MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

**8.323: Relativistic Quantum Field Theory I**

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# PROBLEM SET 10 SOLUTIONS

**Problem 1: Majorana Fermions (Peskin & Schroeder, Problem 3.4)** *(15 points)*†

(a) In a transformed frame the LHS of the Majorana equation of motion reads,

$$i\bar{\sigma}^{\mu}\partial'_{\mu}\chi'(x') - im\sigma^2\chi'^{*}(x') \tag{1.1}$$

where the primed quantities in the transformed frame are related to the unprimed quantities in the original frame by a Lorentz transformation,

$$\begin{aligned} x'^{\mu} &= \Lambda^{\mu}{}_{\nu}x^{\nu} \\ \chi'(x') &= \Lambda_L\chi(x) \\ \partial'_{\mu} &= (\Lambda^{-1})^{\nu}{}_{\mu}\partial_{\nu} \end{aligned} \tag{1.2}$$

We must show that Eq. (1.1) is equal to 0 given the Majorana equation is satisfied in the original frame.

$$i\bar{\sigma}^{\mu}\partial'_{\mu}\chi'(x') - im\sigma^2\chi^{*} = i\bar{\sigma}^{\mu}\Lambda_L(\Lambda^{-1})^{\nu}{}_{\mu}\partial_{\nu}\chi(x) - im\sigma^2\Lambda_L^{*}\chi^{*}(x) \tag{1.3}$$

Now using the transformation property of the gamma matrices,

$$\Lambda_{1/2}^{-1}\gamma^{\mu}\Lambda_{1/2} = \Lambda^{\mu}{}_{\nu}\gamma^{\nu} \tag{1.4}$$

and in the Weyl representation

$$\begin{aligned} \gamma^{\mu} &= \begin{pmatrix} 0 & \sigma^{\mu} \\ \bar{\sigma}^{\mu} & 0 \end{pmatrix} \\ \Lambda^{1/2} &= \begin{pmatrix} \Lambda_L & 0 \\ 0 & \Lambda_R \end{pmatrix} \end{aligned} \tag{1.5}$$

we can calculate the transformation property of the $\bar{\sigma}_{\mu}$

$$\Lambda_R^{-1}\bar{\sigma}^{\mu}\Lambda_L = \Lambda^{\mu}{}_{\nu}\bar{\sigma}^{\nu} \tag{1.6}$$

Also from the explicit form of the Lorentz transformation on Left and Right Weyl spinors,

$$\begin{aligned}\Lambda_L &= \exp\left(-i\vec{\theta}\cdot\vec{\sigma}/2 - \vec{\beta}\cdot\vec{\sigma}/2\right)\\ \Lambda_R &= \exp\left(-i\vec{\theta}\cdot\vec{\sigma}/2 + \vec{\beta}\cdot\vec{\sigma}/2\right)\end{aligned} \tag{1.7}$$

it is easy to show that

$$\sigma^2\Lambda_L^*\sigma^2 = \Lambda_R \tag{1.8}$$

Using Eq. (1.6) and Eq. (1.8) in Eq. (1.3) we get,

$$i\bar{\sigma}^\mu\Lambda_L(\Lambda^{-1})^\nu{}_\mu\partial_\nu\chi(x) - im\sigma^2\Lambda_L^*\chi^*(x) = \Lambda_R\left(i\bar{\sigma}^\mu\partial_\mu\chi(x) - im\sigma^2\chi^*(x)\right) = 0 \tag{1.9}$$

To get the Klein Gordan equation, apply $\sigma^\nu\partial_\nu$ to the Majorana equation and using the fact that $\sigma_2\sigma^\mu\sigma_2 = \bar{\sigma}^{\mu*}$ we get

$$i\sigma^\mu\bar{\sigma}^\nu\partial_\mu\partial_\nu\chi - im\sigma^2\bar{\sigma}^{\mu*}\partial_\mu\chi^* \tag{1.10}$$

The complex conjugate of the Majorana equation is

$$-i\bar{\sigma}^{*\mu}\partial_\mu\chi^* - im\sigma^2\chi = 0 \tag{1.11}$$

Plugging this in Eq. (1.10),

$$\sigma^\mu\bar{\sigma}^\nu\partial_\mu\partial_\nu\chi + m^2\chi = 0 \tag{1.12}$$

As the derivatives are symmetric in $\mu \leftrightarrow \nu$ we may write $\sigma^\mu\bar{\sigma}^\nu\partial_\mu\partial_\nu = \frac{1}{2}(\sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu)\partial_\mu\partial_\nu = g^{\mu\nu}\partial_\mu\partial_\nu = \Box$ so

$$(\Box + m^2)\chi = 0 \tag{1.13}$$

(b) Take the complex conjugate of the action

$$\begin{aligned}S^* &= \int d^4x\left(+\chi^T i\bar{\sigma}^{\mu*}\partial_\mu\chi^* - \frac{im}{2}\left(\chi^\dagger\sigma^2\chi^* - \chi^T\sigma^2\chi\right)\right)\\ &= \int d^4x\left(-i\partial_\mu\chi^\dagger\bar{\sigma}^\mu\chi + \frac{im}{2}\left(\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^*\right)\right)\end{aligned} \tag{1.14}$$

where for the kinetic term we transposed the matrix (note that $\bar{\sigma}$ is hermitian). The mass term is the same. Consider,

$$\begin{aligned}S - S^* &= \int d^4x\left(i\chi^\dagger\bar{\sigma}^\mu\partial_\mu\chi + i\partial_\mu\chi^\dagger\bar{\sigma}^\mu\chi\right)\\ &= \int d^4x i\partial_\mu\left(\chi^\dagger\bar{\sigma^\mu}\chi\right) = 0\end{aligned} \tag{1.15}$$

as this is a total derivative.

To find the equations of motion, vary the action with respect to $\chi$ remembering to treat $\delta\chi$ as a Grassman variable and to treat $\chi$ and $\chi^*$ independently.

$$\begin{aligned}\delta S = \int d^4x \bigg( & \chi^\dagger i\bar{\sigma}^\mu \partial_\mu \delta\chi + \frac{im}{2}\left(\delta\chi^T \sigma^2 \chi + \chi^T \sigma^2 \delta\chi\right) \\ & \delta\chi^\dagger i\bar{\sigma}^\mu \partial_\mu \chi - \frac{im}{2}\left(\delta\chi^\dagger \sigma^2 \chi^* + \chi^\dagger \sigma^2 \delta\chi^*\right)\bigg) \\ = \int d^4x & \left(\left(-i\partial_\mu \chi^\dagger \bar{\sigma}^\mu + im\chi^T\sigma^2\right)\delta\chi + \delta\chi^\dagger\left(i\bar{\sigma}^\mu\partial_\mu\chi - im\sigma^2\chi^*\right)\right)\end{aligned} \tag{1.16}$$

Setting the variation to zero gives the Majorana equation and the h.c. of the Majorana equation.

(c) The Dirac action is

$$\begin{aligned}S_D &= \int d^4x \bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi \\ &= \int d^4x \begin{pmatrix} -i\chi_2^T\sigma_2 & \chi_1^\dagger \end{pmatrix} \begin{pmatrix} -m & i\sigma^\mu\partial_\mu \\ i\bar{\sigma}^\mu\partial_\mu & -m \end{pmatrix} \begin{pmatrix} \chi_1 \\ i\sigma^2\chi_2^* \end{pmatrix} \\ &= \int d^4x \left(i\chi_2^T\bar{\sigma}^{T\mu}\partial_\mu\chi_2^* + i\chi_1^\dagger\bar{\sigma}^\mu\partial_\mu\chi_1 + im\left(\chi_2^T\sigma_2\chi_1 - \chi_1^\dagger\sigma^2\chi_2^*\right)\right)\end{aligned} \tag{1.17}$$

taking the transpose of the first term and integrating by parts,

$$S_D = \int d^4x \left(i\chi_2^\dagger\bar{\sigma}^\mu\partial_\mu\chi_2 + i\chi_1^\dagger\bar{\sigma}^\mu\partial_\mu\chi_1 + im\left(\chi_2^T\sigma_2\chi_1 - \chi_1^\dagger\sigma^2\chi_2^*\right)\right) \tag{1.18}$$

The equations of motion that come from this action are,

$$\begin{aligned} i\bar{\sigma}^\mu\partial_\mu\chi_1 - im\sigma^2\chi_2^* = 0 \\ i\bar{\sigma}^\mu\partial_\mu\chi_2 - im\sigma^2\chi_1^* = 0 \end{aligned} \tag{1.19}$$

(d) The divergence of the current for the Majorana theory is

$$\begin{aligned}\partial_\mu j^\mu &= \partial_\mu\chi^\dagger\bar{\sigma}^\mu\chi + \chi^\dagger\bar{\sigma}^\mu\partial_\mu\chi \\ &= m(\chi^T\sigma^2\chi + \chi^\dagger\sigma^2\chi^*)\end{aligned} \tag{1.20}$$

where we used the Majorana equation of motion and its Hermitian conjugate. We see that this is only a conserved current if $m = 0$.

