MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

8.323: Relativistic Quantum Field Theory I

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PROBLEM SET 7 SOLUTIONS

Problem 1: Particle production by a classical source *(10 points)*†

(a) Denote the complement of Ω in momentum space by $\bar{\Omega}$. We want to compute the probability $P_\Omega(N)$ of N particles being found in Ω ignoring how many are present in $\bar{\Omega}$. Given the probability $P_\Omega(N,M)$ of finding N particles in Ω and M particles in $\bar{\Omega}$, the required probability will be obtained as:

$$P_\Omega(N) = \sum_{M=0}^{\infty} P_\Omega(N,M) \ . \tag{1.1}$$

The first step is therefore to obtain $P_\Omega(N,M)$. This can be written in the following way:

$$P_\Omega(N,M) = \frac{1}{C(N,M)} \int_\Omega \prod_{i=1}^{N} \frac{d^3\vec{p}_i}{(2\pi)^3 2E_{\vec{p}_i}} \int_{\bar{\Omega}} \prod_{\ell=1}^{M} \frac{d^3\vec{q}_\ell}{(2\pi)^3 2E_{\vec{q}_\ell}} \left|\langle \vec{p}_1,...,\vec{p}_N,\vec{q}_1,...,\vec{q}_M,\text{out}|0,\text{in}\rangle\right|^2 \tag{1.2}$$

where $C(N,M)$ is a combinatorial factor counting how many times we sum over the same state in doing the integral. This is easily computed by noticing that given the state

$$\langle \vec{p}_1,...,\vec{p}_N,\vec{q}_1,...,\vec{q}_M,\text{out}| \ ,$$

we also integrate over

$$\langle \vec{p}_{\sigma(1)},...,\vec{p}_{\sigma(N)},\vec{q}_{\tau(1)},...,\vec{q}_{\tau(M)},\text{out}|$$

(where σ and τ are permutations of N and M objects respectively), which represents the same state. Therefore $C(N,M)$ counts the number of the possible permutations, so $C(N,M) = N!M!$.

We thus obtain:

$$P_\Omega(N,M) = \frac{1}{N!M!} e^{-\lambda} \int_\Omega \prod_{i=1}^{N} \frac{d^3\vec{p}_i}{(2\pi)^3 2E_{\vec{p}_i}} |\tilde{\jmath}(\vec{p}_i)|^2 \int_{\bar{\Omega}} \prod_{\ell=1}^{M} \frac{d^3\vec{q}_\ell}{(2\pi)^3 2E_{\vec{q}_\ell}} |\tilde{\jmath}(\vec{q}_\ell)|^2 \tag{1.3}$$

Therefore by defining $\lambda_\Omega \equiv \int_\Omega \frac{d^3\vec{p}}{(2\pi)^3 2E_{\vec{p}}} |\tilde{\jmath}(\vec{p})|^2$ and $\lambda_{\bar{\Omega}} \equiv \int_{\bar{\Omega}} \frac{d^3\vec{p}}{(2\pi)^3 2E_{\vec{p}}} |\tilde{\jmath}(\vec{p})|^2$ we get:

$$P_\Omega(N,M) = \frac{1}{N!M!} e^{-\lambda} \lambda_\Omega^N \lambda_{\bar{\Omega}}^M \ , \tag{1.4}$$

and finally:

$$\boxed{P_\Omega(N) = \sum_{M=0}^{\infty} P_\Omega(N,M) = e^{-\lambda+\lambda_{\bar{\Omega}}} \frac{(\lambda_\Omega)^N}{N!} = e^{-\lambda_\Omega} \frac{(\lambda_\Omega)^N}{N!} \ ,} \tag{1.5}$$

which is the same as the probability of getting N particles in $\Omega \bigcup \bar{\Omega}$ except for the replacement of λ by λ_Ω. This is a Poisson probability distribution with mean value λ_Ω.

This result makes sense because the detector cannot distinguish between $\tilde{\jmath}(\vec{p})$ and $\chi_\Omega(\vec{p})\tilde{\jmath}(\vec{p})$, where $\chi_\Omega(\vec{p})$ is the characteristic function of Ω, as all momenta are independent. (The characteristic function $\chi_\Omega(\vec{p})$ is defined to be 1 if $\vec{p} \in \Omega$, and 0 otherwise.) In the second case there are no particles produced outside Ω and the probability for the detector to detect N particles is the probability for a total of N particles to be produced.

