MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

8.323: Relativistic Quantum Field Theory I

Prof. Alan Guth April 12, 2008

PROBLEM SET 6 SOLUTIONS

Problem 1: Proof of $e^A e^B = e^{A+B+\frac{1}{2}[A\,,B]}$ *(10 points)*†

Following the suggestion, we consider the modified form of the identity

$$e^{\lambda A}\, e^{\lambda B} = e^{\lambda A+\lambda B+\frac{1}{2}\lambda^2[A\,,B]} \ , \tag{1.1}$$

which is obtained from the original identity by replacing A by λA and B by λB. We define

$$\begin{aligned} F_1(\lambda) &\equiv e^{\lambda A}\, e^{\lambda B} \ , \\ F_2(\lambda) &\equiv e^{\lambda A+\lambda B+\frac{1}{2}\lambda^2[A\,,B]} \ , \end{aligned} \tag{1.2}$$

and then our goal is to prove that $F_1(\lambda) = F_2(\lambda)$. Clearly

$$F_1(0) = F_2(0) = I \ , \tag{1.3}$$

where I is the identity operator, and

$$\frac{\mathrm{d}F_1(\lambda)}{\mathrm{d}\lambda} = A\,F_1(\lambda) + F_1(\lambda)\,B \ . \tag{1.4}$$

Now all that is needed is to show that $F_2(\lambda)$ satisfies the same differential equation.

Taking advantage of the fact that $[A,\, B]$ commutes with all the operators mentioned in this problem, we can write

$$\begin{aligned} \frac{\mathrm{d}F_2(\lambda)}{\mathrm{d}\lambda} &= \frac{\mathrm{d}}{\mathrm{d}\lambda}\left\{ e^{\lambda(A+B)}\, e^{\frac{1}{2}\lambda^2[A\,,B]}\right\} \\ &= \{(A+B) + \lambda\,[A,\, B]\}\; e^{\lambda(A+B)}\, e^{\frac{1}{2}\lambda^2[A\,,B]} \\ &= AF_2(\lambda) + \lambda\,[A,\, B]\,F_2(\lambda) + Be^{\lambda(A+B)}\, e^{\frac{1}{2}\lambda^2[A\,,B]} \\ &= AF_2(\lambda) + \lambda\,[A,\, B]\,F_2(\lambda) + e^{\lambda(A+B)}\, e^{-\lambda(A+B)}\, Be^{\lambda(A+B)}\, e^{\frac{1}{2}\lambda^2[A\,,B]} \ . \end{aligned} \tag{1.5}$$

Now we make use of the identity proven in Problem 5 of Problem Set 2, which was written there as

$$e^A B e^{-A} = B + [A,\, B] + \frac{1}{2}\,[A,\, [A,\, B]] + \frac{1}{3!}\,[A,\, [A,\, [A,\, B]]] + \dots \ . \tag{1.6}$$

Applying this identity to the 2nd term on the right-hand side of Eq. (1.5),

$$\begin{aligned} e^{-\lambda(A+B)}\, B e^{\lambda(A+B)} &= B - \lambda\left[(A+B\,,\,B)\right] + \frac{\lambda^2}{2}\left[A+B\,,\,[(A+B)\,,\,B]\right] + \ldots \\ &= B - \lambda\left[(A\,,\,B)\right]\ , \end{aligned} \tag{1.7}$$

where the series terminates because $[A\,,\,B]$ commutes with the other operators. Substituting Eq. (1.7) into Eq. (1.5), one finds the desired result:

$$\frac{\mathrm{d}F_2(\lambda)}{\mathrm{d}\lambda} = AF_2(\lambda) + F_2(\lambda)\, B\ . \tag{1.8}$$

Since a first order differential equation with specified initial conditions (as in Eq. (1.3)) has a unique solution, it follows that $F_2(\lambda) = F_1(\lambda)$, which for $\lambda = 1$ is the identity that we were trying to prove.