The divergence of the current for the Dirac theory is,

$$\begin{aligned}\partial_\mu j^\mu &= \partial_\mu \chi_1^\dagger \bar{\sigma}^\mu \chi_1 + \chi_1^\dagger \bar{\sigma}^\mu \partial_\mu \chi_1 - \partial_\mu \chi_2^\dagger \bar{\sigma}^\mu \chi_2 + \chi_2^\dagger \bar{\sigma}^\mu \partial_\mu \chi_2 \\ &= m(\chi_2^T \sigma^2 \chi_1 + \chi_1^\dagger \sigma^2 \chi_2^* - \chi_1^T \sigma^2 \chi_2 - \chi_2^\dagger \sigma^2 \chi_1^*) = 0\end{aligned} \tag{1.21}$$

where in the last step we used Eqs. (1.19). This is a conserved current and is associated with the global symmetry,

$$\begin{aligned}\chi_1 &\to e^{i\alpha}\chi_1 \\ \chi_2 &\to e^{-i\alpha}\chi_2\end{aligned} \tag{1.22}$$

For the Dirac theory we can see that the $U(1)$ symmetry in Eq. (1.22) can be written as an $0(2)$ symmetry by changing to the fields

$$\begin{aligned}\phi_1 &= (\chi_1 + \chi_2)/\sqrt{2} \\ \phi_2 &= i(\chi_1 - \chi_2)/\sqrt{2}\end{aligned} \tag{1.23}$$

such that for infinitesimal $U(1)$ transformations,

$$\begin{aligned}\phi_1 &\to \phi_1 + \alpha\phi_2 \\ \phi_2 &\to \phi_2 - \alpha\phi_1\end{aligned} \tag{1.24}$$

The Dirac action in terms of $\phi_1$ and $\phi_2$ is,

$$S_D = \int d^4x \left( i\phi_1^\dagger \bar{\sigma}^\mu \partial_\mu \phi_1 + i\phi_2^\dagger \bar{\sigma}^\mu \partial_\mu \phi_2 + \frac{im}{2}\left( \phi_1^T \sigma_2 \phi_1 + \phi_2^T \sigma_2 \phi_2 - \phi_1^\dagger \sigma^2 \phi_1^* - \phi_2^\dagger \sigma^2 \phi_2^* \right)\right) \tag{1.25}$$

We generalize this $O(2)$ theory to an $O(N)$ theory,

$$S_D = \int d^4x \sum_{j=1}^{N} \left( i\chi_j^\dagger \bar{\sigma}^\mu \partial_\mu \chi_j + \frac{im}{2}\left( \chi_j^T \sigma_2 \chi_j - \chi_j^\dagger \sigma^2 \chi_j^* \right)\right) \tag{1.26}$$

The action is invariant under

$$\chi_j \to \chi_j' = O_{ji}\chi_i \tag{1.27}$$

for any $N \times N$ orthogonal matrix $OO^T = 1$. The currents are

$$J^{\mu}_{ij} = i\chi_i^{\dagger}\bar{\sigma}^{\mu}\chi_j + c.c \tag{1.28}$$

where the indicies $ij$ are antisymmetric, so there are $N(N-1)/2$ currents.

(e) To quantize the Majorana theory we simply realize that the Majorana theory is equivalent to the Dirac theory with an extra constraint $\chi_1 = \chi_2$ or

$$\psi^* = -i\gamma_2\psi \tag{1.29}$$

Then the Majorana action is half the Dirac action,

$$S_M = \frac{1}{2}S_D|_{\chi_1=\chi_2} \tag{1.30}$$

The anti-commutation relations follow, and the Hamiltonian,

$$H_M = \frac{1}{2}H_D|_{\chi_1=\chi_2} = -\int d^3x\left(i\chi^{\dagger}\sigma^i\partial_i\chi - \frac{im}{2}\left(\chi^T\sigma^2\chi - \chi^{\dagger}\sigma^2\chi^*\right)\right) \tag{1.31}$$

Consider the expansion for the Dirac field,

$$\psi(x) = \begin{bmatrix}\chi \\ i\sigma^2\chi^*\end{bmatrix} = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(a_p^s u_p^s e^{-ip\cdot x} + b_p^{\dagger s}v_p^s e^{ip\cdot x}\right) \ , \tag{1.32}$$

where $a_p^s$ and $b_p^s$ have the usual anticommutation relations. And

$$u_p^s = \begin{pmatrix}\sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi^s\end{pmatrix} \ , \quad v_p^s = \begin{pmatrix}\sqrt{p\cdot\sigma}\,\eta^s \\ -\sqrt{p\cdot\bar{\sigma}}\,\eta^s\end{pmatrix} \ . \tag{1.33}$$

This yields the constraint,

$$a_p^s\sqrt{p\cdot\sigma}\xi^s = b_p^s(\sigma^2\sqrt{p\cdot\bar{\sigma}^*}\sigma^2)(i\sigma^2\eta^{s*}) \ . \tag{1.34}$$

But $\sigma^2\sqrt{p\cdot\bar{\sigma}^*}\sigma^2 = \sqrt{p\cdot\sigma}$. Therefore we must have

$$a_p^s = b_p^s, \qquad \xi^s = i\sigma^2\eta^{s*} \ . \tag{1.35}$$

Substituting this, we find the full expression for the quantized Majorana field:

$$\chi(x) = \int\frac{d^3p}{(2\pi)^3}\frac{\sqrt{p\cdot\sigma}}{\sqrt{2E_p}}\left(a_p^s i\sigma^2\eta^{s*}e^{-ip\cdot x} + a_p^{s\dagger}\eta^s e^{ip\cdot x}\right) \ . \tag{1.36}$$

The Hamiltonian for the Majorana theory could be found by substituting the above expansion in Eq. (1.31). This is tedious, so we simply note that the result should be one half times the Hamiltonian of the Dirac theory with the constraint $a_p^s = b_p^s$,

$$H_M = \int\frac{d^3p}{(2\pi)^3}E_p\sum_s a_p^{s\dagger}a_p^s \ , \tag{1.37}$$

ignoring the zero-point energy. This is the required diagonalization.

**Problem 2: Supersymmetry (Peskin & Schroeder, Problem 3.5)** *(15 points)*¶

**(a)** We would like to show that the Lagrangian density,

$$\mathscr{L} = \partial_\mu \phi^* \partial^\mu \phi + \chi^\dagger i \bar{\sigma}^\mu \partial_\mu \chi + F^* F \,, \tag{2.1}$$

is invariant (up to a total divergence) under the infinitesimal transformation,

$$\begin{aligned} \delta\phi &= -i\epsilon^T \sigma^2 \chi \\ \delta\chi &= \epsilon F + \sigma^\mu \partial_\mu \phi \, \sigma^2 \epsilon^* \\ \delta F &= -i\epsilon^\dagger \bar{\sigma}^\mu \partial_\mu \chi \end{aligned} \tag{2.2}$$

where $\epsilon$ is a 2-component spinor of Grassmann numbers. (The sign of the second term in the $\delta\chi$ equation was misprinted in the earliest edition of Peskin & Schroeder, but it is shown correctly above and in all recent editions.) Note that this transformation is a supersymmetry, in the sense that it mixes bosons and fermions: the change in each Bose field is proportional to a Fermi field, and the change in each Fermi field is proportional to a Bose field.

