Problem 1: Proof of $e^A e^B = e^{A + B + \frac{1}{2} [A, B]}$ (10 points)†

Following the suggestion, we consider the modified form of the identity

$$e^{\lambda A} e^{\lambda B} = e^{\lambda A + \lambda B + \frac{1}{2} \lambda^2 [A, B]} ,$$  \hspace{1cm} (1.1)

which is obtained from the original identity by replacing $A$ by $\lambda A$ and $B$ by $\lambda B$. We define

$$F_1(\lambda) \equiv e^{\lambda A} e^{\lambda B} ,$$
$$F_2(\lambda) \equiv e^{\lambda A + \lambda B + \frac{1}{2} \lambda^2 [A, B]} ,$$  \hspace{1cm} (1.2)

and then our goal is to prove that $F_1(\lambda) = F_2(\lambda)$. Clearly

$$F_1(0) = F_2(0) = I ,$$  \hspace{1cm} (1.3)

where $I$ is the identity operator, and

$$\frac{dF_1(\lambda)}{d\lambda} = AF_1(\lambda) + F_1(\lambda) B .$$  \hspace{1cm} (1.4)

Now all that is needed is to show that $F_2(\lambda)$ satisfies the same differential equation.

Taking advantage of the fact that $[A, B]$ commutes with all the operators mentioned in this problem, we can write

$$\frac{dF_2(\lambda)}{d\lambda} = \frac{d}{d\lambda} \left\{ e^{\lambda (A+B)} e^{\frac{1}{2} \lambda^2 [A, B]} \right\}$$

$$= \left\{ (A + B) + \lambda [A, B] \right\} e^{\lambda (A+B)} e^{\frac{1}{2} \lambda^2 [A, B]}$$
$$= AF_2(\lambda) + \lambda [A, B] F_2(\lambda) + Be^{\lambda (A+B)} e^{\frac{1}{2} \lambda^2 [A, B]}$$
$$= AF_2(\lambda) + \lambda [A, B] F_2(\lambda) + e^{\lambda (A+B)} e^{-\lambda (A+B)} B e^{\lambda (A+B)} e^{\frac{1}{2} \lambda^2 [A, B]} .$$  \hspace{1cm} (1.5)

Now we make use of the identity proven in Problem 5 of Problem Set 2, which was written there as

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2} [A, [A, B]] + \frac{1}{3!} [A, [A, [A, B]]] + \ldots .$$  \hspace{1cm} (1.6)
Applying this identity to the 2nd term on the right-hand side of Eq. (1.5),

\[ e^{-\lambda(A+B)} Be^{\lambda(A+B)} = B - \lambda [(A + B, B)] + \frac{\lambda^2}{2} [A + B, [(A + B), B]] + \ldots \]

\[ = B - \lambda [(A, B)] , \quad (1.7) \]

where the series terminates because \([ A, B] \) commutes with the other operators.

Substituting Eq. (1.7) into Eq. (1.5), one finds the desired result:

\[ \frac{dF_2(\lambda)}{d\lambda} = AF_2(\lambda) + F_2(\lambda) B . \quad (1.8) \]

Since a first order differential equation with specified initial conditions (as in Eq. (1.3)) has a unique solution, it follows that \( F_2(\lambda) = F_1(\lambda) \), which for \( \lambda = 1 \) is the identity that we were trying to prove.

**Problem 2: A Composite Operator (15 points)**

(a) The normal-ordered \( \phi^2 \) is

\[ \phi^2(x) : = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_q}} \]

\[ \times \left\{ a(\vec{p}) a(\vec{q}) e^{-i(p+q) \cdot x} + 2a^\dagger(\vec{p}) a(\vec{q}) e^{i(p-q) \cdot x} + a^\dagger(\vec{p}) a^\dagger(\vec{q}) e^{i(p+q) \cdot x} \right\} \]

Then

\[ \langle 0 \mid \phi^2(x) : \phi^2(y) : \mid 0 \rangle = \]

\[ \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_q}} \int \frac{d^3r}{(2\pi)^3} \frac{1}{\sqrt{2E_r}} \int \frac{d^3s}{(2\pi)^3} \frac{1}{\sqrt{2E_s}} \]

\[ \times \left\{ a(\vec{p}) a(\vec{q}) e^{-i(p+q) \cdot x} + 2a^\dagger(\vec{p}) a(\vec{q}) e^{i(p-q) \cdot x} + a^\dagger(\vec{p}) a^\dagger(\vec{q}) e^{i(p+q) \cdot x} \right\} \]