Problem 2: A Composite Operator *(15 points)*¶

(a) The normal-ordered ϕ^2 is

$$\begin{aligned} :\phi^2(x):\ \equiv \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}} \int \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_q}} \\ \times \left\{ a(\vec{p})\,a(\vec{q}\,)\,e^{-i(p+q)\cdot x} + 2a^\dagger(\vec{p})\,a(\vec{q}\,)\,e^{i(p-q)\cdot x} + a^\dagger(\vec{p})\,a^\dagger(\vec{q}\,)\,e^{i(p+q)\cdot x} \right\} \end{aligned} \tag{2.1}$$

Then

$$\begin{aligned} &\left\langle 0\,\middle|\,:\phi^2(x):\,:\phi^2(y):\,\middle|\,0\right\rangle = \\ &\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}} \int \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_q}} \int \frac{d^3r}{(2\pi)^3}\frac{1}{\sqrt{2E_r}} \int \frac{d^3s}{(2\pi)^3}\frac{1}{\sqrt{2E_s}} \\ &\quad\times \left\langle 0\,\middle|\left\{ a(\vec{p})\,a(\vec{q}\,)\,e^{-i(p+q)\cdot x} + 2a^\dagger(\vec{p})\,a(\vec{q}\,)\,e^{i(p-q)\cdot x} + a^\dagger(\vec{p})\,a^\dagger(\vec{q}\,)\,e^{i(p+q)\cdot x} \right\}\right. \\ &\quad\quad\times \left.\left\{ a(\vec{r})\,a(\vec{s})\,e^{-i(r+s)\cdot y} + 2a^\dagger(\vec{r})\,a(\vec{s})\,e^{i(r-s)\cdot y} + a^\dagger(\vec{r})\,a^\dagger(\vec{s})\,e^{i(r+s)\cdot y} \right\}\,\middle|\,0\right\rangle \\ &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}} \int \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_q}} \int \frac{d^3r}{(2\pi)^3}\frac{1}{\sqrt{2E_r}} \int \frac{d^3s}{(2\pi)^3}\frac{1}{\sqrt{2E_s}} \\ &\quad\times \left\langle 0\,\middle|\,a(\vec{p})\,a(\vec{q}\,)\,a^\dagger(\vec{r})\,a^\dagger(\vec{s})\,\middle|\,0\right\rangle\ e^{-i(p+q)\cdot x + i(r+s)\cdot y}\ , \end{aligned} \tag{2.2}$$

since $a(\vec{p})|0\rangle = 0$ and $\langle 0|a^\dagger(\vec{p}) = 0$. Now use the commutation relations $[a(\vec{p}), a^\dagger(\vec{q})] = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q})$ to show

$$\begin{aligned}
a(\vec{p})\, a(\vec{q})\, a^\dagger(\vec{r})\, a^\dagger(\vec{s}) = {} & (2\pi)^6\, \delta^{(3)}(\vec{q} - \vec{r})\, \delta^{(3)}(\vec{p} - \vec{s}) + (2\pi)^3\, \delta^{(3)}(\vec{q} - \vec{r})\, a^\dagger(\vec{s})\, a(\vec{p}) \\
& + (2\pi)^6\, \delta^{(3)}(\vec{p} - \vec{r})\, \delta^{(3)}(\vec{q} - \vec{s}) + (2\pi)^3\, \delta^{(3)}(\vec{p} - \vec{r})\, a^\dagger(\vec{s})\, a(\vec{q}) \\
& + (2\pi)^3\, \delta^{(3)}(\vec{q} - \vec{s})\, a^\dagger(\vec{r})\, a(\vec{p}) + a^\dagger(\vec{r})\, a(\vec{p})\, a^\dagger(\vec{s})\, a(\vec{q}) \ ,
\end{aligned} \tag{2.3}$$

and hence,

$$\begin{aligned}
& \left\langle 0 \left| a(\vec{p})\, a(\vec{q})\, a^\dagger(\vec{r})\, a^\dagger(\vec{s}) \right| 0 \right\rangle = \\
& \quad (2\pi)^6\, \delta^{(3)}(\vec{r} - \vec{q})\, \delta^{(3)}(\vec{p} - \vec{s}) + (2\pi)^6\, \delta^{(3)}(\vec{p} - \vec{r})\, \delta^{(3)}(\vec{s} - \vec{q}) \ .
\end{aligned} \tag{2.4}$$

This leads to

$$\begin{aligned}
& \left\langle 0 \left| :\phi^2(x): \, :\phi^2(y): \right| 0 \right\rangle = \\
& \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 q}{(2\pi)^3} \frac{1}{4 E_p E_q} \left\{ e^{-i(p+q)\cdot x + i(q+p)\cdot y} + e^{-i(p+q)\cdot x + i(p+q)\cdot y} \right\} \\
& = \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 q}{(2\pi)^3} \frac{1}{2 E_p E_q}\, e^{-ip\cdot(x-y)} e^{-iq\cdot(x-y)} \\
& = 2 \left[\int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p}\, e^{-ip\cdot(x-y)} \right]^2 \\
& = 2 D^2(x - y) \ .
\end{aligned} \tag{2.5}$$