The variation of the Lagrangian density is then

$$\delta\mathscr{L} = \partial_\mu(\delta\phi^*)\partial^\mu\phi + \partial_\mu\phi^*\partial^\mu(\delta\phi) + (\delta\chi^\dagger) i\bar{\sigma}^\mu \partial_\mu \chi + \chi^\dagger i \bar{\sigma}^\mu \partial_\mu (\delta\chi) + (\delta F)^* F + F^*(\delta F) \,. \tag{2.3}$$

Before evaluating this formula, it will be useful to review a few basic rules for the manipulations. The equations involve $\sigma^\mu$ and $\bar{\sigma}^\mu$, so we recall that $\sigma^\mu = (1, \vec{\sigma})$, $\bar{\sigma}^\mu = (1, -\vec{\sigma})$, and that

$$\{\sigma^\mu \,,\, \bar{\sigma}^\nu\} = \{\sigma^\nu \,,\, \bar{\sigma}^\mu\} = 2g^{\mu\nu} \;, \tag{2.4a}$$

$$[\sigma^\mu \,,\, \bar{\sigma}^\nu] = -[\sigma^\nu \,,\, \bar{\sigma}^\mu] \;, \tag{2.4b}$$

$$\bar{\sigma}^\mu \sigma^\nu = \frac{1}{2}\{\bar{\sigma}^\mu \,,\, \sigma^\nu\} + \frac{1}{2}[\bar{\sigma}^\mu \,,\, \sigma^\nu] = g^{\mu\nu} + \frac{1}{2}[\bar{\sigma}^\mu \,,\, \sigma^\nu] \;, \tag{2.4c}$$

$$\sigma^\nu \bar{\sigma}^\mu = \frac{1}{2}\{\sigma^\nu \,,\, \bar{\sigma}^\mu\} + \frac{1}{2}[\bar{\sigma}^\mu \,,\, \sigma^\nu] = g^{\mu\nu} + \frac{1}{2}[\sigma^\nu \,,\, \bar{\sigma}^\mu] \;, \tag{2.4d}$$

$$\sigma^2 \sigma^{\mu *} \sigma^2 = \bar{\sigma}^\mu \;. \tag{2.4e}$$

We will also need to take the complex conjugate of expressions involving Grassmann variables, so we must remember that the complex conjugate of a Grassmann expression is defined to reverse the order of the variables, simulating the effect of the adjoint of an operator expression. So, if $\alpha$ and $\beta$ are Grassmann vectors, and $M$ is a $c$-number matrix, then

$$(\alpha^T M\beta)^* = \sum_{ij}(\alpha_i M_{ij}\beta_j)^* = \sum_{ij}\beta_j^* M_{ij}^*\alpha_i^* = \sum_{ij}\beta_j^* M_{ji}^\dagger\alpha_i^* = \beta^\dagger M^\dagger\alpha^* \ . \tag{2.5}$$

We can alternatively write

$$(\alpha^T M\beta)^* = \sum_{ij}\beta_j^* M_{ij}^*\alpha_i^* = -\sum_{ij}\alpha_i^* M_{ij}^*\beta_j^* = -\alpha^\dagger M^*\beta^* \ , \tag{2.6}$$

but we will usually find the identity (2.5) more useful, since many of the matrices of interest will be self-adjoint, making Eq. (2.5) easy to evaluate.

Using these rules, we find

$$\begin{aligned}
\delta\phi^* &= i\chi^\dagger\sigma^2\epsilon^* \ ,\\
\delta\chi^\dagger &= F^*\epsilon^\dagger + \epsilon^T\sigma^2\sigma^\mu\partial_\mu\phi^* \ ,\\
\delta F^* &= i\partial_\mu\chi^\dagger\bar\sigma^\mu\epsilon \ .
\end{aligned} \tag{2.7}$$

Inserting Eqs. (2.2) and (2.7) into Eq. (2.3), we generate an 8-term equation, where for reference we label the 8 terms with the letters A through H:

$$\begin{aligned}
\delta\mathcal{L} = &\underbrace{\partial_\mu[i\chi^\dagger\sigma^2\epsilon^*]\partial^\mu\phi}_{A} + \underbrace{\partial_\mu\phi^*\partial^\mu[-i\epsilon^T\sigma^2\chi]}_{B} + \underbrace{F^*\epsilon^\dagger i\bar\sigma^\mu\partial_\mu\chi}_{C} + \underbrace{\epsilon^T\sigma^2\sigma^\nu\partial_\nu\phi^* i\bar\sigma^\mu\partial_\mu\chi}_{D}\\
&+ \underbrace{\chi^\dagger i\bar\sigma^\mu\partial_\mu[\epsilon F]}_{E} + \underbrace{\chi^\dagger i\bar\sigma^\mu\partial_\mu[\sigma^\nu\partial_\nu\phi\sigma^2\epsilon^*]}_{F} + \underbrace{i\partial_\mu\chi^\dagger\bar\sigma^\mu\epsilon F}_{G} + \underbrace{F^*[-i\epsilon^\dagger\bar\sigma^\mu\partial_\mu\chi]}_{H} \ .
\end{aligned} \tag{2.8}$$

Now we must try to combine like terms, noticing for example that the terms $A$ and $F$ are the only terms that depend on $\chi^\dagger$ and $\phi$. So,

$$\begin{aligned}
A + F &= i\partial^\mu\phi\partial_\mu\chi^\dagger\sigma^2\epsilon^* + i\partial^2_{\mu\nu}\phi\chi^\dagger\bar\sigma^\mu\sigma^\nu\sigma^2\epsilon^*\\
&= i\partial^\mu\phi\partial_\mu\chi^\dagger\sigma^2\epsilon^* + ig^{\mu\nu}\partial^2_{\mu\nu}\phi\chi^\dagger\sigma^2\epsilon^*\\
&= \partial_\mu[i\partial^\mu\chi^\dagger\sigma^2\epsilon^*] \ ,
\end{aligned} \tag{2.9}$$

where we have used identity (2.4c), followed by the recognition from identity (2.4b) that the commutator is antisymmetric in $\mu$ and $\nu$, and hence vanishes when contracted with $\partial^2_{\mu\nu}\phi$. Continuing,

$$\begin{aligned}
B + D &= -i\partial^\mu\phi^*\epsilon^T\sigma^2\partial_\mu\chi + i\partial_\nu\phi^*\epsilon^T\sigma^2\sigma^\nu\bar\sigma^\mu\partial_\mu\chi\\
&= -i\partial^\mu\phi^*\epsilon^T\sigma^2\partial_\mu\chi + i\partial^\mu\phi^*\epsilon^T\sigma^2\partial_\mu\chi + \frac{i}{2}\partial_\nu\phi^*\epsilon^T\sigma^2\left[\sigma^\nu\, ,\,\bar\sigma^\mu\right]\partial_\mu\chi\\
&= \partial_\mu\left[\frac{i}{2}\partial_\nu\phi^*\epsilon^T\sigma^2\left[\sigma^\nu\, ,\,\bar\sigma^\mu\right]\chi\right] \ .
\end{aligned} \tag{2.10}$$

$$C + H = iF^*\epsilon^\dagger\bar\sigma^\mu\partial_\mu\chi - iF^*\epsilon^\dagger\bar\sigma^\mu\partial_\mu\chi = 0 \ . \tag{2.11}$$

$$E + G = i\partial_\mu F\chi^\dagger\bar\sigma^\mu\epsilon + iF\partial_\mu\chi^\dagger\bar\sigma^\mu\epsilon = \partial_\mu[iF\chi^\dagger\bar\sigma^\mu\epsilon] \ . \tag{2.12}$$

(Note that the final expression in Eq. (2.10) could just as well have been written as

$$\partial_\nu \left[ \frac{i}{2} \phi^* \epsilon^T \sigma^2 \left[ \sigma^\nu \, , \, \bar{\sigma}^\mu \right] \partial_\mu \chi \right] \, ,$$

so both answers would be correct.) Putting all the terms together,

$$\boxed{\delta \mathcal{L} = \partial_\mu \left\{ i F \chi^\dagger \bar{\sigma}^\mu \epsilon + i \partial^\mu \phi \chi^\dagger \sigma^2 \epsilon^* + \frac{i}{2} \partial_\nu \phi^* \epsilon^T \sigma^2 \left[ \sigma^\nu \, , \, \bar{\sigma}^\mu \right] \chi \right\} \quad .} \tag{2.13}$$

