MASSACHUSETTS INSTITUTE OF TECHNOLOGY  
Physics Department  

8.323: Relativistic Quantum Field Theory I  
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PROBLEM SET 6 SOLUTIONS

Problem 1: Proof of $e^A e^B = e^{A+B+rac{1}{2}[A,B]}$ (10 points)

Following the suggestion, we consider the modified form of the identity

$$e^{\lambda A} e^{\lambda B} = e^{\lambda A + \lambda B + \frac{1}{2} \lambda^2 [A,B]} , \quad (1.1)$$

which is obtained from the original identity by replacing $A$ by $\lambda A$ and $B$ by $\lambda B$. We define

$$F_1(\lambda) \equiv e^{\lambda A} e^{\lambda B} ,$$

$$F_2(\lambda) \equiv e^{\lambda A + \lambda B + \frac{1}{2} \lambda^2 [A,B]} , \quad (1.2)$$

and then our goal is to prove that $F_1(\lambda) = F_2(\lambda)$. Clearly

$$F_1(0) = F_2(0) = I , \quad (1.3)$$

where $I$ is the identity operator, and

$$\frac{dF_1(\lambda)}{d\lambda} = AF_1(\lambda) + F_1(\lambda) B . \quad (1.4)$$

Now all that is needed is to show that $F_2(\lambda)$ satisfies the same differential equation.

Taking advantage of the fact that $[A, B]$ commutes with all the operators mentioned in this problem, we can write

$$\frac{dF_2(\lambda)}{d\lambda} = \frac{d}{d\lambda} \left\{ e^{\lambda (A+B)} e^{\frac{1}{2} \lambda^2 [A,B]} \right\}$$

$$= \left\{ (A + B) + \lambda [A,B] \right\} e^{\lambda (A+B)} e^{\frac{1}{2} \lambda^2 [A,B]}$$

$$= AF_2(\lambda) + \lambda [A,B] F_2(\lambda) + Be^{\lambda (A+B)} e^{\frac{1}{2} \lambda^2 [A,B]}$$

$$= AF_2(\lambda) + \lambda [A,B] F_2(\lambda) + e^{\lambda (A+B)} B e^{\lambda (A+B)} e^{\frac{1}{2} \lambda^2 [A,B]} . \quad (1.5)$$

Now we make use of the identity proven in Problem 5 of Problem Set 2, which was written there as

$$e^{A} B e^{-A} = B + [A, B] + \frac{1}{2} [A, [A, B]] + \frac{1}{3!} [A, A, [A, B]] + \ldots . \quad (1.6)$$

Applying this identity to the 2nd term on the right-hand side of Eq. (1.5),

$$e^{-\lambda(A+B)} B e^{\lambda(A+B)} = B - \lambda [(A+B, B)] + \frac{\lambda^2}{2} [A + B, [(A+B, B)], + \ldots$$

$$= B - \lambda [(A, B)] , \quad (1.7)$$

where the series terminates because $[A, B]$ commutes with the other operators. Substituting Eq. (1.7) into Eq. (1.5), one finds the desired result:

$$\frac{dF_2(\lambda)}{d\lambda} = AF_2(\lambda) + F_2(\lambda) B . \quad (1.8)$$

Since a first order differential equation with specified initial conditions (as in Eq. (1.3)) has a unique solution, it follows that $F_2(\lambda) = F_1(\lambda)$, which for $\lambda = 1$ is the identity that we were trying to prove.

Problem 2: A Composite Operator (15 points)

(a) The normal-ordered $\phi^2$ is

$$:\phi^2(x): = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_q}} \times \left\{ a(p) a(q) e^{-i(p+q) \cdot x} + 2a^\dagger(\vec{p}) a(\vec{q}) e^{i(p-q) \cdot x} + a^\dagger(\vec{p}) a^\dagger(\vec{q}) e^{i(p+q) \cdot x} \right\}$$