\[ \times \left\{ a(\vec{r}) a(\vec{s}) e^{-i(r+s) \cdot y} + 2a^\dagger(\vec{r}) a(\vec{s}) e^{i(r-q) \cdot y} + a^\dagger(\vec{r}) a^\dagger(\vec{s}) e^{i(r+s) \cdot y} \right\} \]

\[ = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_q}} \int \frac{d^3r}{(2\pi)^3} \frac{1}{\sqrt{2E_r}} \int \frac{d^3s}{(2\pi)^3} \frac{1}{\sqrt{2E_s}} \]

\[ \times \langle 0 \mid a(\vec{p}) a(\vec{q}) a^\dagger(\vec{r}) a^\dagger(\vec{s}) \mid 0 \rangle e^{-i(p+q) \cdot x + i(r+s) \cdot y} , \]

\[ (2.2) \]
since $a(\vec{p})|0\rangle = 0$ and $\langle 0|a^\dagger(\vec{p}) = 0$. Now use the commutation relations $[a(\vec{p}), a^\dagger(\vec{q})] = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q})$ to show

$$a(\vec{p}) a(\vec{q}) a^\dagger(\vec{r}) a^\dagger(\vec{s}) = (2\pi)^6 \delta^{(3)}(\vec{q} - \vec{r}) \delta^{(3)}(\vec{p} - \vec{s}) + (2\pi)^3 \delta^{(3)}(\vec{q} - \vec{r}) a^\dagger(\vec{s}) a(\vec{p})$$

$$+ (2\pi)^6 \delta^{(3)}(\vec{p} - \vec{r}) \delta^{(3)}(\vec{q} - \vec{s}) + (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{r}) a^\dagger(\vec{s}) a(\vec{q})$$

$$+ (2\pi)^3 \delta^{(3)}(\vec{q} - \vec{s}) a^\dagger(\vec{r}) a(\vec{p}) + a^\dagger(\vec{r}) a(\vec{p}) a^\dagger(\vec{s}) a(\vec{q}) ,$$

and hence,

$$\langle 0|a(\vec{p}) a(\vec{q}) a^\dagger(\vec{r}) a^\dagger(\vec{s})|0\rangle =$$

$$(2\pi)^6 \delta^{(3)}(\vec{r} - \vec{q}) \delta^{(3)}(\vec{p} - \vec{s}) + (2\pi)^6 \delta^{(3)}(\vec{p} - \vec{r}) \delta^{(3)}(\vec{s} - \vec{q}) .$$

This leads to

$$\langle 0| :\phi^2(x) : :\phi^2(y) :|0\rangle =$$

$$\int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{1}{4EpEq} \left\{ e^{-i(p+q)\cdot x + i(q+p)\cdot y} + e^{-i(p+q)\cdot x + i(p+q)\cdot y} \right\}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{1}{2EpEq} e^{-ip\cdot (x-y)} e^{-iq\cdot (x-y)}$$

$$= 2 \left[ \int \frac{d^3p}{(2\pi)^3} \frac{1}{2Ep} e^{-ip\cdot (x-y)} \right]^2$$

$$= 2D^2(x - y) .$$

(b) We know from Problem Set 4 that for spacelike separations $D(x - y) = \frac{m}{4\pi^2r} K_1(mr)$ where $r^2 = -(x - y)^2$. Therefore,

$$\sigma^2 = 2 \int d^3x d^3y w(\vec{x}) w(\vec{y}) D^2(\vec{x} - \vec{y})$$

$$= 2 \int d^3x d^3y w(\vec{x}) w(\vec{y}) \frac{m^2}{16\pi^4r^2} K_1^2(mr) .$$

For small $r$, $K_1(r) \rightarrow \frac{1}{r}$ (see, for example, Jackson, p. 116). We see that the above integral diverges when $\vec{x} = \vec{y}$. 
\[ \langle 0 | O_4^2[w] | 0 \rangle = \left\{ \int d^4 x w(x^\mu) : \phi^2(x^\mu) : \right\}^2 \langle 0 | 0 \rangle \]

\[ \int d^4 x d^4 y w(x^\mu) w(y^\nu) \langle 0 | : \phi^2(x^\mu) : : \phi^2(y^\nu) : | 0 \rangle \]

\[ \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \int \frac{d^3 q}{(2\pi)^3} \frac{1}{\sqrt{2E_q}} \int \frac{d^3 r}{(2\pi)^3} \frac{1}{\sqrt{2E_r}} \int \frac{d^3 s}{(2\pi)^3} \frac{1}{\sqrt{2E_s}} \]