(b) We want to compute the probability $P_{d\Omega}(1|N)$ of finding a particle in $d^3\vec{p} = d\Omega$ given the fact that there are a total of N particles. Denote by $P(N)$ the probability of finding N particles in total. Then we have

$$P_{d\Omega}(1|N) = \frac{P_{d\Omega}(1,N-1)}{P(N)} \ , \tag{1.6}$$

where $P_{d\Omega}(1,N-1)$ is defined as in Eq. (1.1). We can then use Eq. (1.4) to write:

$$P_{d\Omega}(1,N-1) = e^{-\lambda} \frac{\lambda^{N-1}}{(N-1)!} |\tilde{\jmath}(\vec{p})|^2 \frac{d^3\vec{p}}{(2\pi)^3 2E_{\vec{p}}} \ , \tag{1.7}$$

$$P(N) = e^{-\lambda} \frac{\lambda^N}{N!} \ . \tag{1.8}$$

We thus obtain:

$$\boxed{P_{d\Omega}(1|N) = \frac{N}{\lambda} \frac{d^3\vec{p}}{(2\pi)^3 2E_{\vec{p}}} |\tilde{\jmath}(\vec{p})|^2 \ .} \tag{1.9}$$

This result makes sense because as N increases the probability of finding a particle has to increase: the probability of finding a particle if I know there are 10000 of them must be bigger than in the case where there are only 10! Also it makes sense that the probability goes to 0 for $N = 0$. In addition, as we will see in the next part, the fact that N is Poisson-distributed implies that the expectation value of N is $\langle N \rangle = \lambda$. Thus the expectation value of the above expression matches the answer that we already knew for the case where N is not observed. Finally, we can see from the definition of λ that the integral of this expression is N, so we should check to make sure that this makes sense. At first it may seem surprising that the integral of the probability function is not 1, but remember that we are not adding the probabilities of mutually exclusive alternatives. It is perfectly possible to find one particle in one interval $\mathrm{d}^3\vec{p}_1$ centered on a momentum $\vec{p}_1$, and another particle in an inteval $\mathrm{d}^3\vec{p}_2$ centered on momentum $\vec{p}_2$. Since $\mathrm{d}^3\vec{p}$ is infinitesimally small, we can rigorously assume that the probability of finding two particles in this interval can be neglected. Thus, the probability of finding a particle in an interval $\mathrm{d}^3\vec{p}$ is equal to the expectation value of the number of particles found in $\mathrm{d}^3\vec{p}$. The integral should therefore give the expectation value for the total number of particles observed, which is precisely N.

(c) First the probability distribution is normalized because:

$$\sum_{n=0}^{\infty} p(n) = e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} = e^{-\lambda+\lambda} = 1 \; . \tag{1.10}$$

For the mean value we have:

$$\overline{n} = \sum_{n=0}^{\infty} n p(n) = e^{-\lambda} \sum_{n=0}^{\infty} n \frac{\lambda^n}{n!} = e^{-\lambda} \lambda \frac{d}{d\lambda} e^{\lambda} = \lambda \; . \tag{1.11}$$

The standard deviation can be found by first computing the mean value of $n(n-1)$:

$$\overline{n(n-1)} = \sum_{n=0}^{\infty} n(n-1) p(n) = e^{-\lambda} \sum_{n=0}^{\infty} n(n-1) \frac{\lambda^n}{n!} = e^{-\lambda} \lambda^2 \frac{d^2}{d\lambda^2} e^{\lambda} = \lambda^2 \; . \tag{1.12}$$

Therefore $\overline{n^2} = \lambda^2 + \overline{n} = \lambda^2 + \lambda$, and then the variance σ^2 is calculated as

$$\sigma^2 = \overline{(n-\bar{n})^2} = \overline{n^2} - \overline{n}^2 = \lambda^2 + \lambda - \lambda^2 = \lambda \; . \tag{1.13}$$

Problem 2: Stationary phase approximation *(10 points)*¶

In the limit $n \longrightarrow \infty$ the integral

$$J_n(n) = \frac{1}{\pi} \operatorname{Re} \left\{ \int_0^{\pi} \mathrm{d}t e^{i n g(t)} \right\} \; , \tag{2.1}$$

where

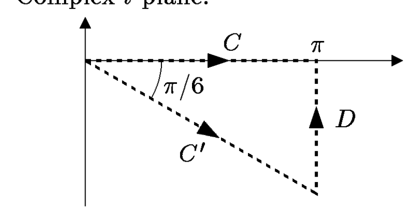
$$g(t) = \sin(t) - t \; , \tag{2.2}$$

is dominated by contributions in the vicinity of the stationary phase points: where $g'(t) = 0$. On the interval $[0, \pi]$ there is one stationary phase point at $t = 0$. Around this point $g(t)$ has a Taylor series

$$g(t) = -t^3/3! + t^5/5! - t^7/7! + \ldots \; . \tag{2.3}$$

The leading term in the asymptotic series is found by using only the first term in the Taylor series for $g(t)$. In order to write the answer in terms of a Γ function we can rotate the contour and integrate along the contour $t = s e^{-i\pi/6}$, where s is real. To do this consider the deformed contour in the complex t plane,

