MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

8.323: Relativistic Quantum Field Theory I

Prof. Alan Guth May 20, 2008

PROBLEM SET 9 SOLUTIONS

Problem 1: The Dirac representation of the Lorentz group *(10 points)*†

(a) By definition,

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \ , \tag{1.1}$$

and

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] = \frac{i}{2}\{\gamma^\mu\gamma^\nu\}_{\mu\nu} \ , \tag{1.2}$$

where we adopt the convention that $\{\}_{\mu\nu}$ denotes antisymmetrization. Hence,

$$[S^{\mu\nu}, S^{\rho\sigma}] = -\frac{1}{4}\{\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma - \gamma^\rho\gamma^\sigma\gamma^\mu\gamma^\nu\}_{\substack{\mu\nu\\ \rho\sigma}} \ . \tag{1.3}$$

Now rewrite the commutation relation (1.1) as

$$\gamma^\sigma\gamma^\mu = -\gamma^\mu\gamma^\sigma + 2g^{\sigma\mu} \ . \tag{1.4}$$

So we can interchange two gamma matrices by changing the sign, and adding $2g^{\sigma\mu}$, where σ and μ are the indices on the two gamma matrices that were interchanged. So, in the second term on the right-hand side of Eq. (1.3), we can bring the γ^σ factor to the right by two interchanges:

$$\begin{aligned}\{\gamma^\rho\gamma^\sigma\gamma^\mu\gamma^\nu\}_{\substack{\mu\nu\\ \rho\sigma}} &= \{\gamma^\rho\gamma^\mu\gamma^\nu\gamma^\sigma + 2g^{\sigma\mu}\gamma^\rho\gamma^\nu - 2g^{\sigma\nu}\gamma^\rho\gamma^\mu\}_{\substack{\mu\nu\\ \rho\sigma}} \\ &= \{\gamma^\rho\gamma^\mu\gamma^\nu\gamma^\sigma + 4g^{\sigma\mu}\gamma^\rho\gamma^\nu\}_{\substack{\mu\nu\\ \rho\sigma}} \ ,\end{aligned} \tag{1.5}$$

where in the second line we used the antisymmetrization in μ and ν to collect the last two terms. Using the same technique to bring γ^ρ through the factors of γ^μ and γ^ν on the right-hand side of Eq. (1.5), we have

$$\begin{aligned}\{\gamma^\rho\gamma^\sigma\gamma^\mu\gamma^\nu\}_{\substack{\mu\nu\\ \rho\sigma}} &= \{\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma + 4g^{\sigma\mu}\gamma^\rho\gamma^\nu + 4g^{\rho\mu}\gamma^\nu\gamma^\sigma\}_{\substack{\mu\nu\\ \rho\sigma}} \\ &= \{\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma + 4g^{\rho\nu}\left(\gamma^\sigma\gamma^\mu - \gamma^\mu\gamma^\sigma\right)\}_{\substack{\mu\nu\\ \rho\sigma}} \\ &= \{\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma + 16ig^{\rho\nu}S^{\mu\sigma}\}_{\substack{\mu\nu\\ \rho\sigma}} \ .\end{aligned} \tag{1.6}$$

Using this result to replace the second term in Eq. (1.3), one has immediately

$$\begin{aligned}[S^{\mu\nu}, S^{\rho\sigma}] &= -\frac{1}{4}\{-16ig^{\rho\nu}S^{\mu\sigma}\}_{\substack{\mu\nu\\ \rho\sigma}} \\ &= \{4ig^{\nu\rho}S^{\mu\sigma}\}_{\substack{\mu\nu\\ \rho\sigma}} \ ,\end{aligned} \tag{1.7}$$

which is the same commutation relation as for the generators of the Lorentz group.

(b)

$$\begin{aligned}[\gamma^\mu, S^{\rho\sigma}] &= \frac{i}{2}[\gamma^\mu, \gamma^\rho\gamma^\sigma] \\ &= \frac{i}{2}\left(\gamma^\mu\gamma^\rho\gamma^\sigma + \gamma^\rho\gamma^\mu\gamma^\sigma - \gamma^\rho\gamma^\sigma\gamma^\mu - \gamma^\rho\gamma^\mu\gamma^\sigma\right) \\ &= i\left(g^{\mu\rho}\gamma^\sigma - g^{\sigma\mu}\gamma^\rho\right) \\ &= i\left(g^{\rho\mu}\delta^\sigma{}_\nu - g^{\sigma\mu}\delta^\rho{}_\nu\right)\gamma^\nu \\ &= (J^{\rho\sigma})^\mu{}_\nu\gamma^\nu \ .\end{aligned} \tag{1.8}$$

Problem 2: Explicit tranformation matrices *(10 points)*†

A Lorentz transformation is generated in the four-vector representation by

$$(\mathcal{J}^{\mu\nu})_{\alpha\beta} \equiv i\left(\delta^\mu_\alpha\,\delta^\nu_\beta - \delta^\mu_\beta\,\delta^\nu_\alpha\right) \ , \tag{2.1}$$

or equivalently

$$(\mathcal{J}^{\mu\nu})^\alpha{}_\beta = i\left(g^{\mu\alpha}\,\delta^\nu_\beta - g^{\nu\alpha}\,\delta^\mu_\beta\right) \ . \tag{2.2}$$

If we want to use matrix notation to suppress the α and β indices, then we must use the form (2.2), so that the sum over repeated indices is always performed with one index raised and the other lowered. Thus, the generator for boosts in the z-direction is given by

$$\mathcal{J}^{03} = i\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \ , \tag{2.3}$$

and a boost transformation matrix is given by

$$\begin{aligned}B_3(\eta) &\equiv e^{-i\eta\mathcal{J}^{03}} \\ &= \exp\left[\eta\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\right] \\ &= \begin{pmatrix} \cosh\eta & 0 & 0 & \sinh\eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\eta & 0 & 0 & \cosh\eta \end{pmatrix} \ .\end{aligned} \tag{2.4}$$

Applying $B_3(\eta)$ to the momentum four-vector of a particle at rest, $p^\mu = (m, 0, 0, 0)$, gives

$$p^\mu = (m\cosh\eta, 0, 0, m\sinh\eta) \ , \tag{2.5}$$

so the velocity v is related to η by

$$v = \frac{p}{E} = \frac{m\sinh\eta}{m\cosh\eta} = \tanh\eta \ . \tag{2.6}$$

We can check that the rapidity is additive, as it should be. Relativistic velocities add as

$$v_{\rm tot} = \frac{v_1 + v_2}{1 + v_1\, v_2} \ , \tag{2.7}$$

so replacing v_i by $\tanh\eta_i$ gives

$$\tanh\eta_{\rm tot} = \frac{\tanh\eta_1 + \tanh\eta_2}{1 + \tanh\eta_1\,\tanh\eta_2} = \tanh(\eta_1 + \eta_2) \ , \tag{2.8}$$

as desired.