(b) We know from Problem Set 4 that for spacelike separations $D(x-y) = \frac{m}{4\pi^2 r} K_1(mr)$ where $r^2 = -(x-y)^2$. Therefore,

$$\begin{aligned}
\sigma^2 & = 2 \int d^3x\, d^3y\, w(\vec{x})\, w(\vec{y}) D^2(\vec{x} - \vec{y}) \\
& = 2 \int d^3x\, d^3y\, w(\vec{x})\, w(\vec{y}) \frac{m^2}{16\pi^4 r^2}\, K_1^2(mr) \ .
\end{aligned} \tag{2.6}$$

For small r, $K_1(r) \to \frac{1}{r}$ (see, for example, Jackson, p. 116). We see that the above integral diverges when $\vec{x} = \vec{y}$.

(c)

$$
\begin{aligned}
O_4[w] &\equiv \int d^4x\, w(x^\mu) : \phi^2(x^\mu) : \\
&= \int d^4x\, w(x^\mu) \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_q}} \\
&\quad \times \left\{ a(\vec{p})\, a(\vec{q})\, e^{-i(p+q)\cdot x} + 2\, a^\dagger(\vec{p})\, a(\vec{q})\, e^{i(p-q)\cdot x} + a^\dagger(\vec{p})\, a^\dagger(\vec{q})\, e^{i(p+q)\cdot x} \right\} \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_q}} \left\{ a(\vec{p})\, a(\vec{q}) \int d^4x\, w(x^\mu)\, e^{i(-p-q)\cdot x} \right. \\
&\quad \left. + \, 2\, a^\dagger(\vec{p})\, a(\vec{q}) \int d^4x\, w(x^\mu)\, e^{i(p-q)\cdot x} + a^\dagger(\vec{p})\, a^\dagger(\vec{q}) \int d^4x\, w(x^\mu)\, e^{i(p+q)\cdot x} \right\} \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_q}} \\
&\quad \times \left\{ a(\vec{p})\, a(\vec{q})\, \tilde{w}(-p-q) + 2\, a^\dagger(\vec{p})\, a(\vec{q})\, \tilde{w}(p-q) + a^\dagger(\vec{p})\, a^\dagger(\vec{q})\, \tilde{w}(p+q) \right\} .
\end{aligned}
\tag{2.7}
$$

(d)

$$
\begin{aligned}
\left\langle 0 \left| O_4^2[w] \right| 0 \right\rangle &= \left\langle 0 \left| \left[\int d^4x\, w(x^\mu) : \phi^2(x^\mu) : \right]^2 \right| 0 \right\rangle \\
&= \int d^4x\, d^4y\, w(x^\mu)\, w(y^\nu) \, \left\langle 0 \left| : \phi^2(x^\mu) : \, : \phi^2(y^\nu) : \right| 0 \right\rangle \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_q}} \int \frac{d^3r}{(2\pi)^3} \frac{1}{\sqrt{2E_r}} \int \frac{d^3s}{(2\pi)^3} \frac{1}{\sqrt{2E_s}} \\
&\quad \times \left\langle 0 \left| \left\{ a(\vec{p})\, a(\vec{q})\, \tilde{w}(-p-q) + 2\, a^\dagger(\vec{p})\, a(\vec{q})\, \tilde{w}(p-q) + a^\dagger(\vec{p})\, a^\dagger(\vec{q})\, \tilde{w}(p+q) \right\} \right. \right. \\
&\qquad \times \left. \left. \left\{ a(\vec{r})\, a(\vec{s})\, \tilde{w}(-r-s) + 2\, a^\dagger(\vec{r})\, a(\vec{s})\, \tilde{w}(r-s) + a^\dagger(\vec{r})\, a^\dagger(\vec{s})\, \tilde{w}(r+s) \right\} \right| 0 \right\rangle \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_q}} \int \frac{d^3r}{(2\pi)^3} \frac{1}{\sqrt{2E_r}} \int \frac{d^3s}{(2\pi)^3} \frac{1}{\sqrt{2E_s}} \\
&\qquad \times \left\langle 0 \left| a(\vec{p})\, a(\vec{q})\, a^\dagger(\vec{r})\, a^\dagger(\vec{s}) \right| 0 \right\rangle \, \tilde{w}(-p-q)\, \tilde{w}(r+s) \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_q}} \int \frac{d^3r}{(2\pi)^3} \frac{1}{\sqrt{2E_r}} \int \frac{d^3s}{(2\pi)^3} \frac{1}{\sqrt{2E_s}} \, (2\pi)^6 \times \\
&\qquad \left\{ \delta^{(3)}(\vec{p}-\vec{r})\, \delta^{(3)}(\vec{q}-\vec{s}) + \delta^{(3)}(\vec{p}-\vec{s})\, \delta^{(3)}(\vec{q}-\vec{r}) \right\} \tilde{w}(-p-q)\, \tilde{w}(r+s)
\end{aligned}
$$