Then

$$\langle 0 | :\phi^2(x): :\phi^2(y): | 0 \rangle =$$

$$\int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_q}} \int \frac{d^3r}{(2\pi)^3} \frac{1}{\sqrt{2E_r}} \int \frac{d^3s}{(2\pi)^3} \frac{1}{\sqrt{2E_s}}$$

$$\times \left\{ a(\vec{p}) a(q) e^{-i(p+q) \cdot y} + 2a^\dagger(\vec{p}) a(q) e^{i(p-q) \cdot y} + a^\dagger(\vec{p}) a^\dagger(\vec{q}) e^{i(p+q) \cdot y} \right\}$$

$$\times \left\{ a(\vec{r}) a(\vec{s}) e^{-i(r+s) \cdot y} + 2a^\dagger(\vec{r}) a(\vec{s}) e^{i(r-s) \cdot y} + a^\dagger(\vec{r}) a^\dagger(\vec{s}) e^{i(r+s) \cdot y} \right\}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_q}} \int \frac{d^3r}{(2\pi)^3} \frac{1}{\sqrt{2E_r}} \int \frac{d^3s}{(2\pi)^3} \frac{1}{\sqrt{2E_s}}$$

$$\times \langle 0 | a(p) a(q) a^\dagger(\vec{r}) a^\dagger(\vec{s}) | 0 \rangle e^{i(p+q) \cdot x + i(r+s) \cdot y} , \quad (2.1)$$

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since \(a(\vec{p})|0\rangle = 0\) and \(\{0\rangle a^\dagger(\vec{p}) = 0\). Now use the commutation relations 
\[ [a(\vec{p}), a^\dagger(\vec{q})] = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}) \] 
to show
\[
a(\vec{p}) a(\vec{q}) a^\dagger(\vec{r}) a^\dagger(\vec{s}) = (2\pi)^6 \delta^{(3)}(\vec{q} - \vec{r}) \delta^{(3)}(\vec{p} - \vec{s}) + (2\pi)^3 \delta^{(3)}(\vec{q} - \vec{r}) a^\dagger(\vec{s}) a(\vec{p}) + (2\pi)^6 \delta^{(3)}(\vec{p} - \vec{r}) a^\dagger(\vec{q}) a(\vec{s}) + (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{r}) a^\dagger(\vec{q}) a^\dagger(\vec{s}) a(\vec{p}) \]
and hence,
\[
\langle 0 | a(\vec{p}) a(\vec{q}) a^\dagger(\vec{r}) a^\dagger(\vec{s}) | 0 \rangle = \]
\[
(2\pi)^6 \delta^{(3)}(\vec{r} - \vec{q}) \delta^{(3)}(\vec{p} - \vec{s}) + (2\pi)^6 \delta^{(3)}(\vec{p} - \vec{r}) \delta^{(3)}(\vec{s} - \vec{q}) .
\]
This leads to
\[
\langle 0 | : \phi^2(x) : \phi^2(y) : | 0 \rangle =
\int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{1}{4E_p E_q} \left\{ e^{-i(p+q) \cdot x + i(q+q) \cdot y} + e^{-i(p+q) \cdot x + i(p+q) \cdot y} \right\}
\int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{1}{2E_p E_q} e^{-ip \cdot (x-y)} e^{-iq \cdot (x-y)}
\]
\[
= 2 \left[ \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip \cdot (x-y)} \right]^2
= 2D^2(x-y).
\]
(b) We know from Problem Set 4 that for spacelike separations \(D(x-y) = \frac{mr}{4\pi^2} K_1 (mr)\) where \(r^2 = -(x-y)^2\). Therefore,
\[
\sigma^2 = 2 \int d^3x d^3y w(\vec{x}) w(\vec{y}) D^2(\vec{x} - \vec{y})
= 2 \int d^3x d^3y w(\vec{x}) w(\vec{y}) \frac{m^2}{16\pi^4 r^2} K_1^2(mr).
\]
For small \(r\), \(K_1(r) \rightarrow \frac{1}{r}\) (see, for example, Jackson, p. 116). We see that the above integral diverges when \(\vec{x} = \vec{y}\).
\[\langle 0 | O_4^2 | w \rangle = 2 \frac{\delta}{(2\pi)^3} \int \frac{d^3p}{2E_p} \int \frac{d^3q}{2E_q} |\tilde{w}(p + q)|^2 . \] 

(2.9)

(e) (This argument is due to Lilian Childress.)

\[w(x^\mu) = \frac{1}{\pi^{3/2} a^3} e^{-|x|^2/a^2} \frac{\Gamma(3)}{\sqrt{\pi}} e^{-(x^\mu)^2/a^2} . \]

(2.10)