We want to find the corresponding matrix in the chiral representation, $B_3(\eta) \equiv e^{-i\eta S^{03}}$. Use

$$\begin{aligned} \gamma^i &= \begin{pmatrix} 0 & \sigma^i \\ \bar\sigma^i & 0 \end{pmatrix} \ , \\ S^{0i} &= \frac{i}{4}[\gamma^0, \gamma^i] \\ &= \frac{i}{4}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] \\ &= \frac{i}{2}\begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} \ . \end{aligned} \tag{2.9}$$

Thus,

$$\begin{aligned} B_3(\eta) = e^{-i\eta S^{03}} &= \exp\left[\frac{\eta}{2}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\right] \\ &= \begin{pmatrix} e^{-\frac{\eta}{2}} & 0 & 0 & 0 \\ 0 & e^{\frac{\eta}{2}} & 0 & 0 \\ 0 & 0 & e^{\frac{\eta}{2}} & 0 \\ 0 & 0 & 0 & e^{-\frac{\eta}{2}} \end{pmatrix} \ . \end{aligned} \tag{2.10}$$

Clearly our answer implies that $B_3(\eta_1)B_3(\eta_2) = B_3(\eta_1 + \eta_2)$, as desired. To express $e^{\pm\frac{1}{2}\eta}$ in terms of p^3 and E, note that Eq. (2.5) implies that

$$E = m\cosh\eta \ , \qquad p^3 = m\sinh\eta \ , \tag{2.11}$$

so

$$\begin{aligned} e^{\frac{1}{2}\eta} &= \sqrt{\cosh\eta + \sinh\eta} = \sqrt{\frac{E + p^3}{m}} \\ e^{-\frac{1}{2}\eta} &= \sqrt{\cosh\eta - \sinh\eta} = \sqrt{\frac{E - p^3}{m}} \ . \end{aligned} \tag{2.12}$$

Similarly, for $R_3(\theta) \equiv e^{-i\theta S^{12}}$,

$$\begin{aligned} S^{ij} &= \frac{i}{4}[S^i, S^j] \\ &= \frac{i}{2}\left[\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}\right]_{ij} \\ &= \frac{\epsilon^{ijk}}{2}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \ . \end{aligned} \tag{2.13}$$

$$\Rightarrow R_3(\theta) = e^{-i\theta S^{12}} = \begin{pmatrix} e^{-\frac{i\theta}{2}} & 0 & 0 & 0 \\ 0 & e^{\frac{i\theta}{2}} & 0 & 0 \\ 0 & 0 & e^{-\frac{i\theta}{2}} & 0 \\ 0 & 0 & 0 & e^{\frac{i\theta}{2}} \end{pmatrix} \ . \tag{2.14}$$

Problem 3: Wigner rotations and the transformation of helicity *(15 points)*†

(a) In the vector representation, the Lorentz matrix $B_x(\eta)$ is:

$$\begin{aligned} B_x(\eta(\beta)) = e^{-i\eta(\beta)K'} = e^{-i\eta(\beta)J^{01}} &= \exp\left[-i\eta(\beta)\begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right] \\ &= \begin{pmatrix} \cosh\eta(\beta) & \sinh\eta(\beta) & 0 & 0 \\ \sinh\eta(\beta) & \cosh\eta(\beta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ . \end{aligned} \tag{3.1}$$

In a similar fashion,

$$B_z(\eta(p)) = e^{-i\eta(\beta)J^{03}} = \exp\left[-i\eta(p)\begin{pmatrix}0&0&0&i\\0&0&0&0\\0&0&0&0\\i&0&0&0\end{pmatrix}\right]$$
$$= \begin{pmatrix}\cosh\eta(p)&0&0&\sinh\eta(p)\\0&1&0&0\\0&0&1&0\\\sinh\eta(p)&0&0&\cosh\eta(p)\end{pmatrix} . \tag{3.2}$$

$$\Lambda^\mu{}_\nu = B_x(\eta(\beta'))B_z(\eta(p))$$
$$= \begin{pmatrix}\cosh\eta(\beta)\cosh\eta(p)&\sinh\eta(\beta)&0&\cosh\eta(\beta)\sinh\eta(p)\\\sinh\eta(\beta)\cosh\eta(p)&\cosh\eta(\beta)&0&\sinh\eta(\beta)\sinh\eta(p)\\0&0&1&0\\\sinh\eta(p)&0&0&\cosh\eta(p)\end{pmatrix} , \tag{3.3}$$

$$p = m\sinh\eta(p) , \quad \beta = \tanh\eta(\beta) . \tag{3.4}$$

The transformation $B_x(\eta(\beta))B_z(\eta(p))$ transforms a momentum vector $(m,0,0,0)$ into:

$$\begin{pmatrix}E'\\p'_x\\p'_y\\p'_z\end{pmatrix} = \Lambda\begin{pmatrix}m\\0\\0\\0\end{pmatrix} = m\begin{pmatrix}\cosh\eta(\beta)\cosh\eta(p)\\\sinh\eta(\beta)\cosh\eta(p)\\0\\\sinh\eta(p)\end{pmatrix} = \begin{pmatrix}\sqrt{\frac{p^2+m^2}{1-\beta^2}}\\\beta\sqrt{\frac{p^2+m^2}{1-\beta^2}}\\0\\p\end{pmatrix} . \tag{3.5}$$

If we define θ as the angle between the z–axis and $\vec{p}'$,

$$\sin\theta = \frac{p'_x}{|\vec{p}'|} = \frac{\sinh\eta(\beta)\cosh\eta(p)}{\sqrt{\cosh^2\eta(\beta)\cosh^2\eta(p)-1}} ,$$
$$\cos\theta = \frac{p'_z}{|\vec{p}'|} = \frac{\sinh\eta(p)}{\sqrt{\cosh^2\eta(\beta)\cosh^2\eta(p)-1}} . \tag{3.6}$$

(b) We must compute

$$B_{\vec{p}'} = R(\hat{p}')B_z(\eta(p')) , \tag{3.7}$$

where

$$B_z(\eta(p')) = \begin{pmatrix}\cosh\eta(p')&0&0&\sinh\eta(p')\\0&1&0&0\\0&0&1&0\\\sinh\eta(p')&0&0&\cosh\eta(p')\end{pmatrix} , \tag{3.8}$$

and $R(\hat{p}')$ is the matrix which rotates from $\hat{z}$ to $\hat{p}'$:

$$R(\hat{p}') = \begin{pmatrix}1&0&0&0\\0&\cos\theta&0&\sin\theta\\0&0&1&0\\0&-\sin\theta&0&\cos\theta\end{pmatrix} , \tag{3.9}$$

where

$$\tan\theta = \frac{p'_x}{p'_z} = \frac{\sqrt{p^2+m^2}}{p}\frac{\beta}{\sqrt{1-\beta^2}} . \tag{3.10}$$