$$= \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \frac{1}{4E_pE_q} \{\tilde{w}(-p-q)\,\tilde{w}(p+q) + \tilde{w}(-p-q)\,\tilde{w}(q+p)\}$$

$$= \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \frac{1}{2E_pE_q} \tilde{w}(p+q)\,\tilde{w}(-p-q)\ . \tag{2.8}$$

This is convergent since $\tilde{w}(p)$ is assumed to be very well-behaved. Since $w(x^\mu)$ is real, $\tilde{w}(-p-q) = \tilde{w}^*(p+q)$, so the last line can be rewritten as

$$\left\langle 0 \left| O_4^2[w] \right| 0 \right\rangle = \frac{2}{(2\pi)^6} \int \frac{\mathrm{d}^3p}{2E_p} \int \frac{\mathrm{d}^3q}{2E_q} \left|\tilde{w}(p+q)\right|^2\ . \tag{2.9}$$

(e) (This argument is due to Lilian Childress.)

$$w(x^\mu) = \frac{1}{\pi^{3/2}a^3} e^{-|\vec{x}|^2/a^2} \frac{1}{\sqrt{\pi}b} e^{-(x^0)^2/b^2} \tag{2.10}$$

Then

$$\tilde{w}(p^\mu) = \int d^4x\, w(x^\mu) e^{i(p^0x^0 - \vec{p}\cdot\vec{x})} = e^{-\frac{1}{4}b^2(p^0)^2 - \frac{1}{4}a^2|\vec{p}|^2}. \tag{2.11}$$

So

$$\begin{aligned}\left\langle 0 \left| O_4^2[w] \right| 0 \right\rangle &= \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \frac{1}{2E_pE_q} \tilde{w}(p+q)\,\tilde{w}(-p-q) \\ &= \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \frac{1}{2E_pE_q}\, e^{-\frac{1}{2}b^2(E_p+E_q)^2 - \frac{1}{2}a^2|\vec{p}+\vec{q}|^2}\ .\end{aligned} \tag{2.12}$$

As $a \to \infty$,

$$e^{-\frac{1}{2}a^2|\vec{p}+\vec{q}|^2} \to \frac{1}{a^3}(2\pi)^{3/2}\,\delta^{(3)}(\vec{p}+\vec{q}). \tag{2.13}$$

Thus,

$$\left\langle 0 \left| O_4^2[w] \right| 0 \right\rangle \to \frac{1}{a^3} \int \frac{d^3p}{(2\pi)^{3/2}} \frac{1}{2(E_p)^2}\, e^{-\frac{1}{2}b^2(2E_p)^2}. \tag{2.14}$$

We see that $\left\langle 0 \left| O_4^2[w] \right| 0 \right\rangle$ goes like a^β where $\beta = -3$, in agreement with our result from Problem Set 3. In the limit of large a the variance of O_4 samples (averages!) many independent values of $:\phi:$ so one would expect the variance to go as $1/N$ where N is the number of independent sample points. N is proportional to the volume $N \sim a^3$ and the result is as expected.