\[ \times \left\{ \right. \left\{ a(\vec{p}) a(\vec{q}) \tilde{w}(-p-q) + 2 a^\dagger(\vec{p}) a(\vec{q}) \tilde{w}(p-q) + a^\dagger(\vec{p}) a^\dagger(\vec{q}) \tilde{w}(p+q) \right\} \]

\[ \times \left\{ a(\vec{r}) a(\vec{s}) \tilde{w}(-r-s) + 2 a^\dagger(\vec{r}) a(\vec{s}) \tilde{w}(r-s) + a^\dagger(\vec{r}) a^\dagger(\vec{s}) \tilde{w}(r+s) \right\} \left| 0 \right\rangle \]

\[ \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \int \frac{d^3 q}{(2\pi)^3} \frac{1}{\sqrt{2E_q}} \int \frac{d^3 r}{(2\pi)^3} \frac{1}{\sqrt{2E_r}} \int \frac{d^3 s}{(2\pi)^3} \frac{1}{\sqrt{2E_s}} \]

\[ \times \langle 0 | a(\vec{p}) a(\vec{q}) a^\dagger(\vec{r}) a^\dagger(\vec{s}) | 0 \rangle \tilde{w}(-p-q) \tilde{w}(r+s) \]

\[ \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \int \frac{d^3 q}{(2\pi)^3} \frac{1}{\sqrt{2E_q}} \int \frac{d^3 r}{(2\pi)^3} \frac{1}{\sqrt{2E_r}} \int \frac{d^3 s}{(2\pi)^3} \frac{1}{\sqrt{2E_s}} \]

\[ \delta(3)(\vec{p} - \vec{r}) \delta(3)(\vec{q} - \vec{s}) + \delta(3)(\vec{p} - \vec{s}) \delta(3)(\vec{q} - \vec{r}) \tilde{w}(-p-q) \tilde{w}(r+s) \]
(f) Since \( \langle 0 \mid \phi(x) \mid 0 \rangle = 0 \), it makes sense try to construct a state with a negative expectation value by considering a small perturbation of the vacuum. If we write such a perturbation as

\[
|\psi\rangle = |0\rangle + \delta |\psi_1\rangle ,
\]

This is convergent since \( \tilde{w}(p) \) is assumed to be very well-behaved. Since \( w(x^\mu) \) is real, \( \tilde{w}(-p - q) = \tilde{w}^\ast(p + q) \), so the last line can be rewritten as
then
\[
\langle \psi | O_4[w] | \psi \rangle = \langle 0 | O_4[w] | 0 \rangle + \delta \left[ \langle 0 | O_4[w] | \psi_1 \rangle + \langle \psi_1 | O_4[w] | 0 \rangle \right]
\]
\[
= 2 \delta \operatorname{Re} \langle \psi_1 | O_4[w] | 0 \rangle .
\]

If we can find any state \( |\psi'_1\rangle \) for which \( \langle \psi'_1 | O_4[w] | 0 \rangle \neq 0 \), then we can arrange for the right-hand side of Eq. (2.16) to be negative by redefining the phase. That is, we can choose \( |\psi_1\rangle = |\psi'_1\rangle e^{i\theta} \), with \( \theta \) chosen to make \( \langle \psi_1 | O_4[w] | 0 \rangle \) real and negative. \( O_4[w] | 0 \rangle \) is clearly a two-particle state, so we can choose \( |\psi_1\rangle \) to be the simplest two-particle state that we know:
\[
|\psi_1\rangle = \hat{p}' \hat{q}' \ e^{i\theta} = 2 \sqrt{E_{p'} E_{q'}} \ a^{\dagger}(\hat{p}') a^{\dagger}(\hat{q}') | 0 \rangle e^{i\theta} . \tag{2.17}
\]