Thus

$$B_{\vec{p}'} = \begin{pmatrix}\cosh\eta(p')&0&0&\sinh\eta(p')\\\sin\theta\sinh\eta(p')&\cos\theta&0&\cosh\eta(p')\sin\theta\\0&0&1&0\\\cos\theta\sinh\eta(p')&-\sin\theta&0&\cos\theta\cosh\eta(p')\end{pmatrix} ,$$
$$B^{-1}_{\vec{p}'} = \begin{pmatrix}\cosh\eta(p')&-\sin\theta\sinh\eta(p')&0&-\cos\theta\sinh\eta(p')\\0&\cos\theta&0&-\sin\theta\\0&0&1&0\\-\sinh\eta(p')&\sin\theta\cosh\eta(p')&0&\cos\theta\cosh\eta(p')\end{pmatrix} . \tag{3.11}$$

Therefore,

$$R_W = \begin{pmatrix}1&0&0&0\\0&\frac{p}{\sqrt{p^2+\beta^2m^2}}&0&-\frac{\beta m}{\sqrt{p^2+\beta^2m^2}}\\0&0&1&0\\0&\frac{\beta m}{\sqrt{p^2+\beta^2m^2}}&0&\frac{p}{\sqrt{p^2+\beta^2m^2}}\end{pmatrix} , \tag{3.12}$$

where we used

$$\tanh\eta(\beta) = \beta ,$$
$$\sinh\eta(p) = \frac{p}{m} ,$$
$$\sinh\eta(p') = \frac{\sqrt{p^2+\beta^2m^2}}{m\sqrt{1-\beta^2}} , \tag{3.13}$$
$$\tan\theta = \frac{p^2+m^2}{p}\frac{\beta}{\sqrt{1-\beta^2}} .$$

A counterclockwise rotation about the y-axis through an angle ψ is described by the matrix

$$R_y(\psi) = \begin{pmatrix}1&0&0&0\\0&\cos\psi&0&\sin\psi\\0&0&1&0\\0&-\sin\psi&0&\cos\psi\end{pmatrix} , \tag{3.14}$$

which matches R_W provided that

$$\cos\psi = \frac{p}{\sqrt{p^2+\beta^2 m^2}}$$
$$\sin\psi = -\frac{\beta m}{\sqrt{p^2+\beta^2 m^2}} \,. \tag{3.15}$$

Thus the Wigner rotation R_W is a clockwise rotation about the y-axis by an angle $\sin^{-1}(\beta m/\sqrt{p^2+\beta^2 m^2})$.

The Wigner rotation on the rest frame states of the spin-$\frac{1}{2}$ particle (i.e. the basis $\{|0, h=\pm\frac{1}{2}\rangle\}$) is then

$$U(R_y(\psi)) = e^{-i\sigma_2\psi/2} = \begin{pmatrix} \cos(\psi/2) & -\sin(\psi/2) \\ \sin(\psi/2) & \cos(\psi/2) \end{pmatrix} \tag{3.16}$$

where,

$$\cos(\psi/2) = \left(\frac{1}{2}+\frac{1}{2}\frac{p}{\sqrt{p^2+\beta^2 m^2}}\right)^{1/2}$$
$$\sin(\psi/2) = \left(\frac{1}{2}-\frac{1}{2}\frac{p}{\sqrt{p^2+\beta^2 m^2}}\right)^{1/2} \tag{3.17}$$

(c) As $m \to 0$, $\cos\psi \to 1$ and $\sin\psi \to 0$, thus $\psi \to 0$, and $R_W \to I$. Therefore

$$\langle \vec{p}', h'|\psi\rangle = \langle \vec{p}=0, h'|U(R_W)|\vec{p}=0,\pm\rangle = \delta_{h',\pm} \,. \tag{3.18}$$

Thus the helicity of a massless particle is Lorentz invariant.

Problem 4: Useful tricks with Dirac matrices *(10 points)*¶

$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ $\quad \gamma^\mu$ is 4 by 4 $\quad \mathrm{Tr}\, 1 = 4$

or $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$ $\quad *$

$\mathrm{Tr}\,\gamma^\mu\gamma^\nu + \mathrm{Tr}\,\gamma^\nu\gamma^\mu = 8g^{\mu\nu}$

so $\mathrm{Tr}\,\gamma^\mu\gamma^\nu = 4g^{\mu\nu}$ which is (ii)

Rewrite $*$ $\quad \gamma^\mu\gamma^\nu = 2g^{\mu\nu} - \gamma^\nu\gamma^\mu$

Contract with $p_\mu q_\nu$ and get (vi)

Now $\gamma_\mu\gamma^\nu\gamma^\mu = \gamma_\mu(2g^{\nu\mu} - \gamma^\mu\gamma^\nu)$

$= 2\gamma^\nu - \gamma_\mu\gamma^\mu\gamma^\nu$

Since $\gamma_\mu\gamma^\mu = 4$ we get

$\gamma_\mu\gamma^\nu\gamma^\mu = -2\gamma^\nu$ which is (xi)

Look at $\gamma_\mu\gamma^\alpha\gamma^\beta\gamma^\mu$

$= (2\delta^\alpha_\mu - \gamma^\alpha\gamma_\mu)(2g^{\beta\mu} - \gamma^\mu\gamma^\beta)$

$= 4g^{\alpha\beta} - 2\gamma^\alpha\gamma^\beta - 2\gamma^\alpha\gamma^\beta + \gamma^\alpha\gamma_\mu\gamma^\mu\gamma^\beta$

$$= 4 g^{\alpha\beta} \quad \text{which is (xii)}$$

Look at $\gamma_\mu \gamma^\alpha \gamma^\nu \gamma^\beta \gamma^\mu$

$$= (2\delta^\alpha_\mu - \gamma^\alpha \gamma_\mu)\gamma^\nu (2g^{\beta\mu} - \gamma^\mu \gamma^\beta)$$

$$= 4g^{\alpha\beta}\gamma^\nu - 2\gamma^\nu\gamma^\alpha\gamma^\beta - 2\gamma^\alpha\gamma^\beta\gamma^\nu - 2\gamma^\alpha\gamma^\nu\gamma^\beta$$

Combine last two

$$= 4g^{\alpha\beta}\gamma^\nu - 2\gamma^\nu\gamma^\alpha\gamma^\beta - 4\gamma^\alpha g^{\beta\nu}$$

$$= 4g^{\alpha\beta}\gamma^\nu - 2\gamma^\nu(2g^{\alpha\beta} - \gamma^\beta\gamma^\alpha) - 4\gamma^\alpha g^{\beta\nu}$$

$$= 2\gamma^\nu\gamma^\beta\gamma^\alpha - 4\gamma^\alpha g^{\beta\nu}$$

$$= 2(2g^{\nu\beta} - \gamma^\beta\gamma^\nu)\gamma^\alpha - 4\gamma^\alpha g^{\beta\nu}$$

$$= -2\gamma^\beta\gamma^\nu\gamma^\alpha \quad \text{which is (xiii)}$$

We now turn to the trace identities.