(f) Since $\left\langle 0 \left| :\phi^2(x): \right| 0 \right\rangle = 0$, it makes sense try to construct a state with a negative expectation value by considering a small perturbation of the vacuum. If we write such a perturbation as

$$|\psi\rangle = |0\rangle + \delta\,|\psi_1\rangle\ , \tag{2.15}$$

then

$$\begin{aligned} \langle\psi\,|O_4[w]|\,\psi\rangle &= \langle 0\,|O_4[w]|\,0\rangle + \delta\Big[\langle 0\,|O_4[w]|\,\psi_1\rangle + \langle\psi_1\,|O_4[w]|\,0\rangle\Big] \\ &= 2\,\delta\,\mathrm{Re}\,\langle\psi_1\,|O_4[w]|\,0\rangle\ . \end{aligned} \tag{2.16}$$

If we can find any state $|\psi_1'\rangle$ for which $\langle\psi_1'\,|O_4[w]|\,0\rangle \neq 0$, then we can arrange for the right-hand side of Eq. (2.16) to be negative by redefining the phase. That is, we can choose $|\psi_1\rangle = |\psi_1'\rangle\,e^{i\theta}$, with θ chosen to make $\langle\psi_1\,|O_4[w]|\,0\rangle$ real and negative. $O_4[w]\,|0\rangle$ is clearly a two-particle state, so we can choose $|\psi_1\rangle$ to be the simplest two-particle state that we know:

$$|\psi_1\rangle = |\vec{p}\,', \vec{q}\,'\rangle\,e^{i\theta} = 2\sqrt{E_{p'}\,E_{q'}}\;a^\dagger(\vec{p}\,')\,a^\dagger(\vec{q}\,')\,|0\rangle\,e^{i\theta}\ . \tag{2.17}$$

Then

$$\begin{aligned} \langle\psi\,|O_4[w]|\,\psi\rangle &= 2\,\delta\,\mathrm{Re}\,\{\langle\psi_1\,|O_4[w]|\,0\rangle\} \\ &= 2\,\delta\,\mathrm{Re}\left\{\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\int\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_q}}\right. \\ &\quad\left.\times\ \left\langle\psi_1\,\Big|a(\vec{p})\,a(\vec{q})\,\tilde{w}(-p-q) + 2\,a^\dagger(\vec{p})\,a(\vec{q})\,\tilde{w}(p-q) + a^\dagger(\vec{p})\,a^\dagger(\vec{q})\,\tilde{w}(p+q)\Big|\,0\right\rangle\right\} \\ &= 2\,\delta\,\mathrm{Re}\left\{e^{-i\theta}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\int\frac{\mathrm{d}^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_q}}2\sqrt{E_{p'}\,E_{q'}}\right. \\ &\quad\times\ \Big\langle 0\,\Big|a(\vec{q'})\,a(\vec{p'})\,\{a(\vec{p})\,a(\vec{q})\,\tilde{w}(-p-q) \\ &\qquad\left. +2\,a^\dagger(\vec{p})\,a(\vec{q})\,\tilde{w}(p-q) + a^\dagger(\vec{p})\,a^\dagger(\vec{q})\,\tilde{w}(p+q)\}\,\big|\,0\big\rangle\right\} \\ &= 2\,\delta\,\mathrm{Re}\left\{e^{-i\theta}\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\int\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_q}}2\sqrt{E_{p'}\,E_{q'}}\right. \\ &\quad\left.\times\ \Big\langle 0\,\Big|a(\vec{q'})\,a(\vec{p'})\,a^\dagger(\vec{p})\,a^\dagger(\vec{q})\Big|\,0\Big\rangle\ \tilde{w}(p+q)\right\} \\ &= 2\,\delta\,\mathrm{Re}\left\{e^{-i\theta}\int d^3p\int d^3q\right. \\ &\quad\left.\times\ \Big\langle 0\,\Big|\delta^{(3)}(\vec{q}\,'-\vec{p})\,\delta^{(3)}(\vec{p}\,'-\vec{q}) + \delta^{(3)}(\vec{q}\,'-\vec{q})\,\delta^{(3)}(\vec{p}\,'-\vec{p})\Big|\,0\Big\rangle\ \tilde{w}(p+q)\right\} \\ &= 4\,\delta\,\mathrm{Re}\,\{e^{-i\theta}\tilde{w}(p'+q')\} \\ &= 4\,\delta\,\cos\theta\,e^{-\frac{1}{4}b^2(E_{p'}+E_{q'})^2-\frac{1}{4}a^2|\vec{p'}+\vec{q'}|^2} < 0, \end{aligned} \tag{2.18}$$

where we have the desired sign if we choose $\theta = \pi$, so $e^{i\theta} = -1$.