Complex t-plane:



$$\int_C \mathrm{d}t\, e^{i n g(t)} = \int_{C'} \mathrm{d}t\, e^{i n g(t)} + \int_D \mathrm{d}t\, e^{i n g(t)} \; . \tag{2.4}$$

The integral along the D contour will be ignored for now. Then

$$J_n(n) \sim \frac{1}{\pi} \operatorname{Re} \left\{ e^{-i\pi/6} \int_0^{\frac{2\pi}{\sqrt{3}}} \mathrm{d}s \, \exp(-n s^3/3!) \right\} \; . \tag{2.5}$$

Changing variables of integration using $s = (6n)^{1/3} x^{1/3}$, this becomes

$$J_n(n) \sim n^{-1/3} \left(\pi^{-1}\, 2^{1/3}\, 3^{-2/3} \right) \operatorname{Re} \left\{ e^{-i\pi/6} \right\} \int_0^{\infty} \mathrm{d}x\, x^{-2/3} e^{-x} \; , \tag{2.6}$$

where we are justified in extending the integration to infinity because the correction is proportional to $\exp(-\mathrm{const}\, n)$ which is negligible in the limit $n \to \infty$ compared to any terms that we will keep. Using the gamma function $\Gamma(n) = \int_0^\infty dx x^{n-1} e^{-x}$, we find the answer

$$\boxed{J_n(n) \sim \frac{1}{\pi} 2^{-2/3} 3^{-1/6} \Gamma\left(\frac{1}{3}\right) n^{-1/3} .} \tag{2.7}$$

Extra Credit Problem *(5 points)*:

To get higher order contributions we need to include more terms from the Taylor series (2.3) in Eq. (2.5). In fact the next-to-leading-order term vanishes so we must include two more terms. Integrating along the C' contour with the integration variable x,

$$\begin{aligned} J_n(n) \sim c n^{-1/3} \,\mathrm{Re} \Big\{ e^{-i\pi/6} \int_0^\infty dx x^{-2/3} e^{-x} \exp\Big(a\, e^{-i\pi/3} x^{5/3} n^{-2/3} \\ + b\; e^{i\pi/3} x^{7/3} n^{-4/3} \Big) \Big\} , \end{aligned} \tag{2.8}$$

where a, b, c are just the numbers $a = 2^{-4/3}\, 3^{2/3}\, 5^{-1}$, $b = 2^{-5/3}\, 3^{1/3}\, 5^{-1}\, 7^{-1}$ and $c = \pi^{-1}\, 2^{1/3}\, 3^{-2/3}$.

Expanding the last exponential in Eq. (2.8) and keeping all terms up to $\mathcal{O}(n^{-5/3})$ gives

$$\begin{aligned} J_n(n) \sim c \int_0^\infty dx e^{-x} \,\mathrm{Re} \Big\{ e^{-i\pi/6} n^{-1/3} x^{-2/3} + e^{-i\pi/2} n^{-1} a x \\ + e^{i\pi/6} n^{-5/3} (b x^{5/3} - a^2/2x^{8/3}) \Big\} . \end{aligned} \tag{2.9}$$

As promised the next-to-leading-order contribution vanishes as it is purely imaginary. The $n^{-5/3}$ term simplifies to give

$$\boxed{\begin{aligned} J_n(n) &\sim \frac{\Gamma(1/3)}{2^{2/3} \cdot 3^{1/6} \cdot \pi \cdot n^{1/3}} - \frac{3^{1/6} \cdot \Gamma(2/3)}{35 \cdot 2^{4/3} \cdot \pi \cdot n^{5/3}} && (2.10) \\ &\sim 0.447\, n^{-1/3} - 0.00587\, n^{-5/3} . && (2.11) \end{aligned}}$$

Finally we consider the D contour. To carry out this integration we change variables of integration by using $t = \pi + i\tau$, so the integral $J_n^D(n)$ along the D contour can be written as

$$J_n^D(n) = \frac{1}{\pi} \mathrm{Re} \left\{ i e^{-in\pi} \int_{-\frac{\pi}{\sqrt{3}}}^0 d\tau\, e^{n(\tau + \sinh\tau)} \right\} . \tag{2.12}$$