Because of the algebra we know that γ^μ transforms as a vector. P+S 3.29

$$\Lambda^{-1}_{\frac{1}{2}} \gamma^\mu \Lambda_{\frac{1}{2}} = \Lambda^\mu{}_\nu \gamma^\nu$$

Take the Trace:

$$\mathrm{Tr}\, \Lambda^{-1}_{\frac{1}{2}} \gamma^\mu \Lambda_{\frac{1}{2}} = \Lambda^\mu{}_\nu \mathrm{Tr}\,\gamma^\nu$$

$$\mathrm{Tr}\,\gamma^\mu = \Lambda^\mu{}_\nu \mathrm{Tr}\,\gamma^\nu$$

For this to hold for arbitrary $\Lambda^\mu{}_\nu$ we need

$$\mathrm{Tr}\,\gamma^\mu = 0 \quad ✓ \quad \text{(i)}$$

Similarly $\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho) = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \Lambda^\rho{}_\sigma \mathrm{Tr}(\gamma^\alpha\gamma^\beta\gamma^\sigma)$

There are only two invariant Lorentz tensors we can use as building blocks

$$g^{\mu\nu} \text{ and } \epsilon_{\mu\nu\rho\sigma}$$

No way to make a 3 index invariant tensor so we get (iii) and also (viii)

Note this argument says that

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = C g^{\mu\nu}$$

To determine the constant take $\mu = \nu = 0$ and we get $C = 4$

which also gives (ii)

Now $\mathrm{Tr}\, \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma$

$$= A g^{\mu\nu} g^{\rho\sigma} + B g^{\mu\rho} g^{\nu\sigma} + C g^{\mu\sigma} g^{\nu\rho} + D \epsilon^{\mu\nu\rho\sigma}$$

by the Lorentz Argument.

Take $\mu = \nu = 0$ $\rho = \sigma = 1$

$$\mathrm{Tr}\, \gamma^0 \gamma^0 \gamma^1 \gamma^1 = A g^{00} g^{11}$$

$$-4 = -A$$
$$A = 4$$

Take $\mu = \rho = 0$ $\nu = \sigma = 1$ get $B = -4$

$\mu = \sigma = 0$ $\nu = \rho = 1$ get $C = 4$

For D $\mathrm{Tr}\, \gamma^0 \gamma^1 \gamma^2 \gamma^3 = D \epsilon^{0123}$

However $\mathrm{Tr}\, \gamma^0 \gamma^1 \gamma^2 \gamma^3 = -\mathrm{Tr}\, \gamma^3 \gamma^0 \gamma^1 \gamma^2$ if you push γ^3 through $\gamma^0\gamma^1\gamma^2$

$= \mathrm{Tr}\, \gamma^3 \gamma^0 \gamma^1 \gamma^2$ by cyclicity

so $\mathrm{Tr}\, \gamma^0\gamma^1\gamma^2\gamma^3 = 0$ or $D = 0$

we get (ix) and also (v)

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

$$\begin{aligned}
\gamma^5\gamma^5 &= -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3 \\
&= \gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3 \\
&= -\gamma^2\gamma^3\gamma^2\gamma^3 \\
&= -\gamma^3\gamma^3 \\
&= 1
\end{aligned}$$

which is (iv)

Consider $\mathrm{Tr}(\gamma^5 \gamma^\mu \gamma^\nu)$

Let $\mu = \nu$, get $\mathrm{Tr}\,\gamma^5$ which is 0

Let $\mu \neq \nu$ e.g. $\mu = 0$ $\nu = 3$

$$\begin{aligned}
&i\,\mathrm{Tr}(\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^3) \\
&= -i\,\mathrm{Tr}(\gamma^1\gamma^2\gamma^3\gamma^3) \\
&= i\,\mathrm{Tr}(\gamma^1\gamma^2) = 0
\end{aligned}$$

we have (x)

Finally $\mathrm{Tr}[\gamma^5 \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma]$

Use previous result to show that it vanishes if any two indices are =

$$\mathrm{Tr}[\gamma^5\gamma_0\gamma_1\gamma_2\gamma_3] = i\,\mathrm{Tr}[\gamma^0\gamma^1\gamma^2\gamma^3\gamma_0\gamma_1\gamma_2\gamma_3]$$

$= 4i$ This gives (xiv)

Problem 5 (Extra Credit): Wigner's Symmetry Representation Theorem *(10 points extra credit)*§

(a) Since T is probability-preserving and

$$|\tilde{\psi}_k\rangle \in T(\psi_k)\ , \tag{5.1}$$

it follows that

$$|\langle\tilde{\psi}_k\,|\tilde{\psi}_\ell\,\rangle| = |\langle\psi_k\,|\psi_\ell\,\rangle| = \delta_{k\ell}\ . \tag{5.2}$$

But $\langle\tilde{\psi}_k\,|\tilde{\psi}_k\,\rangle$ is necessarily real and positive, so $\langle\tilde{\psi}_k\,|\tilde{\psi}_k\,\rangle = 1$. For $k \neq \ell$,

$$|\langle\tilde{\psi}_k\,|\tilde{\psi}_\ell\,\rangle| = 0 \quad \Longrightarrow \quad \langle\tilde{\psi}_k\,|\tilde{\psi}_\ell\,\rangle = 0\ , \tag{5.3}$$

so

$$\langle\tilde{\psi}_k\,|\tilde{\psi}_\ell\,\rangle = \delta_{k\ell}\ . \tag{5.4}$$

Thus the $|\tilde{\psi}_k\rangle$ are orthonormal. To show that they are complete, suppose to the contrary that there exists a state $|\Psi\rangle$ with $\langle\Psi\,|\Psi\,\rangle = 1$ and $\langle\tilde{\psi}_k\,|\Psi\,\rangle = 0$ for all k. Then choose a vector $|\Psi'\rangle \in T^{-1}(\Psi)$. This vector will necessarily have the following properties:

$$\langle\Psi'\,|\Psi'\,\rangle = \langle\Psi\,|\Psi\,\rangle = 1\ , \tag{5.5a}$$

$$\langle\psi_k\,|\Psi'\,\rangle = \langle\tilde{\psi}_k\,|\Psi\,\rangle = 0\ . \tag{5.5b}$$

Thus $|\Psi'\rangle$ is a normalized vector orthogonal to all the $|\psi_k\rangle$, which contradicts the completeness of the $|\psi_k\rangle$ vectors. Thus the state $|\Psi\rangle$ cannot exist, and the $|\tilde{\psi}_k\rangle$ vectors are complete.