Problem 3: Application of path integrals: The harmonic oscillator in thermal equilibrium *(15 points)*[‡]

(a) Clearly,

$$\left[a\,,\,a^\dagger\right] = \frac{1}{2\omega}[p - i\omega q, p + i\omega q] = \frac{i}{2}\,[p\,,\,q] - \frac{i}{2}\,[q\,,\,p] = 1\ . \tag{3.1}$$

(b) Note that

$$\begin{aligned} a^\dagger a &= \frac{1}{2\omega}\left(p^2 + \omega^2 q^2 + i\omega\,[q\,,\,p]\right) \\ \Longrightarrow\quad H &= \left(a^\dagger a + \frac{1}{2}\right)\omega\ . \end{aligned} \tag{3.2}$$

It can be shown that the number operator $N \equiv a^\dagger a$ can only have non-negative integer eigenvalues:

$$N\,|n\rangle = n\,|n\rangle\ \ , \tag{3.3}$$

where $|n\rangle$ is the nth excited state. The energy levels are

$$E_n = \left(n + \frac{1}{2}\right)\omega\ . \tag{3.4}$$

Furthermore,

$$q = \frac{i\left(a - a^\dagger\right)}{\sqrt{2\omega}}\ , \tag{3.5}$$

so

$$\begin{aligned} \langle n|\,q^2\,|n\rangle &= -\frac{1}{2\omega}\,\langle n|\,(a - a^\dagger)(a - a^\dagger)\,|n\rangle \\ &= \frac{1}{2\omega}\,\langle n|\,aa^\dagger + a^\dagger a\,|n\rangle \\ &= \frac{1}{2\omega}\,\langle n|\,2N + 1\,|n\rangle = \frac{1}{\omega}\left(n + \frac{1}{2}\right)\ . \end{aligned} \tag{3.6}$$

This result also follows from the virial theorem.

(c) Consider first the partition function Z,

$$\begin{aligned} Z &= \sum_{n=0}^{\infty} e^{-\beta E_n} = e^{-\beta\omega/2}\sum_{n=0}^{\infty}\left(e^{-\beta\omega}\right)^n = \frac{e^{-\beta\omega/2}}{1 - e^{-\beta\omega}} \\ &= \frac{1}{2\sinh(\beta\omega/2)}\ . \end{aligned} \tag{3.7}$$

Now,

$$\begin{aligned}\left\langle q^2\right\rangle &= \frac{1}{Z}\sum_{n=0}^{\infty}\frac{1}{\omega}\left(n+\frac{1}{2}\right)e^{-\beta\omega\left(n+\frac{1}{2}\right)} = -\frac{1}{\omega^2}\frac{\partial \ln Z}{\partial\beta} \\ &= \frac{\coth(\beta\omega/2)}{2\omega}\ .\end{aligned} \tag{3.8}$$

(d) The Schrödinger wave function may be written $\psi_n(\bar{q}) = \langle\bar{q}\,|n\,\rangle$, so

$$\begin{aligned}P(\bar{q}) &= \frac{1}{Z}\sum_{n=0}^{\infty}e^{-\beta E_n}\langle\bar{q}\,|n\,\rangle\langle n\,|\bar{q}\,\rangle \\ &= \frac{1}{Z}\sum_{n=0}^{\infty}\left\langle\bar{q}\,\middle|e^{-\beta H}\middle|\,n\right\rangle\langle n\,|\bar{q}\,\rangle \\ &= \frac{1}{Z}\left\langle\bar{q}\,\middle|e^{-\beta H}\middle|\,\bar{q}\right\rangle\ ,\end{aligned} \tag{3.9}$$

where we have used the completeness relation $\sum_{n=0}^{\infty}|n\rangle\langle n| = I$.

(e) The variation of the action gives

$$\begin{aligned}\delta S_E[q(\tau)] &= \int_{-\beta/2}^{\beta/2}\mathrm{d}\tau\left(\frac{dq}{d\tau}\frac{d\delta q}{d\tau}+\omega^2 q\delta q\right) \\ &= \int_{-\beta/2}^{\beta/2}d\tau\,\delta q\left(-\frac{d^2q}{d\tau^2}+\omega^2 q\right)\ ,\end{aligned} \tag{3.10}$$

where we have integrated by parts and dropped the boundary term because the variation δq is necessarily fixed at $\tau = \pm\beta/2$.