Then

\[\tilde{w}(p^\mu) = \int d^4x w(x^\mu) e^{i(p^\mu x^\rho - \bar{p} \cdot \bar{x})} = e^{-\frac{i}{2} p^\mu (p^\rho) - \frac{i}{4} a^2 |\bar{p}|^2} . \]

(2.11)

So

\[\langle 0 | O_4^2 | w \rangle = 2 \frac{\delta}{(2\pi)^3} \int \frac{d^3p}{2E_p} \int \frac{d^3q}{2E_q} \tilde{w}(p + q) \tilde{w}(-p - q) \]

\[= \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \frac{\Gamma(3)}{\sqrt{\pi}} \delta^{(3)}(\bar{p} + \bar{q}) e^{\frac{i}{2} \bar{p} \cdot \bar{q} + \frac{i}{4} a^2 |\bar{p}|^2 |\bar{q}|^2} . \]

(2.12)

As \(a \to \infty\),

\[e^{-\frac{i}{2} a^2 |\bar{p}|^2 |\bar{q}|^2} \to \frac{\Gamma(3)}{a^3} (2\pi)^{3/2} \delta^{(3)}(\bar{p} + \bar{q}) . \]

(2.13)

Thus,

\[\langle 0 | O_4^2 | w \rangle = \frac{\Gamma(3)}{a^3} (2\pi)^{3/2} \int \frac{d^3p}{(2\pi)^3} \frac{\Gamma(3)}{\sqrt{\pi}} e^{\frac{i}{2} \bar{p} \cdot \bar{q}} e^{-\frac{i}{2} \frac{\Gamma(3)}{a^3} (2\pi)^{3/2} \delta^{(3)}(\bar{p} + \bar{q})} . \]

(2.14)

We see that \(\langle 0 | O_4^2 | w \rangle \) goes like \(a^3\) where \(\beta = -3\), in agreement with our result from Problem Set 3. In the limit of large \(a\) the variance of \(O_4\) samples (averages!) many independent values of :\(\phi:\) so one would expect the variance to go as \(1/N\) where \(N\) is the number of independent sample points. \(N\) is proportional to the volume \(N \sim a^3\) and the result is as expected.

(f) Since \(\langle 0 | :\phi^2(x) : | 0 \rangle = 0\), it makes sense try to construct a state with a negative expectation value by considering a small perturbation of the vacuum. If we write such a perturbation as

\[|\psi\rangle = |0\rangle + \delta |\psi_1\rangle , \]

(2.15)

it then

\[\langle \psi | O_4[w] | \psi \rangle = \langle 0 | O_4[w] | 0 \rangle + \delta \langle 0 | O_4[w] | \psi_1 \rangle + \langle \psi_1 | O_4[w] | 0 \rangle \]

\[= 2 \delta \Re \langle \psi_1 | O_4[w] | 0 \rangle . \]

(2.16)

If we can find any state \(|\psi_1\rangle\) for which \(\langle \psi_1 | O_4[w] | 0 \rangle \neq 0\), then we can arrange for the right-hand side of Eq. (2.16) to be negative by redefining the phase. That is, we can choose \(|\psi_1\rangle = |\psi_1\rangle e^{i\theta}\), with \(\theta\) chosen to make \(\langle \psi_1 | O_4[w] | 0 \rangle\) real and negative. \(O_4[w] | 0 \rangle\) is clearly a two-particle state, so we can choose \(|\psi_1\rangle\) to be the simplest two-particle state that we know:

\[|\psi_1\rangle = |\bar{p}', \bar{q}'\rangle e^{i\theta} = 2 \sqrt{E_{p'} E_{q'}} a^\dagger (\bar{p}') a^\dagger (\bar{q}') |0\rangle e^{i\theta} . \]

(2.17)

Then

\[\langle \psi | O_4[w] | \psi \rangle = 2 \delta \Re \langle \psi_1 | O_4[w] | 0 \rangle \]

\[= 2 \delta \Re \left\{ \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_q}} \right. \]

\[\times \left. \langle \psi_1 | a(\bar{p}) a(\bar{q}) \tilde{w}(-p - q) + 2 a^\dagger (\bar{p}) a(\bar{q}) \tilde{w}(p + q) + a^\dagger (\bar{p}) a^\dagger (\bar{q}) \tilde{w}(p + q) | 0 \rangle \right\} \]

\[= 2 \delta \Re \left\{ e^{-i\theta} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_q}} 2 \sqrt{E_{p'} E_{q'}} \right. \]