(b) Choose some vector $|\tilde{\phi}_k\rangle \in T(\phi_k)$, which will then necessarily have the following properties:

$$|\langle\tilde{\psi}_1\,|\tilde{\phi}_k\,\rangle| = |\langle\psi_1\,|\phi_k\,\rangle| = \frac{1}{\sqrt{2}}\ , \tag{5.6a}$$

$$|\langle\tilde{\psi}_k\,|\tilde{\phi}_k\,\rangle| = |\langle\psi_k\,|\phi_k\,\rangle| = \frac{1}{\sqrt{2}}\ . \tag{5.6b}$$

For $\ell \neq k$,

$$|\langle\tilde{\psi}_\ell\,|\tilde{\phi}_k\,\rangle| = |\langle\psi_\ell\,|\phi_k\,\rangle| = 0\ . \tag{5.6c}$$

Since the $|\tilde{\psi}_k\rangle$ are complete, we can expand $|\tilde{\phi}_k\rangle$ in this basis, using Eqs. (5.6) to determine the magnitude of the expansion coefficients:

$$|\tilde{\phi}_k\rangle = \frac{1}{\sqrt{2}}\left(e^{i\theta_k^{(a)}}\,|\tilde{\psi}_1\rangle + e^{i\theta_k^{(b)}}\,|\tilde{\psi}_k\rangle\right) \tag{5.7}$$

for some real $\theta_k^{(a)}$ and $\theta_k^{(b)}$. But if $|\tilde{\phi}_k\rangle \in T(\phi_k)$, then $e^{-i\theta_k^{(a)}}\,|\tilde{\phi}_k\rangle$ must also be in $T(\phi_k)$, so

$$T(\phi_k) \ni \frac{1}{\sqrt{2}}\left(|\tilde{\psi}_1\rangle + e^{i\theta_k}\,|\tilde{\psi}_k\rangle\right)\ , \tag{5.8}$$

where $\theta_k = \theta_k^{(b)} - \theta_k^{(a)}$.

(c) Following the problem set, we define

$$\begin{aligned} |\psi_1'\rangle &= |\tilde{\psi}_1\rangle \\ |\psi_k'\rangle &= e^{i\theta_k}\,|\tilde{\psi}_k\rangle \quad \text{for } k = 2,3,\ldots\ , \end{aligned} \tag{5.9}$$

so

$$T(\phi_k) \ni |\phi_k'\rangle\ , \quad \text{where} \quad |\phi_k'\rangle = \frac{1}{\sqrt{2}}\big(\,|\psi_1'\rangle + |\psi_k'\rangle\big)\ . \tag{5.10}$$

Now choose some vector $|\tilde{\Phi}(\theta)\rangle \in T\big(\Phi(\theta)\big)$. Then

$$|\langle\psi_k'\,|\tilde{\Phi}(\theta)\,\rangle| = |\langle\psi_k\,|\Phi(\theta)\,\rangle| = \begin{cases} \frac{1}{\sqrt{2}} & \text{if } k = 1,2 \\ 0 & \text{otherwise}\ . \end{cases} \tag{5.11}$$

This implies that the expansion of $|\tilde{\Phi}(\theta)\rangle$ in the basis $|\psi_k'\rangle$ has the form

$$|\tilde{\Phi}(\theta)\rangle = \frac{1}{\sqrt{2}}\left(e^{i\theta_1'}\,|\psi_1'\rangle + e^{i\theta_2'}\,|\psi_2'\rangle\right)\ . \tag{5.12}$$

Then $T\big(\Phi(\theta)\big)$ must also contain the vector

$$|\Phi'(\theta)\rangle = e^{-i\theta_1'}\,|\tilde{\Phi}(\theta)\rangle = \frac{1}{\sqrt{2}}\left(|\psi_1'\rangle + e^{i\theta'}\,|\psi_2'\rangle\right)\ , \tag{5.13}$$

where $\theta' = \theta_2' - \theta_1'$. Furthermore,

$$|\langle\phi_2'\,|\Phi'(\theta)\,\rangle| = |\langle\phi_2\,|\Phi(\theta)\,\rangle|\ , \tag{5.14}$$

which implies that

$$\left|1 + e^{i\theta'}\right| = \left|1 + e^{i\theta}\right|\ . \tag{5.15}$$

At this point it is useful to prove a lemma that is slightly more general than what is needed here, since more generality will be needed later. Suppose that

$$\left|z_1 + z_2 e^{i\theta'}\right| = \left|z_1 + z_2 e^{i\theta}\right|\ , \tag{5.16}$$

where z_1 and z_2 are complex numbers. One then has

$$\mathrm{Re}\left(z_1^* z_2 e^{i\theta'}\right) = \mathrm{Re}\left(z_1^* z_2 e^{i\theta}\right) .$$

Writing $z_1^* z_2 = |z_1 z_2| e^{i\phi}$, where $\phi = \arg(z_1^* z_2)$, we require

$$\cos(\theta' + \phi) = \cos(\theta + \phi) ,$$

which has two solutions:

$$\theta' = \theta \quad \text{and} \quad \theta' = -\theta - 2\arg(z_1^* z_2) . \tag{5.17}$$

Of course one could add $2\pi n$ to either solution, where n is an integer, but I will use the convention that angles are defined modulo 2π, so adding $2\pi n$ is no different from adding zero.

Applying the lemma to Eq. (5.15), we have

$$\theta' = \theta \quad \text{or} \quad \theta' = -\theta . \tag{5.18}$$

These two solutions give the two cases described in the problem. For case A, $T\big(\Phi(\theta)\big) \ni \big|\Phi'_+(\theta)\big\rangle$, where

$$\big|\Phi'_+(\theta)\big\rangle = \frac{1}{\sqrt{2}}\big(\left|\psi'_1\right\rangle + e^{i\theta}\left|\psi'_2\right\rangle\big) , \qquad \text{(case A)} \tag{5.19a}$$

and for case B, $T\big(\Phi(\theta)\big) \ni \big|\Phi'_-(\theta)\big\rangle$, where

$$\big|\Phi'_-(\theta)\big\rangle = \frac{1}{\sqrt{2}}\big(\left|\psi'_1\right\rangle + e^{-i\theta}\left|\psi'_2\right\rangle\big) . \qquad \text{(case B)} \tag{5.19b}$$

(d) We wish to show that a given transformation T must obey case A for all θ, or case B for all θ. It is impossible for case A to apply for one value of θ and for case B to apply for another, as long as we avoid the trivial cases $\theta = 0$ and $\theta = \pi$, for which the two cases are identical. To show this, suppose that case A applies for $\theta = \theta_A$, so $\big|\Phi'_+(\theta_A)\big\rangle \in T\big(\Phi(\theta_A)\big)$. Suppose that case B applies for $\theta = \theta_B$, so $\big|\Phi'_-(\theta_B)\big\rangle \in T\big(\Phi(\theta_B)\big)$.