**(b)** The mass term is described by

$$\Delta \mathcal{L} = \left[ m \phi F + \frac{i}{2} m \chi^T \sigma^2 \chi \right] + \text{c.c.} \tag{2.14}$$

We can express its variation under the supersymmetry transformation of Eq. (2.2) as follows, where again I label the terms for future reference:

$$\begin{aligned} \delta \Delta \mathcal{L} &= \left\{ m \, \delta\phi \, F + m \phi \, \delta F + \frac{i}{2} m (\delta \chi)^T \sigma^2 \chi + \frac{i}{2} m \chi^T \sigma^2 \, \delta\chi \right\} + \text{c.c.} \\ &= \Bigg\{ \underbrace{m[-i \epsilon^T \sigma^2 \chi] F}_{A} + \underbrace{m \phi [-i \epsilon^\dagger \bar{\sigma}^\mu \partial_\mu \chi]}_{B} \\ &\qquad + \underbrace{\frac{i}{2} m [F \epsilon^T] \sigma^2 \chi}_{C} + \underbrace{\frac{i}{2} m \left[ (\sigma^\mu \partial_\mu \phi \, \sigma^2 \epsilon^*)^T \right] \sigma^2 \chi}_{D} \\ &\qquad + \underbrace{\frac{i}{2} m \chi^T \sigma^2 [\epsilon F]}_{E} + \underbrace{\frac{i}{2} m \chi^T \sigma^2 [\sigma^\mu \partial_\mu \phi \, \sigma^2 \epsilon^*]}_{F} \Bigg\} + \text{c.c.} \; . \end{aligned} \tag{2.15}$$

Note that terms $A$, $C$, and $E$ are proportional to $\epsilon$ (or $\epsilon^T$), while $B$, $D$, and $F$ are proportional to $\epsilon^*$ (or $\epsilon^\dagger$), so we should consider these groups separately. $A$ and $C$ are identical in form, both proportional to $\epsilon^T \sigma^2 \chi$. $E$ is proportional to $\chi^T \sigma^2 \epsilon$, but the two expressions can be related:

$$\chi^T \sigma^2 \epsilon = \sum_{ij} \chi_i \sigma^2_{ij} \epsilon_j = - \sum_{ij} \epsilon_j \sigma^2_{ij} \chi_i = -\epsilon^T (\sigma^2)^T \chi = \epsilon^T \sigma^2 \chi \; . \tag{2.16}$$

With this equality, one can see immediately that $A + C + E = 0$. Then $D$ can be simplified by

$$D = \frac{i}{2} m (\sigma^\mu \partial_\mu \phi \sigma^2 \epsilon^*)^T \sigma^2 \chi = \frac{i}{2} m \partial_\mu \phi \epsilon^\dagger (\sigma^2)^T (\sigma^\mu)^T \sigma^2 \chi \; . \tag{2.17}$$

But $(\sigma^2)^T = -\sigma^2$ and $(\sigma^\mu)^T = \sigma^{\mu *}$, so

$$D = -\frac{i}{2} m \partial_\mu \phi \epsilon^\dagger \sigma^2 \sigma^{\mu *} \sigma^2 \chi = -\frac{i}{2} m \partial_\mu \phi \epsilon^\dagger \bar{\sigma}^\mu \chi \ , \tag{2.18}$$

where we have used identity (2.4e). Then we can rewrite $F$ in a form that resembles $B$ and $D$:

$$\begin{aligned} F = \frac{i}{2} m \chi^T \sigma^2 \sigma^\mu \partial_\mu \phi \, \sigma^2 \epsilon^* &= -\frac{i}{2} m \partial_\mu \phi \epsilon^\dagger (\sigma^2)^T (\sigma^\mu)^T (\sigma^2)^T \chi \\ = -\frac{i}{2} m \partial_\mu \phi \epsilon^\dagger \sigma^2 \sigma^{\mu *} \sigma^2 \chi &= -\frac{i}{2} m \partial_\mu \phi \epsilon^\dagger \bar{\sigma}^\mu \chi = D \ . \end{aligned} \tag{2.19}$$

With these simplifications, one can immediately see that

$$\boxed{\delta \Delta \mathcal{L} = B + D + F = \partial_\mu [-im\phi \epsilon^\dagger \bar{\sigma}^\mu \chi] \ .} \tag{2.20}$$

The Euler-Lagrange equations for $F$ and $F^*$ are

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial F} = 0 \qquad &\Longrightarrow \qquad F^* = -m\phi \\ \frac{\partial \mathcal{L}}{\partial F^*} = 0 \qquad &\Longrightarrow \qquad F = -m\phi^* . \end{aligned} \tag{2.21}$$

Substituting these expressions for $F$ and $F^*$ in $\mathcal{L} + \Delta\mathcal{L}$, one finds

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi + \chi^\dagger i \bar{\sigma}^\mu \partial_\mu \chi + m^2 \phi^* \phi - m^2 \phi \phi^* - m^2 \phi^* \phi + \frac{im}{2} \left[ \chi^T \sigma^2 \chi - \chi^\dagger \sigma^2 \chi^* \right]$$

$$\boxed{\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi + \chi^\dagger i \bar{\sigma}^\mu \partial_\mu \chi - m^2 \phi^* \phi + \frac{im}{2} \left[ \chi^T \sigma^2 \chi - \chi^\dagger \sigma^2 \chi^* \right] \ ,} \tag{2.22}$$

which is a Lagrangian with mass terms for both $\phi$ and $\chi$ with the same mass $m$.

**(c)** Now we generalize the theory to include $n$ complex scalar fields $\phi_i(x)$, and $n$ Weyl fermion fields $\chi_i(x)$, and we also modify the Lagrangian density by adding the following expression to the free field Lagrangian density of part (a):

$$\Delta \mathcal{L} = \left\{ F_i \frac{\partial W[\phi]}{\partial \phi_i} + \frac{i}{2} \frac{\partial^2 W[\phi]}{\partial \phi_i \partial \phi_j} \chi_i^T \sigma^2 \chi_j \right\} + \text{c.c.} \ , \tag{2.23}$$

where $W[\phi]$ is an arbitrary function of the $\phi_i$, called the superpotential. At this stage we have a full, interacting supersymmetric field theory. The supersymmetry transformation is still described by $\epsilon$, a single 2-component spinor of Grassmann numbers, with Eq. (2.2) replaced by the generalization

$$\begin{aligned} \delta\phi_i &= -i\epsilon^T\sigma^2\chi_i \\ \delta\chi_i &= \epsilon F_i + \sigma^\mu\partial_\mu\phi_i\,\sigma^2\epsilon^* \\ \delta F_i &= -i\epsilon^\dagger\bar{\sigma}^\mu\partial_\mu\chi_i\ . \end{aligned} \tag{2.24}$$

The free field part of the Lagrangian density is supersymmetric as before since it is simply $n$ copies of the original Lagrangian. For the interacting piece of the Lagrangian density, we calculate the variation under the supersymmetry transformation as follows. For brevity, we introduce the notation $W_i \equiv \partial W/\partial\phi_i$, $W_{ij} \equiv \partial^2 W/\partial\phi_i\partial\phi_j$, etc., which allows us to rewrite the interacting piece of the Lagrangian density as

$$\Delta\mathcal{L} = F_iW_i + \frac{i}{2}W_{ij}\left(\chi_i^T\sigma^2\chi_j\right) + \text{c.c.}\ . \tag{2.25}$$