\[\times \left. \langle 0 | a(\bar{q}) a(\bar{p}) \{ a(\bar{p}) a(\bar{q}) \tilde{w}(-p - q) + 2 a^\dagger (\bar{p}) a(\bar{q}) a^\dagger (\bar{q}) a^\dagger (\bar{q}) \tilde{w}(p + q) | 0 \rangle \} \right\} \]

\[= 2 \delta \Re \left\{ e^{-i\theta} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_q}} 2 \sqrt{E_{p'} E_{q'}} \right. \]

\[\times \left. \langle 0 | a(\bar{q}) a(\bar{p}) a^\dagger (\bar{p}) a^\dagger (\bar{q}) \tilde{w}(p + q) | 0 \rangle \right\} \]

\[= 2 \delta \Re \left\{ e^{-i\theta} \int d^3p \int d^3q \right. \]

\[\times \left. \langle 0 | \delta^{(3)}(\bar{q}' - \bar{p}) \delta^{(3)}(\bar{p}' - \bar{q}) + \delta^{(3)}(\bar{q}' - \bar{q}) \delta^{(3)}(\bar{p}' - \bar{p}) | 0 \rangle \tilde{w}(p + q) \right\} \]

\[= 4 \delta \Re \left\{ e^{-i\theta} \tilde{w}(p' + q) \} \right. \]

\[= 4 \delta \cos \theta e^{-\frac{i}{2} \bar{p} \cdot \bar{q} + \frac{i}{4} a^2 |\bar{p}' + \bar{q}'|^2} < 0, \]

(2.18)

where we have the desired sign if we choose \(\theta = \pi\), so \(e^{i\theta} = -1\).
Problem 3: Application of path integrals: The harmonic oscillator in thermal equilibrium (15 points)

(a) Clearly,
\[
\left[ a, a^\dagger \right] = \frac{1}{2\omega} [p - i\omega q, p + i\omega q] = \frac{i}{2} [p, q] - \frac{i}{2} [q, p] = 1 .
\] (3.1)

(b) Note that
\[
a^\dagger a = \frac{1}{2\omega} \left( p^2 + \omega^2 q^2 + i\omega [q, p] \right)
\implies H = \left( a^\dagger a + \frac{1}{2} \right) \omega .
\] (3.2)

It can be shown that the number operator \( N \equiv a^\dagger a \) can only have non-negative integer eigenvalues:
\[
N |n\rangle = n |n\rangle ,
\] (3.3)
where |\( n \rangle \rangle \) is the \( n \)th excited state. The energy levels are
\[
E_n = \left( n + \frac{1}{2} \right) \omega .
\] (3.4)
Furthermore,
\[
q = \frac{i \left( a - a^\dagger \right)}{\sqrt{2}\omega} ,
\] (3.5)
so
\[
\langle n | q^2 | n \rangle = \frac{1}{2\omega} \langle n | (a - a^\dagger)(a - a^\dagger) | n \rangle
= \frac{1}{2\omega} \langle n | aa^\dagger + a^\dagger a | n \rangle
= \frac{1}{2\omega} \langle n | 2N + 1 | n \rangle = \frac{1}{\omega} \left( n + \frac{1}{2} \right) .
\] (3.6)

This result also follows from the virial theorem.

(c) Consider first the partition function \( Z \),
\[
Z = \sum_{n=0}^{\infty} e^{-\beta E_n} = e^{-\beta\omega/2} \sum_{n=0}^{\infty} \left( e^{-\beta\omega} \right)^n = e^{-\beta\omega/2} \frac{1}{1 - e^{-\beta\omega}}
= \frac{1}{2 \sinh(\beta\omega/2)} .
\] (3.7)

Now,
\[
\langle q^2 \rangle = \frac{1}{Z} \sum_{n=0}^{\infty} \frac{1}{\omega} \left( n + \frac{1}{2} \right) e^{-\beta\omega (n+\frac{1}{2})} = \frac{1}{\omega^2 \partial \ln Z}{\beta} \quad \text{(3.8)}
\]
\[
= \coth(\beta\omega/2) .
\]

(d) The Schrödinger wave function may be written \( \psi_n(\bar{q}) = \langle \bar{q} | n \rangle \), so
\[
P(\bar{q}) = \frac{1}{Z} \sum_{n=0}^{\infty} e^{-\beta E_n} \langle \bar{q} | n \rangle \langle n | \bar{q} \rangle
= \frac{1}{Z} \sum_{n=0}^{\infty} \langle \bar{q} | e^{-\beta H} | n \rangle \langle n | \bar{q} \rangle
= \frac{1}{Z} \langle \bar{q} | e^{-\beta H} | \bar{q} \rangle ,
\] (3.9)
where we have used the completeness relation \( \sum_{n=0}^{\infty} |n\rangle \langle n| = I \).