The classical equations of motion result from extremizing the action,

$$\frac{d^2q_{\rm cl}}{d\tau^2} - \omega^2 q_{\rm cl} = 0\ , \tag{3.11}$$

with boundary conditions $q_{\rm cl}(\pm\beta/2) = \bar{q}$. The general solution to Eq. (3.11) is

$$q_{\rm cl} = A\sinh(\omega\tau) + B\cosh(\omega\tau)\ , \tag{3.12}$$

and the boundary conditions give

$$\begin{aligned}&A = 0\ ,\ B = \frac{\bar{q}}{\cosh(\beta\tau/2)} \\ &\Longrightarrow\quad q_{\rm cl} = \bar{q}\,\frac{\cosh(\omega\tau)}{\cosh(\beta\omega/2)}\ .\end{aligned} \tag{3.13}$$

(f) With $q(\tau) = q_{\rm cl}(\tau) + q'(\tau)$ the action splits up as

$$\begin{aligned} S_E[q(\tau)] &= S_E[q_{\rm cl}(\tau)] + S_E[q'(\tau)] + \int_{-\beta/2}^{\beta/2} \mathrm{d}\tau \left(\frac{dq'}{d\tau}\frac{dq_{\rm cl}}{d\tau} + \omega^2 q' q_{\rm cl} \right) \\ &= S_E[q_{\rm cl}(\tau)] + S_E[q'(\tau)] + \int_{-\beta/2}^{\beta/2} \mathrm{d}\tau\, q'(\tau) \left(-\frac{d^2 q_{\rm cl}}{d\tau^2} + \omega^2 q_{\rm cl} \right) \\ &= S_E[q_{\rm cl}(\tau)] + S_E[q'(\tau)] \ , \end{aligned} \tag{3.14}$$

where we integrated by parts and dropped the boundary term because $q'(\pm\beta/2) = q(\pm\beta/2) - q_{\rm cl}(\pm\beta/2) = 0$. The cross term in Eq. (3.14) vanishes because $q_{\rm cl}$ satisfies the equations of motion.

(g) We can pull out $S_E[q_{\rm cl}(\tau)]$ from the path integral as it does not depend on any of the integration variables $q'(\tau)$, so

$$P(\bar{q}) \propto e^{-S_E[q_{\rm cl}(\tau)]} \int_{\tau=-\beta/2}^{\tau=\beta/2} \mathcal{D}q'(\tau) \Bigg|_{\substack{q'(\beta/2)=0 \\ q'(-\beta/2)=0}} e^{-S_E[q'(\tau)]} \ . \tag{3.15}$$

The remaining path integral over $q'(\tau)$ does not depend on $\bar{q}$ so we may absorb it into the proportionality constant. Thus

$$P(\bar{q}) \propto e^{-S_E[q_{\rm cl}(\tau)]} \ , \tag{3.16}$$

where

$$\begin{aligned} S_E[q_{\rm cl}(\tau)] &= \frac{\bar{q}^2\omega^2}{2\cosh^2(\beta\omega/2)} \int_{-\beta/2}^{\beta/2} \mathrm{d}\tau \left(\sinh^2(\omega\tau) + \cosh^2(\omega\tau) \right) \\ &= \frac{\bar{q}^2\omega^2}{2\cosh^2(\beta\omega/2)} \int_{-\beta/2}^{\beta/2} \mathrm{d}\tau \cosh(2\omega\tau) = \frac{\bar{q}^2\omega\sinh(\beta\omega)}{2\cosh^2(\beta\omega/2)} \\ &= \frac{\bar{q}^2\omega}{\coth(\beta\omega/2)} \ . \end{aligned} \tag{3.17}$$

So the distribution is proportional to a Gaussian with mean zero and standard deviation σ, with

$$\sigma^2 = \frac{1}{2\omega} \coth(\beta\omega/2) \ . \tag{3.18}$$

The correctly normalized probability distribution must be the standard Gaussian distribution,

$$P(\bar{q}) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\bar{q}^2/(2\sigma^2)} \ . \tag{3.19}$$

Because the mean is zero, $\left\langle \bar{q}^2 \right\rangle$ is simply equal to the variance σ^2 which is the same as the result in Eq. (3.8).

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†Solution written by Alan Guth.

¶Solution written Joydip Kundu, Tom Faulkner, and Alan Guth.

‡Solution written by Tom Faulkner.