The variation is then given by

$$\begin{aligned} \delta\Delta\mathcal{L} &= \delta F_iW_i + F_iW_{ij}\delta\phi_j + \frac{i}{2}W_{ij}\left(\delta\chi_i^T\sigma^2\chi_j + \chi_i^T\sigma^2\delta\chi_j\right) + \frac{i}{2}W_{ijk}\delta\phi_k\left(\chi_i^T\sigma^2\chi_j\right) + \text{c.c.} \\ &= \delta F_iW_i + F_iW_{ij}\delta\phi_j + iW_{ij}\left(\delta\chi_i^T\sigma^2\chi_j\right) + \frac{i}{2}W_{ijk}\delta\phi_k\left(\chi_i^T\sigma^2\chi_j\right) + \text{c.c.} \\ &= \underbrace{-iW_i\left(\epsilon^\dagger\bar{\sigma}^\mu\partial_\mu\chi_i\right)}_{A}\underbrace{-iF_iW_{ij}\left(\epsilon^T\sigma^2\chi_j\right)}_{B} + \underbrace{iF_iW_{ij}\left(\epsilon^T\sigma^2\chi_j\right)}_{C} \\ &\quad + \underbrace{iW_{ij}\left(\left[\sigma^\mu\partial_\mu\phi_i\sigma^2\epsilon^*\right]^T\sigma^2\chi_j\right)}_{D} + \underbrace{\frac{1}{2}W_{ijk}\left(\epsilon^T\sigma^2\chi_k\right)\left(\chi_i^T\sigma^2\chi_j\right)}_{E} + \text{c.c.}\ . \end{aligned} \tag{2.26}$$

To proceed, simplify $D$ by

$$\begin{aligned} D &= iW_{ij}\partial_\mu\phi_i\epsilon^\dagger(\sigma^2)^T(\sigma^\mu)^T\sigma^2\chi_j = -iW_{ij}\partial_\mu\phi_i\epsilon^\dagger\sigma^2\sigma^{\mu*}\sigma^2\chi_j \\ &= -iW_{ij}\partial_\mu\phi_i\epsilon^\dagger\bar{\sigma}^\mu\chi_j\ . \end{aligned} \tag{2.27}$$

Now one can see that $A + D = \partial_\mu(-iW_{ij}\phi_i\epsilon^\dagger\bar{\sigma}^\mu\chi_j)$, and obviously $B$ and $C$ cancel. This leaves term $E$, and there are a variety of ways to show that $E$ vanishes identically. The easiest is to imagine writing $\chi_i$ explicitly as a 2-component spinor,

$$\chi_i \equiv \begin{pmatrix} \chi_i^1 \\ \chi_i^2 \end{pmatrix}\ ,$$

and expanding $E$ explicitly in terms of $\chi_i^1$ and $\chi_i^2$. Each term will involve the product of three $\chi$ fields, so at least two of them will always have the same upper index. But $\chi_i^a \chi_j^a = -\chi_j^a \chi_i^a$, so each term will be antisymmetric in two out of the three indices $i$, $j$, and $k$, and will therefore give a vanishing contribution when contracted with the fully symmetric $W_{ijk}$. Thus,

$$\boxed{\delta\Delta\mathcal{L} = \partial_\mu(-iW_{ij}\phi_i\epsilon^\dagger\bar{\sigma}^\mu\chi_j) \ ,} \tag{2.28}$$

so the theory is supersymmetric as advertised.

Finally, let us derive the equations of motions for $\phi_i, \chi_i, F_i$. The Euler-Lagrange equations become

$$\boxed{\begin{aligned} \Box\phi_i^* &= F_j\frac{\partial^2 W}{\partial\phi_i\partial\phi_j} + \frac{i}{2}\frac{\partial^3 W}{\partial\phi_i\partial\phi_j\partial\phi_k}\chi_j^T\sigma^2\chi_k \ , \\ \overline{\sigma}^\mu\partial_\mu\chi_i &= \left[\frac{\partial^2 W}{\partial\phi_i\partial\phi_j}\right]^*\sigma^2\chi_j^* \ , \\ &\text{and} \\ F_i &= -\left(\frac{\partial W}{\partial\phi_i}\right)^* \ . \end{aligned}} \tag{2.29}$$

With the choice $n = 1$ and $W = g\phi^3/3$, we find the field equations:

$$\boxed{\begin{aligned} F &= -\left(\frac{\partial W}{\partial\phi}\right)^* = -(g\phi^2)^* \ , \\ \Box\phi &= F^*\frac{\partial^2 W}{\partial\phi^2} - \frac{i}{2}\frac{\partial^3 W}{\partial\phi^3}\chi^\dagger\sigma^2\chi^* \\ &= -2\phi|g\phi|^2 - ig\chi^\dagger\sigma^2\chi^* \ , \\ i\overline{\sigma}\cdot\partial\chi &= 2ig\phi^*\sigma^2\chi^* \ . \end{aligned}} \tag{2.30}$$

**Problem 3: P, C, and T for Scalar and Dirac theories (Peskin & Schroeder, Problem 3.7)** *(10 points)*§

(a) We assume that $\eta_a = 1$ (see Peskin and Schroeder 3.123) and $P^2 = 1$. Then

$$\begin{aligned} P\bar{\psi}\sigma^{\mu\nu}\psi P &= P\bar{\psi}P\sigma^{\mu\nu}P\psi P \\ &= \bar{\psi}\gamma^0\sigma^{\mu\nu}\gamma^0\psi, \end{aligned} \tag{3.1}$$

using Peskin and Schroeder 3.126 and 3.128. Direct computation reveals that

$$\gamma^0\sigma^{0i}\gamma^0 = -\sigma^{0i} \qquad \text{and} \qquad \gamma^0\sigma^{ij}\gamma^0 = \sigma^{ij}.$$

So

$$P\bar{\psi}\sigma^{\mu\nu}\psi P = (-1)^\mu(-1)^\nu\bar{\psi}\sigma^{\mu\nu}\psi.$$

Next,

$$\begin{aligned} T\bar{\psi}\sigma^{\mu\nu}\psi T &= T\bar{\psi}T(\sigma^{\mu\nu})^*T\psi T \\ &= \bar{\psi}(\gamma^1\gamma^3)(\sigma^{\mu\nu})^*(-\gamma^1\gamma^3)\psi \end{aligned} \tag{3.2}$$

from Peskin and Schroeder 3.139 and 3.140. Using

$$\gamma^1\gamma^3(\gamma^\mu)^* = (-1)^\mu\gamma^\mu\gamma^1\gamma^3, \tag{3.3}$$

we obtain

$$\begin{aligned} (\gamma^1\gamma^3)(\sigma^{\mu\nu})^*(-\gamma^1\gamma^3) &= (\gamma^1\gamma^3)\frac{-i}{2}[(\gamma^\mu)^*,(\gamma^\nu)^*](-\gamma^1\gamma^3) \\ &= (-1)^\mu(-1)^\nu\frac{-i}{2}[\gamma^\mu,\gamma^\nu](\gamma^1\gamma^3)(-\gamma^1\gamma^3) \\ &= (-1)^\mu(-1)^\nu\frac{-i}{2}[\gamma^\mu,\gamma^\nu] \\ &= -(-1)^\mu(-1)^\nu\sigma^{\mu\nu}. \end{aligned} \tag{3.4}$$

Thus,

$$T\bar{\psi}\sigma^{\mu\nu}\psi T = -(-1)^\mu(-1)^\nu\bar{\psi}\sigma^{\mu\nu}\psi. \tag{3.5}$$

Finally, we use Peskin and Schroeder 3.145 and 3.146 to calculate

$$\begin{aligned} C\bar{\psi}\sigma^{\mu\nu}\psi C &= C\bar{\psi}C\sigma^{\mu\nu}C\psi C \\ &= (-i\gamma^0\gamma^2\psi)^T\sigma^{\mu\nu}(-i)(\bar{\psi}\gamma^0\gamma^2)^T \\ &= -\psi^T(\gamma^2)^T(\gamma^0)^T\sigma^{\mu\nu}(\gamma^2)^T(\gamma^0)^T\bar{\psi}^T \\ &= \bar{\psi}\gamma^0\gamma^2(\sigma^{\mu\nu})^T\gamma^0\gamma^2\psi, \end{aligned} \tag{3.6}$$

where we have used the anticommutation of Grassmann numbers in the last line. Now use $\gamma^0\gamma^2(\gamma^\mu)^T = -\gamma^\mu\gamma^0\gamma^2$ to show $\gamma^0\gamma^2(\sigma^{\mu\nu})^T\gamma^0\gamma^2 = -\sigma^{\mu\nu}$, and conclude

$$C\bar{\psi}\sigma^{\mu\nu}\psi C = -\bar{\psi}\sigma^{\mu\nu}\psi.$$

(b)

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}(a(\vec{p})e^{ip\cdot x} + b^\dagger(\vec{p})e^{-ip\cdot x})$$