Now consider the inner product between these two states:

$$\big|\big\langle\Phi'_-(\theta_B)\,\big|\Phi'_+(\theta_A)\big\rangle\big| = \big|\langle\Phi(\theta_B)\,|\Phi(\theta_A)\rangle\big| , \tag{5.20}$$

which implies that

$$\left|1 + e^{i(\theta_A+\theta_B)}\right| = \left|1 + e^{i(\theta_A-\theta_B)}\right| . \tag{5.21}$$

The lemma implies that this equation can hold only if

$$\theta_A + \theta_B = \theta_A - \theta_B \tag{5.22a}$$

or

$$\theta_A + \theta_B = -(\theta_A - \theta_B) . \tag{5.22b}$$

Eq. (5.22a) implies that $2\theta_B = 0$, so θ_B must equal 0 or π (remember that 2π is equivalent to zero). Similarly Eq. (5.22b) implies that θ_A is either 0 or π. Thus, the cases A and B can be mixed only if one of the angles considered is 0 or π, which is exactly the set of angles for which the cases A and B coincide.

(e) Consider first case A. Let $|\tilde\Psi_N(\alpha_2,\ldots,\alpha_N)\rangle$ denote a vector chosen from $T\big(\Psi_N(\alpha_2,\ldots,\alpha_N)\big)$, where

$$|\Psi_N(\alpha_2,\ldots,\alpha_N)\rangle = \frac{1}{\sqrt{N}}\big(\left|\psi_1\right\rangle + e^{i\alpha_2}\left|\psi_2\right\rangle + e^{i\alpha_3}\left|\psi_3\right\rangle + \ldots + e^{i\alpha_N}\left|\psi_N\right\rangle\big) . \tag{5.23}$$

Then

$$\big|\langle\psi'_k\,|\tilde\Psi_N(\alpha_2,\ldots,\alpha_N)\rangle\big| = \big|\langle\psi_k\,|\Psi_N(\alpha_2,\ldots,\alpha_N)\rangle\big| = \begin{cases} \frac{1}{\sqrt{N}} & \text{if } k \le N \\ 0 & \text{otherwise} . \end{cases} \tag{5.24}$$

Now let $|\Psi'_N(\alpha_2,\ldots,\alpha_N)\rangle = e^{i\theta}\,|\tilde\Psi_N(\alpha_2,\ldots,\alpha_N)\rangle$, where θ is chosen so that $\langle\psi'_1\,|\Psi'_N(\alpha_2,\ldots,\alpha_N)\rangle$ is real and positive. One can then write

$$|\Psi'_N(\alpha_2,\ldots,\alpha_N)\rangle = \frac{1}{\sqrt{N}}\Big(\left|\psi'_1\right\rangle + e^{i\alpha'_2}\left|\psi'_2\right\rangle + e^{i\alpha'_3}\left|\psi'_3\right\rangle + \ldots + e^{i\alpha'_N}\left|\psi'_N\right\rangle\Big) , \tag{5.25}$$

where the phases $\alpha'_2, \alpha'_3, \ldots, \alpha'_N$ are as yet undetermined functions of N and $\alpha_2, \alpha_3, \ldots, \alpha_N$.

We next consider the inner product of the vector $|\Psi'_N(\alpha_2,\ldots,\alpha_N)\rangle$ with $|\phi'_k\rangle$, as defined in Eq. (5.10):

$$\big|\langle\phi'_k\,|\Psi'_N(\alpha_2,\ldots,\alpha_N)\rangle\big| = \big|\langle\phi_k\,|\Psi_N(\alpha_2,\ldots,\alpha_N)\rangle\big| , \tag{5.26}$$

so for $k \le N$,

$$\frac{1}{\sqrt{2N}}\left|1 + e^{i\alpha'_k}\right| = \frac{1}{\sqrt{2N}}\left|1 + e^{i\alpha_k}\right| , \tag{5.27}$$

which implies through the lemma of Eq. (5.17) that

$$\alpha'_k = \pm\alpha_k \tag{5.28}$$

for all $k = 1, 2, \ldots, N$.

We can now prove by induction that $\alpha'_k = \alpha_k$ is the only solution. Note that we already know this to be true for $N = 1$, from Eqs. (5.1) and (5.9), and that we know it to be true for $N = 2$ from Eq. (5.19a). For the inductive proof we assume that it is true for $N = N_0$, where $N_0 \geq 2$, and try to prove that it is then necessarily true for $N = N_0 + 1$.

First we consider the inner product of $|\Psi'_N(\alpha_2, \alpha_3, \ldots, \alpha_N)\rangle$ with $\left|\Psi'_{N-1}(\alpha_2, \alpha_3, \ldots, \alpha_{N-1})\right\rangle$. The induction hypothesis allows us to assume that

$$\begin{aligned} &\left|\Psi'_{N-1}(\alpha_2, \ldots, \alpha_{N-1})\right\rangle = \\ &\qquad \frac{1}{\sqrt{N-1}}\Big(|\psi'_1\rangle + e^{i\alpha_2}\,|\psi_2\rangle + e^{i\alpha_3}\,|\psi'_3\rangle + \ldots + e^{i\alpha_{N-1}}\,\left|\psi'_{N-1}\right\rangle\Big) \ . \end{aligned} \tag{5.29}$$

So

$$\begin{aligned} &\left|\left\langle \Psi'_{N-1}(\alpha_2, \ldots, \alpha_{N-1}) \,\middle|\, \Psi'_N(\alpha_2, \ldots, \alpha_N)\right\rangle\right| = \\ &\qquad |\langle \Psi_{N-1}(\alpha_2, \ldots, \alpha_{N-1}) \,|\, \Psi_N(\alpha_2, \ldots, \alpha_N)\rangle| \end{aligned}$$

implies that

$$\left|1 + e^{i(\alpha'_2 - \alpha_2)} + \ldots + e^{i(\alpha'_{N-1} - \alpha_{N-1})}\right| = N - 1 \ . \tag{5.30}$$