(e) The variation of the action gives
\[
\delta S_E[\bar{q}(\tau)] = \int_{-\beta/2}^{\beta/2} d\tau \left( \frac{dq}{d\tau} \delta q + \omega^2 q \delta q \right)
= \int_{-\beta/2}^{\beta/2} d\tau \delta q \left( -\frac{d^2 q}{d\tau^2} + \omega^2 q \right) ,
\] (3.10)
where we have integrated by parts and dropped the boundary term because the variation \( \delta q \) is necessarily fixed at \( \tau = \pm\beta/2 \).

The classical equations of motion result from extremizing the action,
\[
\frac{d^2 q_{cl}}{d\tau^2} - \omega^2 q_{cl} = 0 ,
\] (3.11)
with boundary conditions \( q_{cl}(\pm\beta/2) = \bar{q} \). The general solution to Eq. (3.11) is
\[
q_{cl} = A \sinh(\omega\tau) + B \cosh(\omega\tau) ,
\] (3.12)
and the boundary conditions give
\[
A = 0 , \quad B = \frac{\bar{q}}{\cosh(\beta\tau/2)}
\implies q_{cl} = \bar{q} \frac{\cosh(\omega\tau)}{\cosh(\beta\omega/2)} .
\] (3.13)
(f) With \( q(\tau) = q_{cl}(\tau) + q'(\tau) \) the action splits up as

\[
S_E[q(\tau)] = S_E[q_{cl}(\tau)] + S_E[q'(\tau)] + \int_{-\beta/2}^{\beta/2} d\tau \left( \frac{dq'}{d\tau} \frac{dq_{cl}}{d\tau} + \omega^2 q_{cl} \right)
\]

\[
= S_E[q_{cl}(\tau)] + S_E[q'(\tau)] + \int_{-\beta/2}^{\beta/2} d\tau q'(\tau) \left( -\frac{d^2 q_{cl}}{d\tau^2} + \omega^2 q_{cl} \right)
\]

\[
= S_E[q_{cl}(\tau)] + S_E[q'(\tau)]
\]

where we integrated by parts and dropped the boundary term because \( q'(\pm\beta/2) = q(\pm\beta/2) - q_{cl}(\pm\beta/2) = 0 \). The cross term in Eq. (3.14) vanishes because \( q_{cl} \) satisfies the equations of motion.

(g) We can pull out \( S_E[q_{cl}(\tau)] \) from the path integral as it does not depend on any of the integration variables \( q'(\tau) \), so

\[
P(\bar{q}) \propto e^{-S_E[q_{cl}(\tau)]} \int_{\tau=-\beta/2}^{\tau=\beta/2} Dq' \left| \begin{array}{c}
\frac{dq'}{d\tau} = 0 \\
\frac{dq_{cl}}{d\tau} = 0
\end{array} \right| e^{-S_E[q'(\tau)]}.
\]

The remaining path integral over \( q'(\tau) \) does not depend on \( \bar{q} \) so we may absorb it into the proportionality constant. Thus

\[
P(\bar{q}) \propto e^{-S_E[q_{cl}(\tau)]},
\]

where

\[
S_E[q_{cl}(\tau)] = \frac{\bar{q}^2 \omega^2}{2 \cosh^2(\beta \omega/2)} \int_{-\beta/2}^{\beta/2} d\tau \left( \sinh^2(\omega \tau) + \cosh^2(\omega \tau) \right)
\]

\[
= \frac{\bar{q}^2 \omega^2}{2 \omega} \frac{\sinh(\beta \omega)}{\cosh(\beta \omega/2)}
\]

So the distribution is proportional to a Gaussian with mean zero and standard deviation \( \sigma \), with

\[
\sigma^2 = \frac{1}{2} \coth(\beta \omega/2)
\]

The correctly normalized probability distribution must be the standard Gaussian distribution,

\[
P(\bar{q}) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\bar{q}^2/(2\sigma^2)}.
\]