Then
\[
\langle \psi | O_4[w] | \psi \rangle = 2 \delta \operatorname{Re} \{ \langle \psi_1 | O_4[w] | 0 \rangle \}
\]
\[
= 2 \delta \operatorname{Re} \left\{ \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_q}} \times \langle \psi_1 | a(\hat{p}) a(\hat{q}) \bar{w}(-(p-q)) + 2 a^{\dagger}(\hat{p}) a(\hat{q}) \bar{w}(p-q) + 2 a^{\dagger}(\hat{p}) a^{\dagger}(\hat{q}) \bar{w}(p+q) | 0 \rangle \right\}
\]
\[
= 2 \delta \operatorname{Re} \left\{ e^{-i\theta} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_q}} 2 \sqrt{E_{p'} E_{q'}} \times \langle 0 | a(\hat{q}) a(\hat{p}) \{ a(\hat{p}) a(\hat{q}) \bar{w}(-(p-q)) + 2 a^{\dagger}(\hat{p}) a^{\dagger}(\hat{q}) \bar{w}(p+q) \} | 0 \rangle \right\}
\]
\[
= 2 \delta \operatorname{Re} \left\{ e^{-i\theta} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_q}} 2 \sqrt{E_{p'} E_{q'}} \times \langle 0 | \delta^{(3)}(\hat{q}' - \hat{p}) \delta^{(3)}(\hat{p}' - \hat{q}) + \delta^{(3)}(\hat{q}' - \hat{q}) \delta^{(3)}(\hat{p}' - \hat{p}) | 0 \rangle \bar{w}(p+q) \right\}
\]
\[
= 4 \delta \operatorname{Re} \{ e^{-i\theta} \bar{w}(p' + q') \}
\]
\[
= 4 \delta \cos \theta e^{-\frac{1}{2} b^2(E_{p'} + E_{q'})^2 - \frac{1}{4} a^2 |p'|^2 + |q'|^2} < 0, \tag{2.18}
\]
where we have the desired sign if we choose \( \theta = \pi \), so \( e^{i\theta} = -1 \).
Problem 3: Application of path integrals: The harmonic oscillator in thermal equilibrium (15 points)‡

(a) Clearly,

\[ \left[ a, a^\dagger \right] = \frac{1}{2\omega} [p - i\omega q, p + i\omega q] = \frac{i}{2} [p, q] - \frac{i}{2} [q, p] = 1. \]  

(3.1)

(b) Note that

\[ a^\dagger a = \frac{1}{2\omega} (p^2 + \omega^2 q^2 + i\omega [q, p]) \]

\[ \implies H = \left( a^\dagger a + \frac{1}{2} \right) \omega. \]  

(3.2)

It can be shown that the number operator \( N \equiv a^\dagger a \) can only have non-negative integer eigenvalues:

\[ N |n\rangle = n |n\rangle, \]  

(3.3)

where \( |n\rangle \) is the \( n \)th excited state. The energy levels are

\[ E_n = \left( n + \frac{1}{2} \right) \omega. \]  

(3.4)

Furthermore,

\[ q = \frac{i}{\sqrt{2\omega}} (a - a^\dagger), \]  

(3.5)

so

\[ \langle n | q^2 | n \rangle = -\frac{1}{2\omega} \langle n | (a - a^\dagger)(a - a^\dagger) | n \rangle \]

\[ = \frac{1}{2\omega} \langle n | a a^\dagger + a^\dagger a | n \rangle \]

\[ = \frac{1}{2\omega} \langle n | 2N + 1 | n \rangle = \frac{1}{\omega} \left( n + \frac{1}{2} \right). \]  

(3.6)

This result also follows from the virial theorem.

(c) Consider first the partition function \( Z \),

\[ Z = \sum_{n=0}^{\infty} e^{-\beta E_n} = e^{-\beta\omega/2} \sum_{n=0}^{\infty} (e^{-\beta\omega})^n = \frac{e^{-\beta\omega/2}}{1 - e^{-\beta\omega}} \]

\[ = \frac{1}{2 \sinh(\beta\omega/2)}. \]  

(3.7)
Now,
\[
\langle q^2 \rangle = \frac{1}{Z} \sum_{n=0}^{\infty} \frac{1}{\omega} \left( n + \frac{1}{2} \right) e^{-\beta \omega (n + \frac{1}{2})} = -\frac{1}{\omega^2} \frac{\partial \ln Z}{\partial \beta}
\]
\[= \frac{\coth(\beta \omega/2)}{2 \omega}. \tag{3.8}\]

(d) The Schrödinger wave function may be written \( \psi_n(\bar{q}) = \langle \bar{q} | n \rangle \), so
\[
P(\bar{q}) = \frac{1}{Z} \sum_{n=0}^{\infty} e^{-\beta E_n} \langle \bar{q} | n \rangle \langle n | \bar{q} \rangle
\]
\[= \frac{1}{Z} \sum_{n=0}^{\infty} \langle \bar{q} | e^{-\beta H} | n \rangle \langle n | \bar{q} \rangle \tag{3.9}\]
\[= \frac{1}{Z} \langle \bar{q} | e^{-\beta H} | \bar{q} \rangle,
\]
where we have used the completeness relation \( \sum_{n=0}^{\infty} | n \rangle \langle n | = I \).