The natural definitions of $C$, $P$, and $T$ are

$$\begin{aligned} Ca(\vec{p})C &= b(\vec{p}), \\ Pa(\vec{p})P &= a(-\vec{p}), \\ Ta(\vec{p})T &= a(-\vec{p}), \\ Cb(\vec{p})C &= a(\vec{p}), \\ Pb(\vec{p})P &= b(-\vec{p}), \end{aligned} \tag{3.7}$$

and

$$Tb(\vec{p})T = b(-\vec{p}),$$

where $T$ is anti-unitary. Then

$$\begin{aligned} C\phi(t,\vec{x})C &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (b(\vec{p})e^{ip\cdot x} + a^\dagger(\vec{p})e^{-ip\cdot x}) \\ &= \phi^*(t,\vec{x}) \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} P\phi(t,\vec{x})P &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (a(-\vec{p})e^{ip\cdot x} + b^\dagger(-\vec{p})e^{-ip\cdot x}) \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (a(\vec{p})e^{i(p^0t-\vec{p}\cdot(-\vec{x}))} + b^\dagger(\vec{p})e^{-i(p^0t-\vec{p}\cdot(-\vec{x}))} \\ &= \phi(t,-\vec{x}) \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} T\phi(t,\vec{x})T &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (a(-\vec{p})e^{-ip\cdot x} + b^\dagger(-\vec{p})e^{ip\cdot x}) \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (a(\vec{p})e^{-i(p^0t-\vec{p}\cdot(-\vec{x}))} + b^\dagger(\vec{p})e^{i(p^0t-\vec{p}\cdot(-\vec{x}))} \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (a(\vec{p})e^{i(p^0(-t)-\vec{p}\cdot\vec{x})} + b^\dagger(\vec{p})e^{-i(p^0(-t)-\vec{p}\cdot\vec{x})} \\ &= \phi(-t,\vec{x}). \end{aligned} \tag{3.10}$$

Good. Now check

$$\begin{aligned} C:J^\mu(t,\vec{x}):C &= C:i(\phi^*\partial^\mu\phi - \partial^\mu\phi^*\phi)(t,\vec{x}):C \\ &= i(C\phi^*C:C\partial^\mu\phi C - C\partial^\mu\phi^*C:C\phi C)(t,\vec{x}) \\ &= i(\phi\partial^\mu\phi^* - \partial^\mu\phi\phi^*)(t,\vec{x}) \\ &= -J^\mu(t,\vec{x}). \end{aligned} \tag{3.11}$$

And using $P\partial^\mu P = (-1)^\mu$ we get

$$\begin{aligned} P:J^\mu(t,\vec{x}):P &= i(\phi^*(-1)^\mu\partial^\mu\phi - (-1)^\mu\partial^\mu\phi^*\phi)(t,-\vec{x}) \\ &= (-1)^\mu J^\mu(t,-\vec{x}). \end{aligned} \tag{3.12}$$

And using $T\partial^\mu T = -(-1)^\mu$ we get

$$\begin{aligned} T:J^\mu(t,\vec{x}):T &= -i(\phi^*(-1)(-1)^\mu\partial^\mu\phi - (-1)(-1)^\mu\partial^\mu\phi^*\phi)(-t,\vec{x}) \\ &= (-1)^\mu J^\mu(-t,\vec{x}) \ . \end{aligned} \tag{3.13}$$

.

(c) A Hermitian, Lorentz-invariant, local operator containing $\phi$ will always also contain $\phi^\dagger$. Likewise, $\psi$ will always be accompanied by $\bar{\psi}$. Any Lorentz index $\mu$, as in $\gamma_\mu$ or $\partial_\mu$, will always be contracted with another $\mu$. We see from the table on page 71 of Peskin and Schroeder that this will always give something even under CPT.

**Problem 4 (Extra Credit): Decay of a Scalar Particle (Peskin & Schroeder, Problem 4.2)** *(10 points extra credit)*[‡]

The interaction Hamiltonian is

$$H_I = \int d^3x \ \mu\Phi_I\phi_I\phi_I \ .$$

It is useful to remember that the free Hamiltonian being bilinear in the derivatives of the fields sets the dimension (in powers of mass) of the scalar fields to 1. Therefore the parameter $\mu$ must be of dimension 1 too. The lowest order Feynman diagram is a vertex with one $\Phi$ leg and two $\phi$ legs, which yields $-2i\mu$. Thus the matrix element $\mathcal{M} = -2\mu$. From Peskin and Schroeder Eq. (4.86),

$$\Gamma = \frac{1}{2}\frac{1}{2M}\int \frac{d^3p_1}{(2\pi)^3}\frac{d^3p_2}{(2\pi)^3}\frac{1}{2E_1}\frac{1}{2E_2}(2\mu)^2(2\pi)^4 \ \delta^{(4)}(P-p_1-p_2) \ ,$$

where the factor $1/2$ in front arises because the product particles are identical. Let us assume that the decaying particle is at rest, with 4-momentum $P = (M,0,0,0)$. Then the momenta and energies of the product particles are given by $\vec{p}_2 = -\vec{p}_1$, $E_1 = E_2 = \sqrt{m^2+p_1^2}$. Now the calculation is straightforward:

$$\begin{aligned} \Gamma &= \frac{\mu^2}{4M}\int \frac{d^3p}{(2\pi)^2}\frac{1}{m^2+p^2}\delta(M-2\sqrt{m^2+p^2}) \\ &= \frac{\mu^2}{4\pi M}\int dp\frac{p^2}{m^2+p^2}\delta(M-2\sqrt{m^2+p^2}) \\ &= \frac{\mu^2}{8\pi M}\sqrt{1-\frac{4m^2}{M^2}} \ . \end{aligned}$$

We see that $\Gamma$ has the dimension of a mass as it should, the lifetime is given by $\tau = 1/\Gamma$. The fact that the formula gives an imaginary value for $M < 2m$ reflects the kinematical fact that a particle cannot decay in two particles whose combined rest mass is greater than its mass.

**Problem 5 (Extra Credit): The Linked Cluster Theorem** *(10 points extra credit)*$^{\otimes}$

**(a)** There are four proper partitions of $\{1, 2, 3\}$:

$$\{1\}, \{2, 3\}; \qquad \{2\}, \{1, 3\}; \qquad \{3\}, \{1, 2\}; \qquad \{1\}, \{2\}, \{3\} \ .$$

Writing out the definition explicitly,

$$\langle ABC \rangle_c = \langle ABC \rangle - \langle A \rangle_c \langle BC \rangle_c - \langle B \rangle_c \langle AC \rangle_c - \langle C \rangle_c \langle AB \rangle_c - \langle A \rangle_c \langle B \rangle_c \langle C \rangle_c \ .$$

Now use

$$\langle AB \rangle_c = \langle AB \rangle - \langle A \rangle \langle B \rangle \qquad \text{and} \qquad \langle A \rangle_c = \langle A \rangle \ ,$$

so

$$\boxed{\langle ABC \rangle_c = \langle ABC \rangle - \langle A \rangle \langle BC \rangle - \langle B \rangle \langle AC \rangle - \langle C \rangle \langle AB \rangle + 2 \langle A \rangle \langle B \rangle \langle C \rangle \ .} \tag{5.1}$$

We obtain an expression for $\langle A^3 \rangle_c$ by setting $B = A$ and $C = A$ in the above expression:

$$\boxed{\langle A^3 \rangle_c = \langle A^3 \rangle - 3 \langle A \rangle \langle A^2 \rangle + 2 \langle A \rangle^3 \ .} \tag{5.2}$$

**(b)** We will use mathematical induction on the sum

$$J \equiv m + n \ . \tag{5.3}$$

We start with $J = 2$, for which the only choice with $m \geq 1$ and $n \geq 1$ is $m = 1$, $n = 1$. The statement we are trying to prove,

$$\langle A_{i_1} \ldots A_{i_m} \, B_{j_1} \ldots B_{j_n} \rangle_c = 0 \ , \tag{5.4}$$

reduces to

$$\langle AB \rangle_c = \langle AB \rangle - \langle A \rangle \langle B \rangle = 0 \ , \tag{5.5}$$

which is certainly true by Eq. (5.4) of the problem set. To use induction, we now assume that Eq. (5.4) of these solutions is valid for all $J < J_0$, and we show that it must then hold for $J = J_0$. So we consider the quantity $\langle A_{i_1} \ldots A_{i_m} \, B_{j_1} \ldots B_{j_n} \rangle_c$, where $m + n = J_0$, which must be shown to vanish. By definition,

$$\langle A_{i_1} \ldots A_{i_m} \, B_{j_1} \ldots B_{j_n} \rangle_c = \langle A_{i_1} \ldots A_{i_m} \, B_{j_1} \ldots B_{j_n} \rangle - \sum_{\substack{\text{all proper} \\ \text{partitions of} \\ \{1, \ldots, J_0\}}} \left\langle \prod_{i,j \in S_1} A_i B_j \right\rangle_c \cdots \left\langle \prod_{i,j \in S_k} A_i B_j \right\rangle_c \ . \tag{5.6}$$