If n is an integer, this term vanishes identically, because the quantity in curly brackets is imaginary. However, if one wishes to consider nonintegral values of n (which was not required for this problem), then this term makes a contribution that is larger than the second term shown in Eq. (2.10):

$$\begin{aligned} J_n^D(n) &\sim \frac{\sin n\pi}{\pi} \int_0^\infty d\tau\, e^{-n\left(2\tau + \frac{\tau^3}{3!} + \frac{\tau^5}{5!} + \ldots\right)} && (2.13) \\ &\sim \frac{\sin n\pi}{2\pi n} + \mathcal{O}\left(\frac{1}{n^3}\right) . && (2.14) \end{aligned}$$

So, for $n \in Z$ Eq. (2.10) gives the first two terms in the asymptotic series of the integral defined by Eq. (2.1), but for $n \notin Z$ the second term is given by Eq. (2.14). Note that the Bessel function agrees with Eq. (2.1) only for integer n, so its asymptotic behavior for noninteger n cannot be determined from Eq. (2.1). In this problem we used Eq. (2.1), rather than a general integral expression for the Bessel function, because the asymptotic behavior of Eq. (2.1) is particularly easy to study. The purpose of the problem was to illustrate the determination of the asymptotic properties of integrals, not the properties of Bessel functions.

Optional note on $J_n(n)$ for noninteger n:*

In doing some numerical experiments to test the formulas derived above, I discovered accidentally that the Bessel function $J_n(n)$ seemed to behave for large n more simply than Eq. (2.1), obeying Eq. (2.10) for all n. I decided to pursue this possibility numerically for the case where n is an integer plus 1/2, since for that case there is a simple analytic formula for the Bessel function:†

$$\begin{aligned} J_{n+\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \Bigg\{ \sin\left(z - \frac{\pi}{2} n\right) \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k (n+2k)!}{(2k)!(n-2k)!(2z)^{2k}} \\ + \cos\left(z - \frac{\pi}{2} n\right) \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \frac{(-1)^k (n+2k+1)!}{(2k+1)!(n-2k-1)!(2z)^{2k+1}} \Bigg\} , \end{aligned} \tag{2.15}$$

where n is zero or a positive integer, and $[x]$ denotes the largest integer less than or equal to x. With an explicit expression like this, one would think it would be

* Note by Alan Guth

† See for example Eq. (8.461) of Gradshteyn and Ryzhik, *Table of Integrals, Series, and Products*, by I.S. Gradshteyn and I.M. Ryzhik, edited by Alan Jeffrey, Corrected and Enlarged Fourth Edition.

a snap to program a computer to evaluate $J_{n+\frac{1}{2}}(n+\frac{1}{2})$ for, say, $n = 1000$, where each sum would involve about 500 terms. But it was not so easy!

$$\boxed{\textbf{Beware } (-1)^k \;\;\textbf{!!!}}$$

I discovered, after getting some wierd results, that Eq. (2.15) involves a huge amount of cancellation, so extreme levels of numerical precision are needed to get a result that is meaningful after all the large contributions have cancelled. For $n = 1000$, for example, the sum in the first line of Eq. (2.15) gives 2.06280, but the term with the largest magnitude is the $k = 207$ term, with value -1.89965×10^{201}. Thus one needs to calculate to over 200 significant figures to get this right! The sum in the second line is just as bad, evaluating to -2.88420, where again the term with largest magnitude is the $k = 207$ term, with value -1.89684×10^{201}. Fortunately there are computer algebra programs that allow numbers of arbitrary precision, so these calculations are easily doable as long as one realizes that this precision is necessary. Using 300 significant figures, I find that Eq. (2.15) gives

$$J_{1000\frac{1}{2}}\left(1000\frac{1}{2}\right) = 0.0447232203592 \,, \tag{2.16}$$

while Eq. (2.10) evaluates to

$$J_{1000\frac{1}{2}}\left(1000\frac{1}{2}\right) \underset{\text{Eq. (2.10)}}{\sim} 0.0447232205577 \,. \tag{2.17}$$

Of course there are much more efficient ways to calculate the Bessel functions for large arguments, so *Mathematica* had no trouble evaluating $J_{1000\frac{1}{2}}\left(1000\frac{1}{2}\right)$, nor did the Bessel function subroutine from W.H. Press, S.A. Teukolsky, W.T. Vettering, and B.P. Flannery, *Numerical Recipes in C, Second Edition* (Cambridge University Press, 1992)— a book that I highly recommend. I see that they now have a third edition, so I just ordered it.

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† Solution written by Guido Festuccia and Alan Guth.

¶ Solution written by Tom Faulkner and Alan Guth.