(e) The variation of the action gives
\[
\delta S_E[q(\tau)] = \int_{-\beta/2}^{\beta/2} d\tau \left( dq \frac{d\delta q}{d\tau} + \omega^2 q \delta q \right)
\]
\[= \int_{-\beta/2}^{\beta/2} d\tau \delta q \left( -\frac{d^2 q}{d\tau^2} + \omega^2 q \right), \tag{3.10}\]

where we have integrated by parts and dropped the boundary term because the variation \( \delta q \) is necessarily fixed at \( \tau = \pm \beta/2 \).

The classical equations of motion result from extremizing the action,
\[
\frac{d^2 q_{cl}}{d\tau^2} - \omega^2 q_{cl} = 0, \tag{3.11}\]
with boundary conditions \( q_{cl}(\pm \beta/2) = \bar{q} \). The general solution to Eq. (3.11) is
\[
q_{cl} = A \sinh(\omega \tau) + B \cosh(\omega \tau), \tag{3.12}\]
and the boundary conditions give
\[
A = 0, \ B = \frac{\bar{q}}{\cosh(\beta \tau/2)} \]
\[\implies q_{cl} = \bar{q} \frac{\cosh(\omega \tau)}{\cosh(\beta \omega/2)}. \tag{3.13}\]
(f) With \( q(\tau) = q_{cl}(\tau) + q'(\tau) \) the action splits up as
\[
S_E[q(\tau)] = S_E[q_{cl}(\tau)] + S_E[q'(\tau)] + \int_{-\beta/2}^{\beta/2} d\tau \left( \frac{dq'}{d\tau} \frac{dq_{cl}}{d\tau} + \omega^2 q'q_{cl} \right) \\
= S_E[q_{cl}(\tau)] + S_E[q'(\tau)] + \int_{-\beta/2}^{\beta/2} d\tau q'(\tau) \left( -\frac{d^2 q_{cl}}{d\tau^2} + \omega^2 q_{cl} \right) \tag{3.14}
\]
where we integrated by parts and dropped the boundary term because \( q'(\pm \beta/2) = q(\pm \beta/2) - q_{cl}(\pm \beta/2) = 0 \). The cross term in Eq. (3.14) vanishes because \( q_{cl} \) satisfies the equations of motion.

(g) We can pull out \( S_E[q_{cl}(\tau)] \) from the path integral as it does not depend on any of the integration variables \( q'(\tau) \), so
\[
P(\bar{q}) \propto e^{-S_E[q_{cl}(\tau)]} \int_{\tau=-\beta/2}^{\tau=\beta/2} Dq'(\tau) \left| \begin{array}{c} q'(\beta/2) = 0 \\ q'(-\beta/2) = 0 \end{array} \right. e^{-S_E[q'(\tau)]} . \tag{3.15}
\]
The remaining path integral over \( q'(\tau) \) does not depend on \( \bar{q} \) so we may absorb it into the proportionality constant. Thus
\[
P(\bar{q}) \propto e^{-S_E[q_{cl}(\tau)]} , \tag{3.16}
\]
where
\[
S_E[q_{cl}(\tau)] = \frac{\bar{q}^2 \omega^2}{2 \cosh^2(\beta\omega/2)} \int_{-\beta/2}^{\beta/2} d\tau \left( \sinh^2(\omega\tau) + \cosh^2(\omega\tau) \right) \\
= \frac{\bar{q}^2 \omega^2}{2 \cosh^2(\beta\omega/2)} \int_{-\beta/2}^{\beta/2} d\tau \cosh(2\omega\tau) = \frac{\bar{q}^2 \omega \sinh(\beta\omega)}{2 \cosh^2(\beta\omega/2)} \tag{3.17}
\]
So the distribution is proportional to a Gaussian with mean zero and standard deviation \( \sigma \), with
\[
\sigma^2 = \frac{1}{2\omega} \coth(\beta\omega/2) . \tag{3.18}
\]
The correctly normalized probability distribution must be the standard Gaussian distribution,
\[
P(\bar{q}) = \frac{1}{\sqrt{2\pi} \sigma} e^{-\bar{q}^2/(2\sigma^2)} . \tag{3.19}
\]
Because the mean is zero, $\langle q^2 \rangle$ is simply equal to the variance $\sigma^2$ which is the same as the result in Eq. (3.8).

† Solution written by Alan Guth.
¶ Solution written Joydip Kundu, Tom Faulkner, and Alan Guth.
‡ Solution written by Tom Faulkner.