All the connected expectation values on the right-hand side have fewer than $J_0$ factors, and therefore the induction hypothesis implies that the only nonzero contributions come from terms which separately partition the $A_i$'s and the $B_j$'s. That is, $\left\langle \prod A_i B_j \right\rangle_c = 0$ if the product includes both $A_i$ factors and $B_j$ factors. Thus, Eq. (5.6) reduces to

$$\begin{aligned}
\langle A_{i_1} \dots A_{i_m} B_{j_1} \dots B_{j_n} \rangle_c = \langle A_{i_1} \dots A_{i_m} B_{j_1} \dots B_{j_n} \rangle \\
- \sum_{\substack{\text{all partitions} \\ \text{of } \{1,\dots,m\}}} \left\langle \prod_{i\in S_1} A_i \right\rangle_c \cdots \left\langle \prod_{i\in S_k} A_i \right\rangle_c \\
\times \sum_{\substack{\text{all partitions} \\ \text{of } \{1,\dots,n\}}} \left\langle \prod_{j\in S'_1} B_j \right\rangle_c \cdots \left\langle \prod_{j\in S'_{k'}} B_j \right\rangle_c ,
\end{aligned}$$

which by the definition of connected parts reduces to

$$\begin{aligned}
&= \langle A_{i_1} \dots A_{i_m} B_{j_1} \dots B_{j_n} \rangle - \langle A_{i_1} \dots A_{i_m} \rangle \langle B_{j_1} \dots B_{j_n} \rangle , \\
&= 0 , \qquad (5.7)
\end{aligned}$$

where the last line follows from Eq. (5.4) of the problem set.

**(c)** Expanding the LHS of Eq. (5.8) of the problem set through third order,

$$\left\langle e^{\sum_i \lambda_i A_i} \right\rangle = 1 + \sum_i \lambda_i \langle A_i \rangle + \frac{1}{2!} \sum_{ij} \lambda_i \lambda_j \langle A_i A_j \rangle + \frac{1}{3!} \sum_{ijk} \lambda_i \lambda_j \lambda_k \langle A_i A_j A_k \rangle + \dots . \qquad (5.8)$$

The argument of the exponential on the RHS can be expanded similarly, so

$$\left\langle e^{\sum_i \lambda_i A_i} \right\rangle_{\bar{c}} - 1 = \sum_i \lambda_i \langle A_i \rangle_{\bar{c}} + \frac{1}{2!} \sum_{ij} \lambda_i \lambda_j \langle A_i A_j \rangle_{\bar{c}} + \frac{1}{3!} \sum_{ijk} \lambda_i \lambda_j \lambda_k \langle A_i A_j A_k \rangle_{\bar{c}} + \dots , \qquad (5.9)$$

so the RHS can be rewritten as

$$\begin{aligned}\text{RHS} = \exp\left\{\left\langle e^{\sum_i \lambda_i A_i}\right\rangle_{\bar{c}} - 1\right\} = 1 \\ &+ \left\{\sum_i \lambda_i \langle A_i\rangle_{\bar{c}} + \frac{1}{2!}\sum_{ij}\lambda_i\lambda_j \langle A_i A_j\rangle_{\bar{c}} + \frac{1}{3!}\sum_{ijk}\lambda_i\lambda_j\lambda_k \langle A_i A_j A_k\rangle_{\bar{c}} + \dots\right\} \\ &+ \frac{1}{2!}\left\{\sum_i \lambda_i \langle A_i\rangle_{\bar{c}} + \frac{1}{2!}\sum_{ij}\lambda_i\lambda_j \langle A_i A_j\rangle_{\bar{c}} + \frac{1}{3!}\sum_{ijk}\lambda_i\lambda_j\lambda_k \langle A_i A_j A_k\rangle_{\bar{c}} + \dots\right\}^2 \\ &+ \frac{1}{3!}\left\{\sum_i \lambda_i \langle A_i\rangle_{\bar{c}} + \frac{1}{2!}\sum_{ij}\lambda_i\lambda_j \langle A_i A_j\rangle_{\bar{c}} + \frac{1}{3!}\sum_{ijk}\lambda_i\lambda_j\lambda_k \langle A_i A_j A_k\rangle_{\bar{c}} + \dots\right\}^3 \\ &+ \dots\ .\end{aligned} \tag{5.10}$$

Now rearrange the terms on the RHS to organize them by powers of the $\lambda_i$:

$$\begin{aligned}\text{RHS} = 1 &+ \sum_i \lambda_i \langle A_i\rangle_{\bar{c}} + \frac{1}{2!}\sum_{ij}\lambda_i\lambda_j \left[\langle A_i A_j\rangle_{\bar{c}} + \langle A_i\rangle_{\bar{c}}\langle A_j\rangle_{\bar{c}}\right] \\ &+ \frac{1}{3!}\sum_{ijk}\lambda_i\lambda_j\lambda_k \left[\langle A_i A_j A_k\rangle_{\bar{c}} + 3\langle A_i A_j\rangle_{\bar{c}}\langle A_k\rangle_{\bar{c}} + \langle A_i\rangle_{\bar{c}}\langle A_j\rangle_{\bar{c}}\langle A_k\rangle_{\bar{c}}\right] + \dots\ .\end{aligned} \tag{5.11}$$

By comparing like powers in Eqs. (5.8) and (5.11), one has

$$\begin{aligned}\langle A_i\rangle &= \langle A_i\rangle_{\bar{c}} \\ \langle A_i A_j\rangle &= \langle A_i A_j\rangle_{\bar{c}} + \langle A_i\rangle_{\bar{c}}\langle A_j\rangle_{\bar{c}} \\ \langle A_i A_j A_k\rangle &= \langle A_i A_j A_k\rangle_{\bar{c}} + \langle A_i A_j\rangle_{\bar{c}}\langle A_k\rangle_{\bar{c}} + \langle A_i A_k\rangle_{\bar{c}}\langle A_j\rangle_{\bar{c}} + \langle A_j A_k\rangle_{\bar{c}}\langle A_i\rangle_{\bar{c}} \\ &\quad + \langle A_i\rangle_{\bar{c}}\langle A_j\rangle_{\bar{c}}\langle A_k\rangle_{\bar{c}}\ ,\end{aligned} \tag{5.12}$$

where one must remember to symmetrize $3\langle A_i A_j\rangle_{\bar{c}}\langle A_k\rangle_{\bar{c}}$ on the right-hand side of Eq. (5.11) before equating it with the coefficient of $\lambda_i\lambda_j\lambda_k$ on the other side of the equation. Using these relationships, one has immediately that

$$\langle A_i A_j\rangle_{\bar{c}} = \langle A_i A_j\rangle - \langle A_i\rangle\langle A_j\rangle = \langle A_i A_j\rangle_c\ , \tag{5.13}$$

and

$$\begin{aligned}\langle A_i A_j A_k\rangle_{\bar{c}} &= \langle A_i A_j A_k\rangle - \langle A_i A_j\rangle\langle A_k\rangle - \langle A_i A_k\rangle\langle A_j\rangle - \langle A_j A_k\rangle\langle A_i\rangle + 2\langle A_i\rangle\langle A_j\rangle\langle A_k\rangle \\ &= \langle A_i A_j A_k\rangle_c\ ,\end{aligned} \tag{5.14}$$

as desired.

