{2\langle z|z\rangle}\langle z|(a^\dagger)^2 + a^\dagger a + aa^\dagger + a^2|z\rangle - \langle q\rangle^2 \\
&= \frac{1}{2}((z^*)^2 + 2z^*z + z^2 + 1) - \frac{1}{2}(z+z^*)^2 \\
&= \frac{1}{2}\ ,
\end{aligned}$$

and

$$\begin{aligned}
\Delta p^2 &= \langle p^2\rangle - \langle p\rangle^2 \\
&= \frac{-1}{2\langle z|z\rangle}\langle z|(a^\dagger)^2 - a^\dagger a - aa^\dagger + a^2|z\rangle - \langle p\rangle^2 \\
&= \frac{-1}{2}((z^*)^2 - 2z^*z + z^2 - 1) + \frac{1}{2}(z^*-z)^2 \\
&= \frac{1}{2}\ .
\end{aligned}$$

So $\Delta q \Delta p = \frac{1}{2}$, that is, $|z\rangle$ is a minimum uncertainty state.