The right-hand side is the maximum possible value of the left-hand side, and this maximum is achieved only if all the phases on the left-hand side coincide with the first one, which is zero (i.e., 1 is real and positive). Thus

$$\alpha'_k = \alpha_k \qquad \text{for } k = 2, 3, \ldots, N-1 \ . \tag{5.31}$$

We still need to determine the final phase, α'_N. Eq. (5.28) is sufficient to fix the answer for the special case $\alpha_N = 0$, so we can write

$$\begin{aligned} &|\Psi'_N(\beta_2, \ldots, \beta_{N-1}, 0)\rangle = \\ &\qquad \frac{1}{\sqrt{N}}\Big(|\psi'_1\rangle + e^{i\beta_2}\,|\psi'_2\rangle + e^{i\beta_3}\,|\psi'_3\rangle + \ldots + e^{i\beta_{N-1}}\,\left|\psi'_{N-1}\right\rangle + |\psi'_N\rangle\Big) \ . \end{aligned} \tag{5.32}$$

Then the probability preservation equation

$$\begin{aligned} &|\langle \Psi'_N(\beta_2, \ldots, \beta_{N-1}, 0) \,|\, \Psi'_N(\alpha_2, \ldots, \alpha_N)\rangle| = \\ &\qquad |\langle \Psi_N(\beta_2, \ldots, \beta_{N-1}, 0) \,|\, \Psi_N(\alpha_2, \ldots, \alpha_N)\rangle| \end{aligned}$$

implies that

$$\begin{aligned} &\left|1 + e^{i(\alpha_2 - \beta_2)} + \ldots + e^{i(\alpha_{N-1} - \beta_{N-1})} + e^{i\alpha'_N}\right| = \\ &\qquad \left|1 + e^{i(\alpha_2 - \beta_2)} + \ldots + e^{i(\alpha_{N-1} - \beta_{N-1})} + e^{i\alpha_N}\right| \ . \end{aligned} \tag{5.33}$$

Writing

$$z \equiv 1 + e^{i(\alpha_2 - \beta_2)} + \ldots + e^{i(\alpha_{N-1} - \beta_{N-1})} \equiv |z| e^{i\theta} \ , \tag{5.34}$$

the lemma of Eq. (5.17) can be applied to Eq. (5.33) to give

$$\alpha'_N = \alpha_N \quad \text{or} \quad \alpha'_N = -\alpha_N + 2\theta \ . \tag{5.35}$$

The first case is always consistent with Eq. (5.28), but the second case is consistent only if $\theta = 0$ or $\theta = \pi$. (It is also consistent if $\theta = \alpha_N$ or $\theta = \alpha_N + \pi$, but then it reduces to the first case and becomes irrelevant, since it does not provide an alternative solution.) Recall that α'_N can depend on N and $\alpha_1, \alpha_2, \ldots, \alpha_N$, as discussed after Eq. (5.25), but of course it cannot depend on the β_k that appear in Eq. (5.33). For any fixed values of $\alpha_1, \alpha_2, \ldots, \alpha_N$, one can always arrange for θ to avoid 0 and π by varying the β_k's appropriately. Here it is important that there is always at least one β_k appearing in Eq. (5.33), but since $N \geq 3$ that is always the case. Thus, the only possible solution is

$$\alpha'_N = \alpha_N \ , \tag{5.36}$$

and the inductive proof is complete for case A.

For case B the proof is essentially identical, but with many changes in signs. The only difference in the input is a change in the induction hypothesis, which is required for consistency with $N = 2$, which is now determined by Eq. (5.19b) instead of (5.19a). I will replace Eq. (5.25) by

$$|\Psi'_N(\alpha_2, \ldots, \alpha_N)\rangle = \frac{1}{\sqrt{N}}\Big(|\psi'_1\rangle + e^{-i\alpha'_2}\,|\psi'_2\rangle + e^{-i\alpha'_3}\,|\psi'_3\rangle + \ldots + e^{-i\alpha'_N}\,|\psi'_N\rangle\Big) \ , \tag{5.25$'$}$$

so we are again trying to prove that $\alpha'_k = \alpha_k$. Eq. (5.27) becomes

$$\frac{1}{\sqrt{2N}}\left|1 + e^{-i\alpha'_k}\right| = \frac{1}{\sqrt{2N}}\left|1 + e^{i\alpha_k}\right| \ , \tag{5.27$'$}$$

but the resulting Eq. (5.28) is unchanged. In Eqs. (5.29) and (5.30) every exponent is negated, but the resulting Eq. (5.31) remains unchanged. Eq. (5.32) becomes

$$\begin{aligned} &|\Psi'_N(\beta_2, \ldots, \beta_{N-1}, 0)\rangle = \\ &\qquad \frac{1}{\sqrt{N}}\Big(|\psi'_1\rangle + e^{-i\beta_2}\,|\psi'_2\rangle + e^{-i\beta_3}\,|\psi'_3\rangle + \ldots + e^{-i\beta_{N-1}}\,\left|\psi'_{N-1}\right\rangle + |\psi'_N\rangle\Big) \ , \end{aligned} \tag{5.32$'$}$$

and Eq. (5.33) becomes

$$\begin{aligned} &\left|1 + e^{-i(\alpha_2 - \beta_2)} + \ldots + e^{-i(\alpha_{N-1} - \beta_{N-1})} + e^{-i\alpha'_N}\right| = \\ &\qquad \left|1 + e^{i(\alpha_2 - \beta_2)} + \ldots + e^{i(\alpha_{N-1} - \beta_{N-1})} + e^{i\alpha_N}\right| \ . \end{aligned} \tag{5.33$'$}$$

However, by using $|z| = |z^*|$ to rewrite the left-hand side of Eq. (5.33′), the equation becomes identical to the original Eq. (5.33). The proof is then completed as in the previous case, showing that in this case, also, $\alpha'_N = \alpha_N$, so the proof by induction is complete.

(f) Rewrite $|\Psi\rangle$ as

$$|\Psi\rangle = \sum_{k=1}^{\infty} |C_k| e^{i\alpha_k} |\psi_k\rangle \ . \tag{5.34}$$

Starting first with case A, we can choose a vector $|\Psi'_+\rangle \in T(\Psi)$ and expand it in the basis $|\psi'_k\rangle$:

$$|\Psi'_+\rangle = \sum_{k=1}^{\infty} |C_k| e^{i\alpha'_k} |\psi'_k\rangle \ , \tag{5.35}$$

where I have already made use of the fact that the $|C_k|$ factors are unchanged, as can be shown by considering inner products with the $|\psi_k\rangle$. Since I can choose the phase of $|\Psi'_+\rangle$, I can insist that $\alpha'_k = \alpha_k$ for the first nonvanishing $|C_k|$. Now consider inner products with the states $|\Psi_N(\alpha_2,\ldots,\alpha_N)\rangle$ for any integer N. Preservation of probability implies that

$$\left|\langle \Psi'_{N,+}(\alpha_2,\ldots,\alpha_N) \,|\, \Psi'_+ \rangle\right| = |\langle \Psi_N(\alpha_2,\ldots,\alpha_N) \,|\, \Psi \rangle| \ , \tag{5.36}$$

so

$$\left| |C_1| + |C_2| e^{i(\alpha'_2-\alpha_2)} + \ldots + |C_N| e^{i(\alpha'_N-\alpha_N)} \right| = \sum_{k=1}^{N} |C_k| \ . \tag{5.37}$$