**(d)** Here we assume that Eq. (5.13) of the problem set,

$$\langle A_1 \dots A_N e^B \rangle = \exp\left\{ \left\langle e^B \right\rangle_{\bar{c}} - 1 \right\} \times \sum_{\substack{\text{all partitions} \\ \text{of } \{1,\dots,N\} \\ \text{into subsets}}} \left\langle \left( \prod_{i \in S_1} A_i \right) e^B \right\rangle_{\bar{c}} \cdots \left\langle \left( \prod_{i \in S_k} A_i \right) e^B \right\rangle_{\bar{c}} , \tag{5.15}$$

with

$$B \equiv \sum_{i=1}^{M} \lambda_i A_i \ , \tag{5.16}$$

holds as written for some value of $N$. By differentiating both sides with respect to $\lambda_{N+1}$, we must show that the equation also holds when $N$ is replaced by $N+1$.

Differentiating the left-hand side gives immediately

$$\frac{\partial(\text{LHS})}{\partial \lambda_{N+1}} = \left\langle A_1 \dots A_N A_{N+1} e^B \right\rangle \ , \tag{5.17}$$

as desired.

To differentiate the RHS, note first that there exists a simple iterative method of generating all partitions of $\{1, \dots, N+1\}$. One starts with the partitions of $\{1, \dots, N\}$. For any partition of $\{1, \dots, N\}$ into $k$ subsets $S_1, \dots, S_k$, one can generate $k+1$ partitions of $\{1, \dots, N+1\}$ by appending $(N+1)$ to any of the subsets $S_i$ $(i = 1, \dots, k)$, or by creating an $S_{k+1}$ which contains only $(N+1)$. By this method each partition of $\{1, \dots, N+1\}$ is generated exactly once.

Now apply $(\partial/\partial\lambda_{N+1})$ to the RHS of Eq. (5.15). Consider a term in the sum corresponding to the partition of $\{1, \dots, N\}$ into $k$ subsets $S_1, \dots S_k$. Note that when the derivative acts on any one of the $k$ factors within the sum, it generates a new term which corresponds to appending $(N+1)$ to the corresponding subset $S_i$. When the derivative acts on the prefactor $\exp\left\{ \left\langle e^B \right\rangle_c - 1 \right\}$, it generates a term equal to the original expression multiplied by $\left\langle A_{N+1} e^B \right\rangle_c$, corresponding to the partition in which $(N+1)$ belongs to a new subset $S_{k+1}$. Thus we generate one term corresponding to each partition of $\{1, \dots, N+1\}$, proving the induction hypothesis:

$$\langle A_1 \dots A_{N+1} e^B \rangle = \exp\left\{ \left\langle e^B \right\rangle_{\bar{c}} - 1 \right\} \times \sum_{\substack{\text{all partitions} \\ \text{of } \{1,\dots,N+1\} \\ \text{into subsets}}} \left\langle \left( \prod_{i \in S_1} A_i \right) e^B \right\rangle_{\bar{c}} \cdots \left\langle \left( \prod_{i \in S_k} A_i \right) e^B \right\rangle_{\bar{c}} . \tag{5.18}$$

**(e)** When all the $\lambda_i$ are set to zero, Eq. (5.13) of the problem set, or Eq. (5.15) of these solutions, becomes

$$\langle A_1 \ldots A_N \rangle = \sum_{\substack{\text{all partitions} \\ \text{of } \{1,\ldots,N\} \\ \text{into subsets}}} \left\langle \left( \prod_{i \in S_1} A_i \right) \right\rangle_{\bar{c}} \cdots \left\langle \left( \prod_{i \in S_k} A_i \right) \right\rangle_{\bar{c}} . \qquad (5.19)$$

The RHS includes the partition in which all integers $\{1 \ldots N\}$ are in the same set $S_1$, which gives the term $\langle A_1 \ldots A_N \rangle_{\bar{c}}$. Separating out this term and bringing the rest to the opposite side of the equation, one has

$$\begin{aligned} \langle A_1 \ldots A_N \rangle_{\bar{c}} &= \langle A_1 \ldots A_N \rangle \\ &\quad - \sum_{\substack{\text{all proper} \\ \text{partitions of} \\ \{1,\ldots,N\}}} \left\langle \prod_{i \in S_1} A_i \right\rangle_{\bar{c}} \cdots \left\langle \prod_{i \in S_k} A_i \right\rangle_{\bar{c}} , \end{aligned} \qquad (5.20)$$

which is identical to Eq. (5.3) of the problem set, which defines the connected part $\langle A_1 \ldots A_N \rangle_c$. Thus the $\bar{c}$-connected parts obey the defining equation of the $c$-connected parts, and therefore they are equal.

**(f)** In evaluating

$$\left\langle \phi^2(x_1)\phi^2(x_2) \right\rangle \equiv \left\langle 0 \left| T\{\phi^2(x_1)\phi^2(x_2)\} \right| 0 \right\rangle , \qquad (5.21)$$

there are two forms of Wick contractions. One possibility is to contract each $\phi(x_1)$ with one of the $\phi(x_2)$ factors. There are two ways to do this, since the first $\phi(x_1)$ can contract with either of the $\phi(x_2)$ factors, and then the rest is determined. This contraction therefore gives $2\Delta_F^2(x_1 - x_2)$. The other possibility is to contract the two $\phi(x_1)$ factors, and the two $\phi(x_2)$ factors, which can be done in only one way. This gives $\Delta_F(x_1 - x_1) \times \Delta_F(x_2 - x_2) = \Delta_F^2(0)$ , so the sum is

$$\boxed{\left\langle 0 \left| T\{\phi^2(x_1)\phi^2(x_2)\} \right| 0 \right\rangle = 2\Delta_F^2(x_1 - x_2) + \Delta_F^2(0) .} \qquad (5.22)$$

To evaluate the connected part, we must subtract $\left\langle 0 \left| T\{\phi^2(x_1)\} \right| 0 \right\rangle \left\langle 0 \left| T\{\phi^2(x_2)\} \right| 0 \right\rangle = \Delta_F^2(0)$, which leaves

$$\boxed{\left\langle 0 \left| T\{\phi^2(x_1)\phi^2(x_2)\} \right| 0 \right\rangle_c = 2\Delta_F^2(x_1 - x_2) .} \qquad (5.23)$$

**(g)** We assume that the connected expectation value $\langle A_1 \ldots A_N \rangle_c$ can be evaluated as the sum of all connected graphs for $N < N_0$, and our goal is to show that it must also be true for $N = N_0$. A connected graph is one for which each $A_i$ is connected to every other $A_j$ by a chain of contractions. The definition of a connected expectation value is given by Eq. (5.3) of the problem set:

$$\begin{aligned}\langle A_1 \ldots A_N \rangle_c &= \langle A_1 \ldots A_N \rangle \\ &\quad - \sum_{\substack{\text{all proper} \\ \text{partitions of} \\ \{1,\ldots,N\}}} \left\langle \prod_{i\in S_1} A_i \right\rangle_c \cdots \left\langle \prod_{i\in S_k} A_i \right\rangle_c .\end{aligned} \tag{5.24}$$

The first term on the RHS is the sum of all ways of carrying out the Wick contractions, or equivalently the sum of all graphs, which of course includes all connected graphs. The second term, which contributes negatively, is a sum over all proper partitions of $\{1 \ldots N\}$, where the contribution for each proper partition is a product of connected expectation values. Since each of these expectation values has fewer than $N_0$ factors, our induction hypothesis allows us to identify the connected expectation value with the corresponding sum of connected graphs. Now consider a graph that contributes to $\langle A_1 \ldots A_N \rangle$ that is not connected, but instead consists of $k$ connected pieces, each of which is disconnected from the others. Each $A_i$ must then belong to one and only one connected piece, so the connected pieces define a partition of the integers $\{1 \ldots N\}$. If we let $S_i$ denote the set of integers for which $A_i$ is part of the $i$'th connected piece of the graph, then the graph can be seen to be a contribution to

$$\left\langle \prod_{i\in S_1} A_i \right\rangle_c \cdots \left\langle \prod_{i\in S_k} A_i \right\rangle_c . \tag{5.25}$$

Thus, any disconnected graph that contributes to the first term on the RHS of Eq. (5.24) is canceled by a graph that contributes to the sum that is subtracted on the RHS of Eq. (5.24). Only the disconnected graphs are uncanceled, and hence make a net contribution to $\langle A_1 \ldots A_N \rangle_c$.

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