The right-hand side is the maximum possible value of the left-hand side, and this maximum can be achieved only if the phase of each nonvanishing term on the left-hand side is the same. Since we have chosen $\alpha'_k = \alpha_k$ for the first nonvanishing $|C_k|$, we know that if any $|C_k|$ is nonzero for $k \le N$, there is at least one nonvanishing term on the left for which the phase is zero. Therefore the phase must be zero for all nonvanishing terms, and hence $\alpha'_k = \alpha_k$ for any nonvanishing term. (If all terms vanish for $k = 1,\ldots,N$, then there are no phases to determine until we consider higher values of N.) Since we can choose N as large as we like, the argument applies to all terms, and therefore

$$|\Psi'_+\rangle = \sum_{k=1}^{\infty} C_k |\psi'_k\rangle \ , \tag{5.38}$$

as desired.

The argument can be easily adapted to case B. The modified equations become

$$|\Psi'_-\rangle = \sum_{k=1}^{\infty} |C_k| e^{-i\alpha'_k} |\psi'_k\rangle \ , \tag{5.35′}$$

$$\left|\langle \Psi'_{N,-}(\alpha_2,\ldots,\alpha_N) \,|\, \Psi'_- \rangle\right| = |\langle \Psi_N(\alpha_2,\ldots,\alpha_N) \,|\, \Psi \rangle| \ , \tag{5.36′}$$

$$\left| |C_1| + |C_2| e^{-i(\alpha'_2-\alpha_2)} + \ldots + |C_N| e^{-i(\alpha'_N-\alpha_N)} \right| = \sum_{k=1}^{N} |C_k| \ , \tag{5.37′}$$

and

$$|\Psi'_-\rangle = \sum_{k=1}^{\infty} C^*_k |\psi'_k\rangle \ . \tag{5.38′}$$

(g) For case A, linearity is shown by demonstrating Eq. (4) of the problem set:

$$\begin{aligned} U(\alpha\,|\Psi_1\rangle + \beta\,|\Psi_2\rangle) &= U\left(\alpha \sum_{k=1}^{\infty} C_k^{(1)} |\psi_k\rangle + \beta \sum_{k=1}^{\infty} C_k^{(2)} |\psi_k\rangle\right) \\ &= U \sum_{k=1}^{\infty} \left(\alpha C_k^{(1)} + \beta C_k^{(2)}\right) |\psi_k\rangle \\ &= \sum_{k=1}^{\infty} \left(\alpha C_k^{(1)} + \beta C_k^{(2)}\right) |\psi'_k\rangle \\ &= \alpha\, U\,|\Psi_1\rangle + \beta\, U\,|\Psi_2\rangle \ . \end{aligned} \tag{5.39}$$

For case B, antilinearity is shown by demonstrating Eq. (5) of the problem set:

$$\begin{aligned} U(\alpha\,|\Psi_1\rangle + \beta\,|\Psi_2\rangle) &= U\left(\alpha \sum_{k=1}^{\infty} C_k^{(1)} |\psi_k\rangle + \beta \sum_{k=1}^{\infty} C_k^{(2)} |\psi_k\rangle\right) \\ &= U \sum_{k=1}^{\infty} \left(\alpha C_k^{(1)} + \beta C_k^{(2)}\right) |\psi_k\rangle \\ &= \sum_{k=1}^{\infty} \left(\alpha^* C_k^{(1)*} + \beta^* C_k^{(2)*}\right) |\psi'_k\rangle \\ &= \alpha^*\, U\,|\Psi_1\rangle + \beta^*\, U\,|\Psi_2\rangle \ . \end{aligned} \tag{5.40}$$

For case A, unitarity is shown by demonstrating Eq. (6) of the problem set:

$$\begin{aligned} \langle U\Psi_2 \,|\, U\Psi_1 \rangle &= \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} C_k^{(2)*} C_\ell^{(1)} \langle \psi'_k \,|\, \psi'_\ell \rangle \\ &= \sum_{k=1}^{\infty} C_k^{(2)*} C_k^{(1)} \\ &= \langle \Psi_2 \,|\, \Psi_1 \rangle \ , \end{aligned} \tag{5.41}$$

where we have used the orthonormality of the $|\psi'_k\rangle$.

For case B, antiunitarity is shown by demonstrating Eq. (7) of the problem set:

$$\begin{aligned}\langle U\Psi_2\,|U\Psi_1\,\rangle &= \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty} C_k^{(2)} C_\ell^{(1)*}\,\langle\psi'_k\,|\psi'_\ell\,\rangle \\ &= \sum_{k=1}^{\infty} C_k^{(2)} C_k^{(1)*} \\ &= \langle\Psi_2\,|\Psi_1\,\rangle^* \ .\end{aligned} \tag{5.42}$$

(h) Suppose that U_1 and U_2 are both linear, unitary (or antilinear, antiunitary) transformations that represent T. Then $U_2^{-1}\,U_1$ is a linear, unitary transformation that maps each ray onto itself. Thus

$$U_2^{-1}\,U_1\,|\psi_k\rangle = e^{i\alpha_k}\,|\psi_k\rangle \quad . \tag{5.43}$$

Consider any pair of indices k and ℓ, with $\ell \neq k$. Then

$$U_2^{-1}\,U_1\,\frac{1}{\sqrt{2}}\left(|\psi_k\rangle + |\psi_\ell\rangle\right) = \frac{1}{\sqrt{2}}\left(e^{i\alpha_k}\,|\psi_k\rangle + e^{i\alpha_\ell}\,|\psi_\ell\rangle\right) \ . \tag{5.44}$$

If $\alpha_\ell \neq \alpha_k$, then the vector on the right belongs to a different ray from the vector on the left, which would violate our assumptions. Therefore α_k must be the same for all k, and

$$U_2^{-1}\,U_1 = e^{i\alpha} \ . \tag{5.45}$$

Thus U_2 is equal to U_1, up to an overall phase.

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†Solutions written by Jamie Portsmouth, Tom Faulkner, and Alan Guth.

¶Problem taken from David Tong's *Lectures on Quantum Field Theory*, http://www.damtp.cam.ac.uk/user/dt281/qft.html. Solution by Edward Farhi.

§Solution by Alan Guth.